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ON THE APPROXIMATION PROPERTIES OF CESÀRO MEANS OF NEGATIVE ORDER OF DOUBLE VILENKIN – FOURIER SERIES

ПРО АПРОКСИМАЦІЙНІ ВЛАСТИВОСТІ СЕРЕДНІХ ЧЕЗАРО ВІД'ЄМНОГО ПОРЯДКУ ДЛЯ ПОДВІЙНИХ РЯДІВ ВІЛЕНКІНА – ФУР'Є

We establish approximation properties of Cesàro $(C,-\alpha,-\beta)$ means with $\alpha,\beta \ \epsilon\ (0,1)$ of Vilenkin – Fourier series. This result allows one to obtain a condition which is sufficient for the convergence of the means $\sigma_{n,m}^{-\alpha,-\beta}(x,y,f)$ to $f(x,y)$ in the L^p-metric.

Для рядів Віленкіна – Фур'є встановлено апроксимаційні властивості $(C,-\alpha,-\beta)$ середніх Чезаро, $\alpha,\beta\in(0,1)$. Цей результат дозволяє отримати умову, яка є достатньою для того, щоб середні $\sigma_{n,m}^{-\alpha,-\beta}(x,y,f)$ були збіжними до $f(x,y)$ у метриці L^p.

Let N_+ denote the set of positive integers, $N:=N_+\cup\{0\}$. Let $m:=(m_0,m_1,\ldots)$ denote a sequence of positive integers not lass then 2. Denote by $Z_{m_k}:=\{0,1,\ldots,m_k-1\}$ the additive group of integers modulo m_k. Define the group G_m as the complete direct product of the groups Z_{m_j}, with the product of the discrete topologies of Z_{mj}'s.

The direct product of the measures

$$\mu_k(\{j\}):=\frac{1}{m_k},\quad j\in Z_{m_k},$$

is the Haar measure on G_m with $\mu(G_m)=1$. If the sequence m is bounded, then G_m is called a bounded Vilenkin group. In this paper, we will consider only bounded Vilenkin group. The elements of G_m can be represented by sequences $x:=(x_0,x_1,\ldots,x_j,\ldots)$, $x_j\in Z_{m_j}$. The group operation $+$ in G_m is given by

$$x+y=\big((x_0+y_0)\operatorname{mod}m_0,\ldots,(x_k+y_k)\operatorname{mod}m_k,\ldots\big),$$

where $x:=(x_0,\ldots,x_k,\ldots)$ and $y:=(y_0,\ldots,y_k,\ldots)\in G_m$. The inverse of $+$ will be denoted by $-$.

It is easy to give a base for the neighborhoods of G_m :

$$I_0(x):=G_m,$$

$$I_n(x):=\{y\in G_m|y_0=x_0,\ldots,y_{n-1}=x_{n-1}\}$$

for $x\in G_m$, $n\in N$. Define $I_n:=I_n(0)$ for $n\in N_+$. Set $e_n:=(0,\ldots,0,1,0,\ldots)\in G_m$ the $(n+1)$th coordinate of which is 1 and the rest are zeros $(n\in N)$.

If we define the so-called generalized number system based on m in the following way: $M_0:=1$, $M_{k+1}:=m_kM_k$, $k\in N$, then every $n\in N$ can be uniquely expressed as $n=\sum_{j=0}^{\infty}n_jM_j$, where $n_j\in Z_{m_j}$, $j\in N_+$, and only a finite number of n_j's differ from zero. We use the following notation. Let $|n|$:=max$\{k\in N: n_k\neq 0\}$ (that is, $M_{|n|}\le n<M_{|n|+1}$).

Next, we introduce of G_m an orthonormal system which is called Vilenkin system. At first define the complex valued functions $r_k(x) : G_m \to C$; the generalized Rademacher functions in this way

$$r_k(x) := \exp \frac{2\pi i x_k}{m_k}, \qquad i^2 = -1, \quad x \in G_m, \quad k \in N.$$

Now define the Vilenkin system $\psi := (\psi_n : n \in N)$ on G_m as follows:

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x), \quad n \in N.$$

In particular, we call the system the Walsh – Paley if $m = 2$.

