UDC 511

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REMARKS ON NUMBER THEORY OVER ADDITIVE ARITHMETICAL SEMIGROUPS

ЗАУВАЖЕННЯ ДО ТЕОРІЇ ЧИСЕЛ ДЛЯ АДИТИВНИХ АРИФМЕТИЧНИХ НАПІВГРУП

We deal with additive arithmetical semigroups and present old and new proofs for the distribution of zeros of the corresponding ζ-functions. We use these results to prove prime number theorems and a Selberg formula for such semigroups.

Розглянуто додаткові арифметичні напівгрупи і наведено старі та нові доведення для розподілу нулів відповідних ζ-функцій. Ці результати використано для доведення теорем простих чисел і формули Сельберга для таких напівгруп.

1. Introduction. Abstract analytic number theory has arisen first as a generalization

of the classical number theory on the (semigroup) $\mathbb{N}$ of natural numbers with the special emphasis on the derivation of the famous *Prime Number Theorem:* If $\pi(x)$ denotes the total number of positive rational primes $\leq x$, then $\pi(x) \sim \dfrac{x}{\log x}$ as $x \to \infty$,

and

of Landau's classical *Prime Ideal Theorem*, which extends the Prime Number Theorem to the (semigroup) G_K of integral ideals in an algebraic number field K.

There are a large number of mathematical systems, particularly ones arising in abstract algebra, which have elementary "unique factorization" properties analogous to those of the positive integers.

In the case of many of these systems there is an additional property, which make them more "arithmetical" in a sense, and more treatable by techniques of classical number theory. This property comes from the existence of a function measuring the "size" of an individual object (usually the object's cardinality or "degree" in some sense) with the essential attribute that there are only a finite number of inequivalent objects whose "size" does not exceed any chosen bound.

Motivated by such systems Knopfmacher [17] introduced the formal concept of an *additive arithmetical semigroup*, which he defined to be a commutative semigroup G with an (additive) degree mapping on G. To be more precise, let G be a free commutative semigroup with identity element 1, generated by a countable set P of primes and admitting an integer valued degree mapping $\partial:$ $G \to \mathbb{N} \cup \{0\}$ with the properties

(i) $\partial(1) = 0$ and $\partial(p) > 0$ for all $p \in P$,

(ii) $\partial(ab) = \partial(a) + \partial(b)$ for all $a, b \in G$,

(iii) the total number $G(n)$ of elements $a \in G$ of degree $\partial(a) = n$ is finite for each $n \geq 0$.

Then (G, ∂) is called an *additive arithmetical semigroup*. Obviously, $G(0) = 1$ and G is countable.

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Let

$$\pi(n) := \#\{p \in P : \partial(p) = n\}$$

denote the total number of primes of degree n in G. If we consider $G(n)$ together with its associated *enumeration function*

$$F(y) := 1 + \sum_{n=1}^{\infty} G(n) y^n,$$

then the identity, at least in the formal sense,

$$F(y) = 1 + \sum_{n=1}^{\infty} G(n) y^n = \prod_{n=1}^{\infty} (1 - y^n)^{-\pi(n)}$$

holds.

$F(y)$ is called the *generating function (or zeta function)* of the additive arithmetical semigroup G.

***Remark* 1.** It is worthwhile to mention here some properties of the sequences $\gamma(n)$, $\beta(n) \in \mathbb{R}$, which are formally related by

$$1 + \sum_{n=1}^{\infty} \gamma(n) y^n = \prod_{n=1}^{\infty} (1 - y^n)^{-\beta(n)}$$

and which have been used in [14, p. 448]:

$$\beta(n) \in \mathbb{Z} \quad \text{for all} \quad n \in \mathbb{N} \Leftrightarrow \gamma(n) \in \mathbb{Z} \quad \text{for all} \quad n \in \mathbb{N} \tag{1.1}$$

and

$$\beta(n) \geq 0 \quad \text{for all} \quad n \in \mathbb{N} \Rightarrow \gamma(n) \geq \beta(n) \quad \text{for all} \quad n \in \mathbb{N}. \tag{1.2}$$

Condition (1.2) is obvious. Concerning (1.1) we assume that $\gamma(n)$ is integer-valued for all $n \in \mathbb{N}$. Then

$$1 + \sum_{n=1}^{\infty} \gamma(n) y^n = (1 - y)^{-\beta(1)} (1 + \alpha y^2 + \ldots) = 1 + \beta(1) y + \ldots$$

and $\beta(1) \in \mathbb{Z}$. Now, assume that $k \in \mathbb{N} \setminus \{1\}$ is the smallest number such that $\beta(k) \notin \mathbb{Z}$. Then

$$\left(1 + \sum_{n=1}^{\infty} \gamma(n) y^n\right) \left(\prod_{m=1}^{k-1} (1 - y^m)^{\beta(m)}\right) =$$

$$= (1 - y^k)^{-\beta(k)} (1 + \alpha' y^{k+1} + \ldots) = 1 + \beta(k) y^k + \ldots . \tag{1.3}$$

The left-hand side of (1.3) is a power series with integer coefficients, and, thus, $\beta(k) \in \mathbb{Z}$. This contradiction proves $\beta(n) \in \mathbb{Z}$ for all $n \in \mathbb{N}$. The implication of the opposite direction in (1.1) is obvious. A corresponding result for the formal representation

$$1 + \sum_{n=1}^{\infty} \gamma(n) y^n = \prod_{n=1}^{\infty} (1 + \varrho(n) y^n)$$

may be found in [15] (Proposition 1).

The following examples show that degree functions arise in various forms.

***Example* 1** (monic polynomials over $GF(q)$)**.** A simple example of an additive arithmetical semigroup is provided by the multiplicative semigroup G_q of all monic polynomials in one indeterminate X over the finite field $GF(q)$ with q elements and with $\partial(a) = \deg(a)$ for $a \in G_q$, where $\deg(a)$ denotes the degree of the polynomial a. Here $G(n) = q^n$, and the generating function may be written as

$$F(y) = \sum_{n=0}^{\infty} q^n y^n = \prod_{n=1}^{\infty} (1 - y^n)^{-\pi_q(n)},$$

and

$$\pi_q(n) = \frac{1}{n} \sum_{r|n} \mu(r) q^{n/r}$$

can be deduced as an algebraic consequence of the Euler product for $F(y)$.

***Example* 2** (multisets)**.** Let P be a finite or denumerable set. Following Flajolet and Sedgewick [6], we use the notation

$$SEQ(P) := \big\{(p_1, \ldots, p_l) \colon l \geq 0,\ p_i \in P,\ i = 1, \ldots, l\big\},$$

where the element for $l = 0$ corresponds to the identity element 1 and the size $|a|$ of an object $a \in SEQ(P)$ is a nonnegative integer and is to be taken as the sum of the sizes of its components,

$$a = (p_1, \ldots, p_l) \mapsto |a| = |p_1| + \ldots + |p_l|.$$

Then we define the multiset $MSET(P)$ as the quotient

$$MSET(P) := SEQ(P)/R,$$

where the equivalence relation R is defined by $(p_1, \ldots, p_r)R(q_1, \ldots, q_r)$ if there exists some arbitrary permutation τ of $1, \ldots, r$ such that $q_j = p_{\tau(j)}$ for all j (see [6, p. 26]). Obviously, a multiset $MSET(P)$ (together with a size function $|\cdot|$) is nothing else but an additive arithmetical semigroup G with $G = MSET(P)$, where P denotes the set of primes, and the degree $\partial(p)$ is given by the size $|p|$. Then $\pi(n)$ is the number of objects in P that have size n.

***Example* 3** (partitions)**.** As a special example of a multiset we choose $P = \mathbb{N}$ with $\partial(j) = j$ for $j \in \mathbb{N}$. If $a = (n_1, \ldots, n_l) \in MSET(\mathbb{N})$, then $\partial(a) = n_1 + \ldots + n_l$. Obviously,

$$\pi(n) = \#\{p \in P \colon \partial(p) = n\} = 1$$

and

$$G(n) = p(n) := \text{number of partitions of } n.$$

The generating function $F(y)$ is given by

$$F(y) = 1 + \sum_{n=1}^{\infty} p(n) y^n = \prod_{n=1}^{\infty} (1 - y^n)^{-1}$$

and converges for $|y| < 1$.

***Example* 4** (monic polynomials over $GF(q)$ in several indeterminates)**.** For an integer $k \geq 2$, let $G_{k,q}$ be the multiplicative semigroup of all monic polynomials in k indeterminates $X_1, \ldots, X_k$ over a finite field $GF(q)$, where $\partial(f)$ is the degree of $f \in G_{k,q}$. The prime elements are the (monic) irreducible polynomials in $G_{k,q}$. Carlitz [4] has proven that

$$\lim_{n\to\infty} \frac{\pi(n)}{G(n)} = 1. \tag{1.4}$$

This shows, that "almost all" elements of $G_{k,q}$ are primes in the sense that $G(n) > 0$ for sufficiently large n and $\pi(n) \sim G(n)$ as $n \to \infty$. Further, by a result of Wright (see Theorem 3 in [20]), (1.4) implies that the generating function

$$F(y) = 1 + \sum_{n=1}^{\infty} G(n) y^n = \prod_{n=1}^{\infty} (1 - y^n)^{-\pi(n)}$$

diverges for all $y \in \mathbb{C}$.

In order to develop an arithmetical theory we assume that

$$G(n) \ll q^n n^{\varrho} \quad \text{with some} \quad q > 1, \quad \varrho \in \mathbb{R},$$

so that $F(y)$ is holomorphic for $|y| < q^{-1}$. The logarithmic derivate of F is given by

$$\frac{F'(y)}{F(y)} = \sum_{n=1}^{\infty} \left(\sum_{d|n} d\pi(d) \right) y^{n-1}. \tag{1.5}$$

Putting

$$\lambda(n) = \sum_{d|n} d\pi(d),$$

gives

$$F(y) = 1 + \sum_{n=1}^{\infty} G(n) y^n = \exp\left(\sum_{n=1}^{\infty} \frac{\lambda(n)}{n} y^n \right).$$

By Möbius inversion formula

$$n\pi(n) = \sum_{d|n} \lambda(d) \mu\left(\frac{n}{d}\right).$$

A straightforward calculation (see [9, p. 86]) shows that

$$\lambda(n) = n\pi(n) + O\Big(nq^{n/2} \left(\frac{n}{2}\right)^{\varrho} \log n\Big).$$

An *abstract prime number theorem* (for additive arithmetical semigroups) is a theorem about the asymptotic behavior of $\pi(n)$ and $\lambda(n)$, respectively.

Further, we shall concentrate on additive arithmetical semigroups whose zeta function F can be written in the form

$$F(y) = \frac{H(y)}{(1 - qy)^{\delta}} \quad \text{with some} \quad \delta > 0,$$

where $H(y)$ is holomorphic for $|y| < q^{-1}$ and continuous for $|y| \le q^{-1}$ and $H(q^{-1}) \neq 0$. By a result of Indlekofer [11] this implies $H(q^{-1}) > 0$.

$F(y)$ has no zeros in the disk $\{y \in \mathbb{C} : |y| < q^{-1}\}$ but may have zeros on the circle $\{y \in \mathbb{C} : |y| = q^{-1}\}$. If $q^{-1}e^{2\pi it}$ is a zero of $F(y)$, the number

$$\alpha(t) := \sup\left\{\alpha : \limsup_{r \nearrow q^{-1}} (q^{-1} - r)^{-\alpha} |F(re^{2\pi it})| < \infty\right\} \tag{1.6}$$

or, equivalently, following Beurling [3],

$$\alpha(t) := \liminf_{r \nearrow q^{-1}} \frac{\log |F(re^{2\pi it})|}{\log(q^{-1} - r)} \tag{1.7}$$

is called, by definition, *the order of the zero* $q^{-1}e^{2\pi it}$.

In this paper we deal with the “total number” of zeros of $F(y)$, which is the key to the investigation of the abstract prime number theorem. In the case $\delta = 1$, we assume in addition, that $H(y)$ is holomorphic for $|y| < R$, where $R > q^{-1}$. In the general case $\delta > 0$, we only assume, that $H(y)$ is holomorphic for $|y| < q^{-1}$ and continuous for $|y| \le q^{-1}$, and $H(q^{-1}) > 0$.

As applications we present several abstract prime number theorems and a corresponding Selberg formula.

***Remark* 2.** In several papers we have investigated the mean behavior of additive and multiplicative functions on additive arithmetical semigroups (see [1, 2, 10, 16]). In this context it is recommendable, to deal with probabilistic aspects, as it has been done, for example, in [12] and [13] for functions defined on the (multiplicative) semigroup $\mathbb{N}$ of natural numbers.

2. Zeros of the $\boldsymbol{\zeta}$-function. We put

$$F(y) = \frac{H(y)}{(1 - qy)^{\delta}}, \quad \delta > 0, \tag{2.1}$$

and consider the special case $\delta = 1$ separately from the general case $\delta > 0$.

2.1. The case $\boldsymbol{\delta = 1}$. When Knopfmacher [17] introduced the concept of the additive arithmetical semigroup his investigations are based on the following axiom.

Axiom $\boldsymbol{\mathcal{A}^{\#}}$. *There exist constants* $A > 0$, $q > 1$ *and* ν *with* $0 \le \nu < 1$ (*all depending on* G), *such that*

$$G(n) = Aq^n + O(q^{\nu n}) \quad \textit{as} \quad n \to \infty.$$

If G satisfies Axiom $\mathcal{A}^{\#}$, then the generating function

$$F(y) = \sum_{n=0}^{\infty} G(n) y^n$$

is holomorphic in the disc $|y| < q^{-\nu}$ up to a pole of order one at $y = q^{-1}$, and we get

$$F(y) = \frac{A}{1 - qy} + H_1(y),$$

where

$$H_1(y) = \sum_{n=0}^{\infty} r_n y^n$$

with

$$r_n := G(n) - Aq^n.$$

Putting

$$H(y) := A + (1 - qy)H_1(y),$$

gives

$$F(y) = \frac{H(y)}{1 - qy}$$

with $H(0) = 1$ and $H(q^{-1}) = A$. H and H_1 are holomorphic for $|y| < q^{-\nu}$.

Chapter 8 of [17] deals with the abstract prime number theorem:

If the additive arithmetical semigroup G satisfies Axiom $\mathcal{A}^{\#}$, then

$$\lambda(n) = q^n + O\left(\frac{q^n}{n^{\alpha-1}}\right), \quad n \to \infty,$$

or, equivalently,

$$\pi(n) = \frac{q^n}{n} + O\left(\frac{q^n}{n^{\alpha}}\right), \quad n \to \infty,$$

is true for any $\alpha > 1$.

Note that this result is only valid if $F(-q^{-1}) \neq 0$. In [14], Indlekofer, Manstavičius and Warlimont gave (in a more general setting) much sharper results valid also in the case $F(-q^{-1}) = 0$.