The Dirichlet kernels is defined by

$$D_n := \sum_{k=0}^{n-1} \psi_k, \quad n \in N_+.$$

Recall that (see [3] or [14])

$$D_{M_n}(x) = \begin{cases} M_n, & \text{if} \quad x \in I_n, \\ 0, & \text{if} \quad x \notin I_n. \end{cases} \tag{1}$$

The Vilenkin system is orthonormal and complete in $L^1(G_m)$[1].

Next, we introduce some notation with respect to the theory of two-dimensional Vilenkin system. Let $\tilde{m}$ be a sequence like m. The relation between the sequences $(\tilde{m}_n)$ and $(\tilde{M}_n)$ is the same as between sequences (m_n) and (M_n). The group $G_m \times G_{\tilde{m}}$ is called a two-dimensional Vilenkin group. The normalized Haar measure is denoted by μ as in the one-dimensional case. We also suppose that $m = \tilde{m}$ and $G_m \times G_{\tilde{m}} = G_m^2$.

The norm of the space $L^p\left(G_m^2\right)$ is defined by

$$\|f\|_p := \left(\int\limits_{G_m^2} |f(x,y)|^p \, d\mu(x,y) \right)^{1/p}, \quad 1 \le p < \infty.$$

Denote by $C\left(G_m^2\right)$ the class of continuous functions on the group G_m^2, endoved with the supremum norm. For the sake of brevity in notation, we agree to write $L^\infty\left(G_m^2\right)$ instead of $C\left(G_m^2\right)$.

The two-dimensional Fourier coefficients, the rectangular partial sums of the Fourier series, the Dirichlet kernels with respect to the two-dimensional Vilenkin system are defined as follows:

$$\widehat{f}(n_1, n_2) := \int\limits_{G_m^2} f(x,y) \bar{\psi}_{n_1}(x) \bar{\psi}_{n_2}(y) d\mu(x,y),$$

$$S_{n_1,n_2}(x,y,f) := \sum_{k_1=0}^{n_1-1} \sum_{k_2=0}^{n_2-1} \widehat{f}(k_1,k_2) \psi_{k_1}(x) \psi_{k_2}(y),$$

$$D_{n_1,n_2}(x,y) := D_{n_1}(x) D_{n_2}(y).$$

Denote

$$S_n^{(1)}(x,y,f) := \sum_{l=0}^{n-1} \widehat{f}(l,y)\psi_l(x),$$

$$S_m^{(2)}(x,y,f) := \sum_{r=0}^{m-1} \widehat{f}\left(x,r\right)\psi_r\left(y\right),$$

where

$$\widehat{f}(l,y) = \int\limits_{G_m} f(x,y)\psi_l(x)d\mu(x)$$

and

$$\widehat{f}\left(x,r\right) = \int\limits_{G_m} f(x,y)\psi_r\left(y\right)d\mu\left(y\right).$$

The $(c,-\alpha,-\beta)$ means of the two-dimensional Vilenkin – Fourier series are defined as

$$\sigma_{n,m}^{-\alpha,-\beta}(x,y,f) = \frac{1}{A_n^{-\alpha}A_m^{-\beta}} \sum_{i=0}^{n}\sum_{j=0}^{m} A_{n-i}^{-\alpha}A_{m-j}^{-\beta}\hat{f}\left(i,j\right)\psi_i(u)\psi_j(v),$$

where

$$A_0^\alpha = 1, \qquad A_n^\alpha = \frac{(\alpha+1)\ldots(\alpha+n)}{n!}.$$

It is well-known that [18]

$$A_n^\alpha = \sum_{k=0}^{n} A_k^{\alpha-1}, \tag{2}$$

$$A_n^\alpha - A_{n-1}^\alpha = A_n^{\alpha-1}, \tag{3}$$

$$A_n^\alpha \sim n^\alpha. \tag{4}$$

The dyadic partial moduli of continuity of a function $f \in L^p\left(G_m^2\right)$ in the L^p-norm are defined by

$$\omega_1\left(f,\frac{1}{M_n}\right)_p = \sup_{u\in I_n} \left\|f(\cdot-u,\cdot)-f\left(\cdot,\cdot\right)\right\|_p,$$

$$\omega_2\left(f,\frac{1}{M_n}\right)_p = \sup_{v\in I_n} \left\|f\left(\cdot,\cdot-v\right)-f\left(\cdot,\cdot\right)\right\|_p,$$

while the dyadic mixed modulus of continuity is defined as follows:

$$\omega_{1,2}\left(f,\frac{1}{M_n},\frac{1}{M_m}\right)_p =$$

$$= \sup_{(u,v)\in I_n\times I_m} \|f(\cdot - u, \cdot - v) - f(\cdot - u, \cdot) - f(\cdot, \cdot - v) + f(\cdot,\cdot)\|_p ,$$

it is clear that

$$\omega_{1,2}\left(f, \frac{1}{M_n}, \frac{1}{M_m}\right)_p \le \omega_1\left(f, \frac{1}{M_n}\right)_p + \omega_2\left(f, \frac{1}{M_m}\right)_p .$$