We write $y = re^{2\pi it}$ and assume that there is some $R > q^{-1}$ such that $H(y)$ is holomorphic for $r < R$ and $H(q^{-1}) \neq 0$. If $H(y) \neq 0$ for $|y| \leq q^{-1}$, then there exists some r, $q^{-1} < r < R$, such that $H(y) \neq 0$ for $|y| \leq r$. This is contained in the following theorem.

Theorem 1 ([14], Theorem 1). *If $H(y) \neq 0$ for $|y| = q^{-1}$, then there exists $0 < \vartheta < 1$ such that $H(y) \neq 0$ for $|y| \leq q^{-\vartheta}$.*

If $F(y)$ has zeros on the circle $\{y \in \mathbb{C} \colon |y| = q^{-1}\}$, then the following two theorems hold.

Theorem 2 ([14], Theorem 1). *If $F(y)$ has zeros of modulus q^{-1}, then F has exactly one zero in the disk $\{y \in \mathbb{C} \colon |y| < R\}$. It is located at $y = -q^{-1}$ and has order 1.*

Theorem 3 ([14], Theorem 2). *Asssume $R \geq q^{-1/2}$. If $F(-q^{-1}) = 0$ we have*

$$F(r)F(-r) \gg \left(1 - q^{\frac{1}{2}}r\right)^{-1} \quad \textit{for} \quad r \nearrow q^{-1/2}. \tag{2.2}$$

The main part of the proof of Theorem 3 is to show that

$$F(r)F(-r) \geq F(r^2) \quad \text{for} \quad r \nearrow q^{-1/2}, \tag{2.3}$$

from which the assertion (2.2) follows immediately since $H(q^{-1}) > 0$. Furthermore, (2.3) implies directly the following corollary.

***Corollary* 1** ([14], Corollary 2). *Assume $R \geq q^{-1/2}$. If*

$$\liminf_{r \nearrow q^{-1/2}} (1 - rq^{1/2})F(r)F(-r) \leq 0,$$

then $F(y)$ has no zeros for $r = q^{-1}$.

The investigation continued with the restriction that $H(y)$ is holomorphic for $|y| < q^{-1}$ and continuous for $|y| \le q^{-1}$. By a result in [11] this implies $H(q^{-1}) > 0$. Zhang required assumptions on the coefficients of the ζ-function (see [21]) whereas Indlekofer assumed in [8] a specific boundary behavior of H. This reads as the following axiom.

Axiom A_1 **[8].** *$H(y)$ is continuous for $|y| \le q^{-1}$, $H'(y)$ is bounded for $|y| < q^{-1}$.*

Axiom A_1 is much less restricted than Axiom $A^{\#}$ and seems to be the weakest condition known today which ensures a Chebyshev type upper estimate $\pi(n) \ll \dfrac{q^n}{n}$. Some conclusions which can be derived from Axioam A_1 without appealing to Axiom $A^{\#}$ are given in the following theorem.

Theorem 4 ([8], Theorem 1)**.** *If Axiom A_1 holds, then either $H(y) \neq 0$ for every $|y| \le q^{-1}$ or*

$$f(y) := \frac{H(y)}{1+qy}$$

defines a function f, which is holomorphic for $|y| < q^{-1}$ and continuous and different from zero for all $|y| \le q^{-1}$.

2.2. The case $\delta > 0$. First, we prove the following theorem.

Theorem 5. *Let*

$$g(y) = \frac{h(y)}{(1-y)^{\delta}}, \quad \delta > 0,$$

be holomorphic and different from zero for $|y| < 1$. Assume that $h(y)$ is continuous for $|y| \le 1$ and $h(1) > 0$. Also assume that

$$\log g(y) = \sum_{m=1}^{\infty} a_m y^m, \quad |y| < 1, \tag{2.4}$$

where the coefficients a_m are nonnegative. Let $0 < t_1 < \ldots < t_n < 1$ be given. Then

$$\sum_{j=1}^{n} \liminf_{r \nearrow 1^-} \frac{\log |g(re^{2\pi i t_j})|}{\log(1-r)} \le \delta. \tag{2.5}$$

Remark **3.** This theorem has been proven by Zhang in [21] (Theorem 4.2). Here we give a different proof which is shorter and uses ideas from Beurling's paper [3].

Proof of Theorem **5.** We start with the identity $(0 < r < 1)$

$$\prod_{i=1}^{p} \frac{1-r^2}{1-2r\cos y_i + r^2} = \sum_{\nu_1 \in \mathbb{Z}} \ldots \sum_{\nu_p \in \mathbb{Z}} r^{|\nu_1|+\ldots+|\nu_p|} \cos(\nu_1 y_1 + \ldots + \nu_p y_p) \tag{2.6}$$

(cf. [3, p. 281]). Put

$$\boldsymbol{\nu} := (\nu_1, \ldots, \nu_p) \in \mathbb{Z}^p,$$

$$\|\boldsymbol{\nu}\| := \sum_{i=1}^{p} |\nu_i|$$

and let $0 < t_1 < \ldots < t_n < 1$. Consider a maximal subset of elements, which are linearly independent over $\mathbb{Q}$, of the set $\{t_1, \ldots, t_n\}$. Then there exist $s_1, \ldots, s_p$, which are linearly independent over $\mathbb{Q}$, such that

$$t_j = \sum_{i=1}^{p} c_{ji} s_i, \quad j = 1, \ldots, n, \tag{2.7}$$

with $c_{ji} \in \mathbb{Z},\ j = 1, \ldots, n,\ i = 1, \ldots, p$. If the $t_1, \ldots, t_n$ are linearly independent then choose $s_j = t_j,\ j = 1, \ldots, n$, and

$$c_{ji} = \begin{cases} 1, & j = i, \\ 0, & j \neq i. \end{cases}$$

Put in (2.6)

$$y_i = s_i y, \quad i = 1, \ldots, p.$$

Then for every $\varrho, 0 < \varrho < 1$, there exists $N_0 = N_0(\varrho)$, such that for every $N \geq N_0$

$$\sum_{\substack{\|\boldsymbol{\nu}\| > N \\ \boldsymbol{\nu} \in \mathbb{Z}^p}} \varrho^{\|\boldsymbol{\nu}\|} \leq \varrho^N \sum_{\boldsymbol{v} \in \mathbb{Z}^p} \varrho^{\|\boldsymbol{\nu}\|} \leq \left(\frac{1 - \varrho}{1 + \varrho} \right)^p$$

and, by (2.6),

$$0 \leq P(y) := \sum_{\|\boldsymbol{\nu}\| \leq N} \varrho^{\|\boldsymbol{\nu}\|} \cos(\boldsymbol{\nu}, \boldsymbol{s}) y, \tag{2.8}$$

where

$$\langle \boldsymbol{\nu}, \boldsymbol{s} \rangle = \nu_1 s_1 + \ldots + \nu_p s_p.$$

Let $y = 2\pi m$ with $m \in \mathbb{N}$. We multiply (2.8) by $a_m r^m$ and sum over m. Then, by (2.4), for $0 < r < 1$,

$$0 \leq \sum_{\substack{\|\boldsymbol{\nu}\| \leq N \\ \langle \boldsymbol{\nu}, \boldsymbol{s} \rangle \in \mathbb{Z}}} \varrho^{\|\boldsymbol{\nu}\|} \log g(r) + \sum_{\substack{\|\boldsymbol{\nu}\| \leq N \\ \langle \boldsymbol{\nu}, \boldsymbol{s} \rangle \notin \mathbb{Z}}} \varrho^{\|\boldsymbol{\nu}\|} \log \left| g(re^{2\pi i \langle \boldsymbol{\nu}, \boldsymbol{s} \rangle}) \right|.$$

It follows that

$$\sum_{\substack{\|\boldsymbol{\nu}\| \leq N \\ \langle \boldsymbol{\nu}, \boldsymbol{s} \rangle \notin \mathbb{Z}}} \varrho^{\|\boldsymbol{\nu}\|} \frac{\log |g(re^{2\pi i \langle \boldsymbol{\nu}, \boldsymbol{s} \rangle})|}{\log(1 - r)} \leq \sum_{\substack{\|\boldsymbol{\nu}\| \leq N \\ \langle \boldsymbol{\nu}, \boldsymbol{s} \rangle \in \mathbb{Z}}} \varrho^{\|\boldsymbol{v}\|} \frac{\log g(r)}{\log \dfrac{1}{1 - r}}. \tag{2.9}$$

Now, define

$$T_0 := \left\{ \boldsymbol{\nu} \in \mathbb{Z}^p : \langle \boldsymbol{\nu}, \boldsymbol{s} \rangle \in \mathbb{Z} \right\}$$

and, for $j = 1, \ldots, n$,

$$T_j := \left\{ \boldsymbol{\nu} \in \mathbb{Z}^p : \langle \boldsymbol{\nu}, \boldsymbol{s} \rangle + t_j \in \mathbb{Z} \right\}.$$

Obviously, putting $\boldsymbol{\nu_j} = (c_{j1}, \ldots, c_{jp})$,

$$T_j := \{ \boldsymbol{\nu} + \boldsymbol{\nu_j} : \boldsymbol{\nu} \in T_0 \}.$$

Note that $T_0, T_1, \ldots, T_n$ are mutually disjoint. From (2.9) we have

$$\sum_{j=1}^{n} \sum_{\substack{\|\boldsymbol{\nu}\| \leq N \\ \boldsymbol{\nu} \in T_j}} \varrho^{\|\boldsymbol{\nu}\|} \frac{\log |g(re^{2\pi i t_j})|}{\log(1 - r)} + \sum_{\substack{\|\boldsymbol{\nu}\| \leq N \\ \boldsymbol{\nu} \notin T_0 \cup \left(\bigcup_{j=1}^{n} T_j \right)}} \varrho^{\|\boldsymbol{\nu}\|} \frac{\log |g(re^{2\pi i \langle \boldsymbol{\nu}, \boldsymbol{s} \rangle})|}{\log(1 - r)} \leq \sum_{\substack{\|\boldsymbol{\nu}\| \leq N \\ \boldsymbol{\nu} \in T_0}} \varrho^{\|\boldsymbol{\nu}\|} \frac{\log g(r)}{\log \dfrac{1}{1 - r}}.$$

Note that

$$\lim_{r \to 1^-} \frac{\log g(r)}{\log \dfrac{1}{1 - r}} = \delta$$

and

$$\liminf_{r\to 1^-} \frac{\log|g(re^{2\pi i(\underline{\nu},\underline{s})})|}{\log(1-r)} \geq 0$$

if $\langle \boldsymbol{\nu}, \boldsymbol{s}\rangle \notin \mathbb{Z}$. This implies, letting $N \to \infty$,

$$\sum_{j=1}^{n} \liminf_{r\to 1^-} \frac{\log|g(re^{2\pi i t_j})|}{\log(1-r)} \sum_{\boldsymbol{\nu}\in T_j} \varrho^{\|\boldsymbol{\nu}\|} \leq \delta \sum_{\boldsymbol{\nu}\in T_0} \varrho^{\|\boldsymbol{\nu}\|}. \tag{2.10}$$

Now, either

$$T_0 = \{\mathbf{0}\}$$

or there exists $\boldsymbol{\nu_0} \in \mathbb{Z}^p$ such that

$$T_0 = \{\boldsymbol{\nu} = t\boldsymbol{\nu_0} : t \in \mathbb{Z}\}. \tag{2.11}$$

For the proof of (2.11), assume that, for some $\boldsymbol{\nu} \in T_0$, $\langle \boldsymbol{\nu}, \boldsymbol{s}\rangle = a \in \mathbb{Z}$ and $a \neq 0$. Then the set $A := \big\{a : 0 < a = \langle \boldsymbol{\nu}, \boldsymbol{s}\rangle \text{ for some } \boldsymbol{\nu} \in T_0\big\}$ is non-empty and has a minimal element $a_0 := \min\{a : a \in A\}$ with $a_0 = \langle \boldsymbol{\nu_0}, \boldsymbol{s}\rangle$ for some (unique) $\boldsymbol{\nu_0} \in \mathbb{Z}^p$. Since $s_1, \ldots, s_p$ are linearly independent, $\boldsymbol{\nu} \in T_0$ if and only if there exist $t \in \mathbb{Z}$ such that $\boldsymbol{\nu} = t\boldsymbol{\nu_0}$.

If $T_0 = \{\mathbf{0}\}$, then, by (2.10),

$$\sum_{j=1}^{n} \liminf_{r\to 1^-} \frac{\log|g(re^{2\pi i t_j})|}{\log(1-r)} \varrho^{\|\boldsymbol{\nu}_j\|} \leq \delta\varrho,$$

and letting $\varrho \to 1^-$ gives the assertion (2.5) of Theorem 5. In the case (2.11) we put

$$\sum\nolimits_0 := \sum_{\boldsymbol{\nu}\in T_0} \varrho^{\|\boldsymbol{\nu}\|} = \sum_{t\in\mathbb{Z}} \varrho^{|t|\|\boldsymbol{\nu_0}\|}, \tag{2.12}$$

$$\sum\nolimits_j := \sum_{\boldsymbol{\nu}\in T_j} \varrho^{\|\boldsymbol{\nu}\|} = \sum_{t\in\mathbb{Z}} \varrho^{\|\boldsymbol{\nu}_j + t\boldsymbol{\nu_0}\|} \tag{2.13}$$

and arrive at

$$\sum\nolimits_0 = 1 + 2\sum_{n\in\mathbb{N}} \varrho^{n\|\boldsymbol{\nu_0}\|} = \frac{1}{1-\varrho^{\|\boldsymbol{\nu_0}\|}}\big(1 - \varrho^{\|\boldsymbol{\nu_0}\|} + 2\varrho^{\|\boldsymbol{\nu_0}\|}\big).$$

In the same way we obtain

$$\varrho^{\|\boldsymbol{\nu}_j\|} \sum\nolimits_0 \leq \sum\nolimits_j \leq \varrho^{-\|\boldsymbol{\nu}_j\|} \sum\nolimits_0, \quad j = 1, \ldots, n,$$

which implies

$$\lim_{\varrho\to 1} \frac{\sum_j}{\sum_0} = 1. \tag{2.14}$$

Then (2.14) proves, by (2.10), (2.12) and (2.13), the assertion of Theorem 5.

We apply Theorem 5 to the ζ-function $F(y)$. Observe that, if $y = q^{-1}e^{2\pi i t}$, $0 < t < \dfrac{1}{2}$, is a zero of $F(y)$, then $y = q^{-1}e^{-2\pi i t}$ is a zero of $F(y)$, too. By using the notations (1.6) and (1.7), respectively, we arrive at the following theorem.

Theorem 6. *Let the ζ-function $F(y)$ of G be given by* (2.1), *where $H(y)$ is holomorphic for $|y| < q^{-1}$ and continuous for $|y| \le q^{-1}$, and $H(q^{-1}) = 0$. Then*

$$\alpha\left(\frac{1}{2}\right) + \sum_{0<t<\frac{1}{2}} \alpha(t) \le \delta$$

or

$$2 \sum_{0<t<\frac{1}{2}} \alpha(t) \le \delta$$

according as $y = q^{-1}$ is or is not a zero of $F(y)$.

Theorem 7. *Let $k \in \mathbb{N}$ be defined by*

$$k := \begin{cases} [\delta] + 1, & \text{if} \quad \delta \notin \mathbb{N}, \\ \delta, & \text{if} \quad \delta \in \mathbb{N}. \end{cases}$$

Assume that $H^{(k)}(y) = O(1)$ for $|y| < q^{-1}$. Then the order α of a zero of H is a positive integer.

Proof. All derivatives $H^{(j)}(y)$ with $j < k$ are continuous for $|y| \le q^{-1}$. Hence, we obtain by Taylor's formula for $0 < r < q^{-1}$

$$H(re^{2\pi i\vartheta}) = \sum_{n=1}^{k-1} \frac{1}{n!} H^{(n)}\big(q^{-1}e^{2\pi i\vartheta}\big) e^{2\pi i n\vartheta} (r - q^{-1})^n +$$

$$+\frac{1}{(k-1)!} \int_{q^{-1}}^{r} H^{(k)}\big(te^{2\pi i\vartheta}\big) e^{2\pi i k\vartheta} (r - t)^{k-1} dt. \tag{2.15}$$

The last term of (2.15) can be estimated by $O\left(\frac{1}{k!}(r - q^{-1})^k\right)$.

Suppose that α is not an integer, i.e.,

$$m - 1 < \alpha < m \le k.$$

Then by (2.15), $H^{(j)}\big(q^{-1}e^{2\pi i\vartheta}\big) = 0$ for $j < m$ and

$$\frac{H(re^{2\pi i\vartheta})}{(q^{-1} - r)^{\alpha'}} = O\left((q^{-1} - r)^{m-\alpha'}\right) \quad \text{as} \quad r \to q^{-1}$$

for every α', $\alpha < \alpha' \le m$, which contradicts the assumption on α. Therefore, α must be a positive integer.

By Theorems 6 and 7 we obtain the following corollaries.

***Corollary* 2.** *Let $\delta = 1$ and $H'(y) = O(1)$ for $|y| < q^{-1}$. Then $H(y)$ has either no zeros on the circle $|y| = q^{-1}$, or exactly one zero $y = -q^{-1}$ of order one.*

Corollary 2 is contained in Theorem 4.

***Corollary* 3.** *Let $0 < \delta < 1$ and $H'(y) = O(1)$ for $|y| < q^{-1}$. Then $H(y)$ has no zeros on the circle $|y| = q^{-1}$.*

***Corollary* 4.** *Let $1 < \delta < 2$ and $H''(y) = O(1)$ for $|y| < q^{-1}$. Then $H(y)$ has either no zeros on the circle $|y| = q^{-1}$, or exactly one zero $y = -q^{-1}$ of order one.*

The condition $H'(y) = O(1)$ implies that $\alpha(\nu) \geq 1$ if $y = q^{-1}e^{2\pi i\nu}$ is a zero of $H(y)$. Thus the following result holds.

***Corollary* 5.** *Let $1 < \delta < 2$ and $H'(y) = O(1)$ for $|y| < q^{-1}$. Then $H(y)$ has either no zeros on the circle $|y| = q^{-1}$, or exactly one zero $y = -q^{-1}$ of order $\leq \delta$.*

The following example could be considered in the context of Theorem 6 and Corollary 3, whereas Example 6 illuminates the strength of the condition in Corollary 2.

***Example* 5.** Let r_1, s_1, r_2, s_2 be positive integers such that $\delta = \dfrac{r_1}{s_1}$, $\alpha = \dfrac{r_2}{s_2} \in \mathbb{Q}$ with $0 < \alpha < \delta$. Consider $q = ms_1s_2$, where $m > 2\dfrac{\delta}{r_1s_2 - s_1r_2} + 1$ holds. Put

$$\lambda(n) = q^n\big(\delta + (-1)^{n+1}\alpha\big), \quad n = 1, 2, \ldots .$$

Then the $\lambda(n)$'s are all positive. Furthermore, $\pi(1) = \lambda(1) > 0$ and, for $n \geq 2$,

$$n\pi(n) = \sum_{d|n} \lambda(d)\mu\left(\frac{n}{d}\right) \geq \lambda(n) - \sum_{1\leq d\leq \frac{n}{2}} \lambda(d) \geq$$

$$\geq q^n(\delta - \alpha) - 2\delta \sum_{1\leq d\leq \frac{n}{2}} q^d \geq q^n(\delta - \alpha) - 2\delta\frac{qq^{\frac{n}{2}}}{q-1} =$$

$$= q^{\frac{n}{2}}\left(q^{\frac{n}{2}}(\delta - \alpha) - \frac{2\delta q}{q-1}\right) > (\delta - \alpha)(q - 1) - 2\delta > 0. \tag{2.16}$$

A straightforward calculation shows that

$$y \cdot \frac{F'(y)}{F(y)} = \sum_{n=1}^{\infty} \lambda(n)y^n = \frac{\delta qy}{1 - qy} + \frac{\alpha qy}{1 + qy}$$

and

$$F(y) = \frac{(1 + qy)^{\alpha}}{(1 - qy)^{\delta}}, \quad |y| < q^{-1},$$

where $0 < \alpha < \delta$. Then $G(0) = 1$ and

$$G(n) = q^n \sum_{k=0}^{n} \binom{k + \delta - 1}{\delta - 1}\binom{\alpha}{n - k}, \quad n = 1, 2, \ldots,$$

are positive integers. Therefore, by Remark 1 and (2.16), $\pi(n) \in \mathbb{N}$.

***Example* 6.** This example is motivated by Example 6.5 of [22]. Consider

$$\lambda(n) = (1 - \cos n\theta)\, q^n, \quad n = 1, 2, \ldots, \tag{2.17}$$

where $q = 2 \cdot 5^4$ and $\cos\theta = \dfrac{4}{5}$. Since

$$\sin n\theta = \operatorname{Im}\left(\left(\frac{4 + 3i}{5}\right)^n\right) = \operatorname{Im}\left(\left(\frac{i^n}{5^n}\right)(3 - 4i)^n\right) =$$

$$= \begin{cases} \pm\frac{1}{5^n}\left(3^n - \binom{n}{2}3^{n-2}4^2 + \ldots\right), & \text{if } n \text{ is odd,} \\ \frac{1}{5^n}\left(\binom{n}{1}4^{n-1}3 - \binom{n}{3}4^{n-3}3^3 + \ldots\right), & \text{if } n \text{ is even,} \end{cases}$$

we see that $\sin n\theta \neq 0$. Thus $1 - \cos n\theta > 0$. Furthermore, we have

$$1 - \cos n\theta = \frac{1}{5^n}\left(5^n - \mathrm{Re}(4+3i)^n\right) \geq \frac{1}{5^n}, \quad n = 1, 2, \ldots.$$

Since $5^n - \mathrm{Re}(4+3i)^n \in \mathbb{Z}$ we obtain that $\lambda(n)$ is a positive integer for every $n \in \mathbb{N}$. Now, we shall show that

$$\pi(n) = \frac{1}{n}\sum_{d|n}(1 - \cos d\theta)q^d \mu\left(\frac{n}{d}\right)$$

is a positive integers for every $n \geq 1$, too. Obviously,

$$n\pi(n) = \sum_{d|n}\lambda(d)\mu\left(\frac{n}{d}\right) \geq (1 - \cos n\theta)q^n - 2\sum_{1\leq d\leq \frac{n}{2}} q^d \geq$$

$$\geq q^{\frac{3n}{4}} - \frac{2q^{1+\frac{n}{2}}}{q-1} \geq q^{\frac{3n}{4}} - 4q^{\frac{n}{2}} > 0 \tag{2.18}$$

since $q = 2 \cdot 5^4$. We observe that, by formulae (1.5) and (2.17), the generating function $F(y)$ is given by

$$F(y) = \frac{(1 - e^{i\theta}qy)^{\frac{1}{2}}(1 - e^{-i\theta}qy)^{\frac{1}{2}}}{1 - qy} = \sum_{n=0}^{\infty} G(n)y^n, \tag{2.19}$$

and has two zeros $q^{-1}e^{\pm\theta}$. Squaring both sides of (2.19) we conclude

$$F^2(y) = \frac{(1 - e^{i\theta}qy)(1 - e^{-i\theta}qy)}{(1 - qy)^2} = \sum_{n=0}^{\infty}\left(\sum_{m\leq n} G(m)G(n-m)\right)y^n.$$

By (6.15) in Example 6.5 of [22],

$$\sum_{m\leq n} G(m)G(n-m) = 2(1 - \cos\ \theta)nq^n. \tag{2.20}$$

We show by induction that $G(n)$ is an even integer number for all $n \geq 1$. Obviously,

$$G(1) = 2 \cdot 5^3.$$

Now, suppose that $G(m)$ is even for $1 \leq m \leq n$. Then

$$\sum_{m\leq n+1} G(m)G(n+1-m) = G(0)G(n+1) + G(0)G(n+1) + \sum_{1\leq m\leq n} G(m)G(n+1-m) \Rightarrow$$

$$\Rightarrow 2G(n+1) + \sum_{1\leq m\leq n} G(m)G(n+1-m) = 2(1 - \cos\theta)(n+1)q^{n+1} \Rightarrow$$

$$\Rightarrow 2G(n+1) \equiv 0 \mod 4$$

and $G(n+1)$ is an even integer, too. Then, by Remark 1 and (2.18), $\pi(n) \in \mathbb{N}$ for all $n \geq 1$.

3. Applications. ***3.1. Abstract Prime Number Theorems.*** In the following we use the notation (2.1).

Theorem 8 ([14], Theorem 1)**.** *Let* $\delta = 1$ *in* (2.1) *and assume that* $H(y)$ *is holomorphic for* $|y| < R$, *where* $R > q^{-1}$. *If* F *has no zeros of modulus* q^{-1}, *then for every* r, $\frac{1}{q} < r < R$, *one has*

$$\lambda(n)q^{-n} = 1 - \sum_{j=1}^{l}(qb_j)^{-n} + O((qr)^{-n}),$$

where b_j, $1 \leq j \leq l = l(r)$, *are the zeros of* F *with* $q^{-1} < |b_j| < r$, *counted according their multiplicities.*

Theorem 9 ([14], Theorem 1)**.** *Assume that the assumptions of Theorem* 8 *hold. If* $F(-q^{-1}) = 0$, *then*

$$\lambda(n)q^{-n} = 1 - (-1)^n + O\big((qr)^{-n}\big)$$

for every r, $q^{-1} < r < R$.

Example **7.** In [14] we gave an example by choosing

$$\pi(n) = \begin{cases} [2q^n/n], & \text{if } n \text{ is odd,} \\ 0, & \text{even.} \end{cases}$$

Then $F(y)$ is given by

$$F(y) = \frac{1+qy}{1-qy}\left(\frac{1+qy^2}{1-qy^2}\right)^{1/2} H_1(y),$$

where $H_1(y)$ is holomorphic and different from zero for $|y| \leq q^{-1/2}$.

In the case $\delta = 1$, we assume that Axiom $\mathcal{A}_1$ is fulfilled. Then we have the following theorem.

Theorem 10 ([8], Theorem 2)**.** *Let* $\delta = 1$ *in* (2.1) *and let Axiom* A_1 *holds. If* $H(-q^{-1}) \neq 0$, *then*

$$\frac{\lambda(n)}{q^n} = 1 + o(1) \quad as \quad n \to \infty$$

and, if $H(-q^{-1}) = 0$, *then*

$$\frac{\lambda(n)}{q^n} + \frac{\lambda(n-1)}{q^{n-1}} = 2 + o(1) \quad as \quad n \to \infty.$$

Furthermore, the asymptotic formula

$$\sum_{m \leq n} \lambda(m)q^{-m} = n + o(n^{1/2}) \quad as \quad n \to \infty$$

holds.

Observe that Theorem 10 ensures only a Chebyshev type upper estimate $\lambda(n) \ll q^n$.

A small change of Axiom $\mathcal{A}_1$ leads to the abstract prime number theorem and to the asymptotic formula (3.3) with remainder term $o(1)$. This modification is contained in the following axiom.

Axiom $\boldsymbol{\mathcal{A}_2}$ [8]**.** *The conditions of Axiom* $\mathcal{A}_1$ *hold, and in addition, the power series of* H' *converges absolutely for* $|y| \leq q^{-1}$.

An immediate consequence is given in the following theorem.

Theorem 11 ([8], Theorem 3)**.** *Let* $\delta = 1$ *in* (2.1)*. If Axiom* $\mathcal{A}_2$ *holds and if* $H(-q^{-1}) \neq 0$, *then*

$$\frac{\lambda(n)}{q^n} = 1 + o(1) \quad as \quad n \to \infty \tag{3.1}$$

and, if $H(-q^{-1}) = 0$, *then*

$$\frac{\lambda(n)}{q^n} = 1 - (-1)^n + o(1) \quad as \quad n \to \infty. \tag{3.2}$$

Furthermore, *the asymptotic formula*

$$\sum_{m \le n} \lambda(m) q^{-m} = n + \frac{H'(q^{-1})}{qH(q^{-1})} + o(1) \quad as \quad n \to \infty \tag{3.3}$$

holds.

Theorem 12. *Let* $0 < \delta < 1$ *in* (2.1) *and assume that* (2.1) *holds for* $0 < \delta < 1$, *and* $H'(y) = O(1)$ *for* $|y| < q^{-1}$*. Then*

$$\frac{\lambda(n)}{q^n} = \delta + o(1), \quad n \to \infty.$$

Proof. The equation (1.5) implies

$$\sum_{n=1}^{\infty} \lambda(n) q^{-n} = \frac{\delta}{1 - qy} + y \frac{H'(y)}{H(y)}.$$

Put

$$H(y) := \sum_{n=0}^{\infty} h(n) y^n \qquad \text{and} \qquad \frac{1}{H(y)} = \sum_{n=0}^{\infty} h_1(n) y^n,$$

where $\sum_{n=0}^{\infty} |h(n)| < \infty$. Since $H(y) \neq 0$ for $|y| \le q^{-1}$ the series $\sum_{n=0}^{\infty} h_1(n) y^n$ converges absolutely for $|y| \le q^{-1}$.

We conclude

$$q^{-n} \sum_{m \le n} m h(m) h_1(n-m) = \sum_{m \le n} m h(m) q^{-m} h_1(n-m) q^{-(n-m)} \le$$

$$\le \underbrace{\sum_{m \le \frac{n}{2}} m h(m) q^{-m} h_1(n-m) q^{-(n-m)}}_{\sum_1} + \underbrace{\sum_{\frac{n}{2} < m \le n} m h(m) q^{-m} h_1(n-m) q^{-(n-m)}}_{\sum_2}.$$

Obviously,

$$\sum_1 \le O(1) \sum_{0 < m \le \frac{n}{2}} h_1(n-m) q^{-(n-m)} = o(1), \quad n \to \infty,$$

and

$$\sum_2 \le o(1) \sum_{\frac{n}{2} < m \le n} h_1(n-m) q^{-(n-m)} = o(1), \quad n \to \infty.$$

Thus, the assertion of Theorem 12 holds.

Following Theorem 12 we put

$$f(y) := \sum_{n=0}^{\infty} \gamma(n) y^n = \frac{H(y)}{1+qy}.$$

With this notation we prove the following theorem.

Theorem 13. *Let* (2.1) *holds with* $\delta = 1$. *Assume that Axiom* $\mathcal{A}_1$ *holds and* $H(-q^{-1}) = 0$. *Suppose that the power series of* f *converges absolutely for* $|y| \leq q^{-1}$ *and* $n\gamma(n)q^{-n} = o(1)$ *as* $n \to \infty$. *Then*

$$\frac{\lambda(n)}{q^n} = 1 - (-1)^n + o(1), \quad n \to \infty.$$

Proof. By

$$F(y) = \frac{1+qy}{1-qy} f(y)$$

we have

$$\sum_{n=1}^{\infty} (\lambda(n)q^{-n} - (-1)^n - 1) q^n y^n = y \frac{f'(y)}{f(y)}.$$

Now, consider

$$y\frac{H'(y)}{H(y)} = \left\{ \sum_{n=1}^{\infty} nh(n)y^n \right\} \left\{ \sum_{n=0}^{\infty} h_1(n)y^n \right\} = \sum_{n=0}^{\infty} \left(\sum_{m \leq n} mh(m)h_1(n-m) \right) y^n.$$

Let

$$\frac{1}{f(y)} =: \sum_{n=0}^{\infty} \gamma_1(n) y^n.$$

Since $f(y) \neq 0$ for $|y| \leq q^{-1}$ the series $\sum_{n=0}^{\infty} \gamma_1(n)y^n$ converges absolutely for $|y| \leq q^{-1}$ and $n\gamma(n)q^{-n} = o(1)$ as $n \to \infty$, the proof of Theorem 12 immediately implies the assertion of Theorem 13.

***Remark* 4.** If Axiom $\mathcal{A}_2$ holds and $H(-q^{-1}) = 0$ is, then the power series of f converges absolutely for $|y| = q^{-1}$ (see for details [8]). Furthermore, by the absolute convergence of the power series of H' and f we obtain the absolute convergence of the power series of f'. This means that Axiom $\mathcal{A}_2$ implies the assumptions of Theorem 13.

More generally, Indlekofer showed in his paper [9] the following quantitive results for (3.1) and (3.2).

Theorem 14 ([9], Theorem 1)**.** *Let* $\delta = 1$ *in* (2.1). *Assume that, for some* $k \in \mathbb{N}$, *the* k*th derivative* $H^{(k)}(y)$ *of* $H(y)$ *converges absolutely for* $|y| \leq q^{-1}$. *Then the following two assertions hold*:

(i) *if* $H(-q^{-1}) \neq 0$, *then*

$$\frac{\lambda(n)}{q^n} = 1 + O(n^{-(k+1)}) + O\left(n \max_{\frac{n}{4} \leq m \leq n} |h(m)| q^{-m} \right) \quad as \quad n \to \infty;$$

(ii) *if* $H(-q^{-1}) = 0$, *then*

$$\frac{\lambda(n)}{q^n} = 1 - (-1)^n + O\left(n \sum_{\frac{n}{8} \le m} |h(m)|q^{-m}\right) \quad \textit{as} \quad n \to \infty.$$

Observe that if

$$\sum_{n=1}^{\infty} n^k |h(n)|q^{-n} < \infty,$$

then

$$n \max_{\frac{n}{4} \le m \le n} |h(m)|q^{-m} = o(n^{1-k})$$

and, obviously,

$$\sum_{m \ge n} |h(m)|q^{-m} = o(n^{-k}).$$

Now, we give abstract prime number theorems for additive arithmetical semigroups, which satisfy the condition of Theorem 7 in the case $0 < \delta < 2$. First of all, we have the following theorem.

Theorem 15. *Assume* (2.1) *with* $1 < \delta < 2$. *If* $H''(y) = O(1)$ *for* $|y| < q^{-1}$, *then the following assertions hold*:

(i) *if* $H(-q^{-1}) \neq 0$, *then*

$$\frac{\lambda(n)}{q^n} = \delta + o(n^{-1}) \quad \textit{as} \quad n \to \infty;$$

(ii) *if* $H(-q^{-1}) = 0$, *then*

$$\frac{\lambda(n)}{q^n} = \delta + (-1)^n + o(n^{-3/2}) \quad \textit{as} \quad n \to \infty.$$

Proof. Since $H''(y) = O(1)$ for $|y| < q^{-1}$ we conclude, by Parseval's equation,

$$\sum_{n=1}^{\infty} n|h(n)|q^{-n} \le \left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right)^{1/2} \left(\sum_{n=1}^{\infty} n^4 |h(n)|^2 q^{-2n}\right)^{1/2} < \infty \tag{3.4}$$

and

$$|h(n)|q^{-n} = o(n^{-2}) \quad \text{as} \quad n \to \infty. \tag{3.5}$$

By (1.5) and (2.1), we have

$$\sum_{n=1}^{\infty} \lambda(n) y^n = \delta \frac{qy}{1 - qy} + y \frac{H'(y)}{H(y)}.$$

If $H(-q^{-1}) \neq 0$, then, by (3.4),

$$y \frac{H'(y)}{H(y)} = \left(\sum_{n=1}^{\infty} n h(n) y^n\right) \left(\sum_{n=0}^{\infty} h_1(n) y^n\right) =: \sum_{n=0} r_n y^n, \tag{3.6}$$

where

$$\frac{1}{H(y)} = \sum_{n=0}^{\infty} h_1(n) y^n$$

is absolutely convergent for $|y| \le q^{-1}$. Obviously,

$$y \left(\frac{1}{H(y)} \right)' = -\frac{H'(y)}{H^2(y)} = \sum_{n=1}^{\infty} n h_1(n) y^n. \tag{3.7}$$

Let us consider

$$\frac{1}{H^2(y)} =: \sum_{n=1}^{\infty} h_2(n) y^n.$$

Then $\dfrac{1}{H^2(y)}$ and $\left(\dfrac{1}{H^2(y)}\right)'$ are absolutely convergent for $|y| \le q^{-1}$, and thus $h_2(n) = O(n^{-1}|h_3(n)|)$, where

$$\sum_{n=1}^{\infty} |h_3(n)| q^{-n} < \infty.$$

By (3.5) and (3.7), we conclude

$$n|h_1(n)| \le \sum_{m \le \frac{n}{2}} m|h(m)|q^{-m}|h_2(n-m)|q^{-(n-m)} +$$

$$+ \sum_{\frac{n}{2} < m \le n} m|h(m)|q^{-m}|h_2(n-m)|q^{-(n-m)} \ll$$

$$\ll \max_{\frac{n}{2} < m \le n} |h_2(m)|q^{-m} + n \max_{\frac{n}{2} < m \le n} h(m) q^{-m} = o(n^{-1}) \quad \text{as} \quad n \to \infty.$$

From (3.6) we deduce in the same way

$$|r_n| q^{-n} = o(n^{-1}),$$

which proves (i) of Theorem 15.

In the other case, we have

$$\sum_{n=1}^{\infty} \lambda(n) y^n = \delta \frac{qy}{1-qy} + \frac{qy}{1+qy} + y \frac{f'(y)}{f(y)},$$

where

$$f(y) = \frac{H(y)}{1+qy} = \sum_{n=0}^{\infty} \gamma(n) y^n.$$

Then (see [8, p. 192])

$$|\gamma(n)| q^{-n} \le \delta_1(n) := \sum_{n \le m} |h(m)| q^{-m},$$

and by Parseval's equation (cf. (3.4))

$$\delta_1(n) \le \frac{1}{n} \sum_{n \le m} |h(m)| q^{-m} = o(n^{-3/2}) \quad \text{as} \quad n \to \infty.$$

Thus $\dfrac{1}{f(y)}$ is absolutely convergent for $|y| \leq q^{-1}$ and $\gamma(n)q^{-n} = o(n^{-3/2})$. As in the proof of Theorem 13 we obtain the assertion (ii), which ends the proof of Theorem 15.

3.2. Abstract Selberg formula.

Theorem 16. *Let* $\delta = 1$ *in* (2.1). *Suppose that the power series of* $H''(y)$ *converges absolutely for* $|y| \leq q^{-1}$. *Then the following assertions hold*:

(i) *if* $H(-q^{-1}) \neq 0$, *then*

$$n\lambda(n) + \sum_{k=1}^{n-1} \lambda(k)\lambda(n-k) = 2nq^n + c_1 q^n + o(q^n) \quad as \quad n \to \infty, \tag{3.8}$$

where

$$c_1 = 2q^{-1}\frac{H'(q^{-1})}{H(q^{-1})} - 1;$$

(ii) *if* $H(-q^{-1}) = 0$, *then*

$$\begin{gathered} n\lambda(n) + \sum_{k=1}^{n-1} \lambda(k)\lambda(n-k) = 2nq^n + c_2 q^n + o(q^n) \quad as \quad n \to \infty, \\ c_2 = \frac{-1}{2} + \frac{3}{2}(-1)^n + 2q^{-1}\frac{f'(q^{-1})}{f(q^{-1})} + q^{-2}\frac{H''(-q^{-1})}{f(-q^{-1})}(-1)^n. \end{gathered} \tag{3.9}$$

Proof. The left-hand sides of (3.8) and (3.9) is the nth coefficients of the power series of

$$y\left(y\frac{F'(y)}{F(y)}\right)' + \left(y\frac{F'(y)}{F(y)}\right)^2.$$

The equation

$$y^2\frac{F''(y)}{F(y)} + y\frac{F'(y)}{F(y)} = y\left(y\frac{F'(y)}{F(y)}\right)' + \left(y\frac{F'(y)}{F(y)}\right)^2$$

describes the analogue of the *Selberg identity* of classical number theory. By (2.1) we deduce

$$\begin{aligned} y^2\frac{F''(y)}{F(y)} + y\frac{F'(y)}{F(y)} &= \frac{2q^2y^2}{(1-qy)^2} + \frac{qy}{1-qy} + \\ &+ \frac{2qy}{1-qy}y\frac{H'(y)}{H(y)} + y\frac{H'(y)}{H(y)} + y^2\frac{H''(y)}{H(y)} =: \\ &=: \sum_1 + \sum_2 + \sum_3 + \sum_4 + \sum_5, \end{aligned} \tag{3.10}$$

where $\sum_1 + \sum_2 + \sum_3 + \sum_4 + \sum_5$ denote the corresponding power series.

Obviously,

$$\text{the } n\text{th} \quad (n \geq 2) \quad \text{coefficient of} \quad \sum_1 + \sum_2 \quad \text{is} \quad 2nq^n - q^n. \tag{3.11}$$

Let us assume that $H(-q^{-1}) \neq 0$. Then the power series of $\dfrac{1}{H(y)}$ (and $H'(y)$) converge absolutely for $|y| \leq q^{-1}$ so that, as $n \to \infty$,

$$\text{the}\quad n\text{th}\quad (n\geq 2)\quad \text{coefficient of}\quad \sum_3 \quad \text{is}\quad 2q^nq^{-1}\frac{H'(q^{-1})}{H(q^{-1})}+o(q^n). \tag{3.12}$$

Further, since $H''(y)$ is absolutely convergent for $|y|\leq q^{-1}$

$$\text{the}\quad n\text{th}\quad (n\geq 2)\quad \text{coefficient for}\quad \sum_4+\sum_5 \quad \text{is}\quad o(q^n)\quad \text{as}\quad n\to\infty. \tag{3.13}$$

Collecting (3.11), (3.12) and (3.13) gives assertion (i).

For the proof of (ii) assume that $H(-q^{-1})=0$. Then, putting $H(y)=(1+qy)f(y)$, by (3.10) we obtain

$$y^2\frac{F''(y)}{F(y)}+y\frac{F'(y)}{F(y)}=\frac{2q^2y^2}{(1-qy)^2}+\frac{qy}{1-qy}+\frac{2qy}{1-qy}\left(\frac{qy}{1+qy}+y\frac{f'(y)}{f(y)}\right)+$$

$$+\frac{qy}{1+qy}+y\frac{f'(y)}{f(y)}+y^2\frac{H''(y)}{(1+qy)f(y)}=$$

$$=\frac{2q^2y^2}{(1-qy)^2}+\frac{qy}{1-qy}+\frac{2qy}{1-qy}\frac{qy}{1+qy}+\frac{qy}{1+qy}+$$

$$+\frac{2qy}{1-qy}y\frac{f'(y)}{f(y)}+8y\frac{f'(y)}{f(y)}+y^2\frac{H''(y)}{(1+qy)f(y)}. \tag{3.14}$$

Putting $f(y)=\sum_{n=0}^{\infty}\gamma(n)y^n$ we know

$$|\gamma(n)|q^{-n}\leq\delta_1(n)=\sum_{n\leq m}|h(m)|q^{-m}$$

which implies

$$\sum_{n=0}^{\infty}n|\gamma(n)|q^{-n}\leq\sum_{n=1}^{\infty}n\sum_{n\leq m}|h(m)|q^{-m}\ll\sum_{n=1}^{\infty}n^2|h(n)|q^{-n}<\infty.$$

Thus the power series of $\dfrac{1}{f(y)}$, $f'(y)$ and $H''(y)$ are absolutely convergent for $|y|\leq q^{-1}$. By this the sum of the nth coefficient of the power series for the right-hand side in (3.14) is given by

$$2(n-1)q^n+q^n+\frac{q^n+(-1)^nq^n}{2}+(-1)^nq^n+2q^nq^{-1}\frac{f'(q^{-1})}{f(q^{-1})}+q^{-2}\frac{H''(-q^{-1})}{f(-q^{-1})}(-1)^nq^n+o(q^n)$$

which proves (ii) of Theorem 16.

***Remark* 5.** In [18] (Chapter 3.7), Zhang proved a weaker form of Selberg's formula

$$n\lambda(n)+\sum_{k=1}^{n-1}\lambda(k)\lambda(n-k)=2nq^n+O(q^n)$$

under stronger conditions. He assumed that $G(n)=Aq^n+O(q^nn^{-\gamma})$ with $\gamma>3$, which implies $|h(n)|q^{-n}=O(n^{-\gamma})$, so that the absolute convergence of $H''(y)$ follows immediately.

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