The dyadic total modulus of continuity is defined by

$$\omega\left(f, \frac{1}{M_n}\right)_p = \sup_{(u,v)\in I_n\times I_n} \|f(\cdot - u, \cdot - v) - f(\cdot,\cdot)\|_p .$$

The problems of summability of partial sums and Cesàro means for Walsh – Fourier series were studied in [2, 4 – 13, 16]. In his monography [17], Zhizhiashvili investigated the behavior of Cesàro method of negative order for trigonometric Fourier series in detail. Goginava [5] studied analogical question in case of the Walsh system. In particular, the following theorem is proved.

Theorem G [5]. *Let f belongs to $L^p(G_2)$ for some $p \in [1,\infty]$ and $\alpha \in (0,1)$. Then, for any $2^k \le n < 2^{k+1}$, $k, n \in N$, the inequality*

$$\left\|\sigma_n^{-\alpha}(f) - f\right\|_p \le c(p,\alpha)\left\{2^{k\alpha}\omega\left(1/2^{k-1}, f\right)_p + \sum_{r=0}^{k-2} 2^{r-k}\omega(1/2^r, f)_p\right\}$$

holds true.

In [15], the present author investigated analogous question in the case of Vilenkin system.

Theorem T. *Let f belongs to $L^p(G_m)$ for some $p \in [1,\infty]$ and $\alpha \in (0,1)$. Then, for any $M_k \le n < M_{k+1}$, $k, n \in N$, the inequality*

$$\left\|\sigma_n^{-\alpha}(f) - f\right\|_p \le c(p,\alpha)\left\{M_k^{\alpha}\omega(1/M_{k-1}, f)_p + \sum_{r=0}^{k-2} \frac{M_r}{M_k}\omega(1/M_r, f)_p\right\}$$

holds true.

Goginava [7] studied approximation properties of Cesàro $(c, -\alpha, -\beta)$ means with $\alpha, \beta \in (0,1)$ question in the case of double Walsh – Furier series. The following theorem was proved.

Theorem G2. *Let f belongs to $L^p\left(G_2^2\right)$ for some $p \in [1,\infty]$ and α, $\beta \in (0,1)$. Then, for any $2^k \le n < 2^{k+1}$, $2^l \le m < 2^{l+1}$, $k, n \in N$, the inequality*

$$\left\|\sigma_{n,m}^{-\alpha,-\beta}(f) - f\right\|_p \le c(\alpha,\beta)\Bigg(2^{k\alpha}\omega_1\left(f, 1/2^{k-1}\right)_p + 2^{l\beta}\omega_2\left(f, 1/2^{l-1}\right)_p +$$

$$+2^{k\alpha}2^{l\beta}\omega_{1,2}\left(f, 1/2^{k-1}, 1/2^{l-1}\right)_p +$$

$$+\sum_{r=0}^{k-2} 2^{r-k}\omega_1(f, 1/2^r)_p + \sum_{s=0}^{l-2} 2^{s-l}\omega_2(f, 1/2^s)_p\Bigg)$$

holds true.

In this paper, we establish analogous question in the case of double Vilenkin – Fouries series.

Theorem 1. *Let f belongs to $L^p\left(G_m^2\right)$ for some $p\in[1,\infty]$ and $\alpha\in(0,1)$. Then, for any $M_k\leq n<M_{k+1},\ M_l\leq m<M_{l+1},\ k,n,m,l\in N,$ the inequality*

$$\left\|\sigma_{n,m}^{-\alpha,-\beta}(f)-f\right\|_p\leq c(\alpha,\beta)\Bigg(\omega_1(f,1/M_{k-1})_pM_k^{\alpha}+\omega_2(f,1/M_{l-1})_pM_l^{\beta}+$$

$$+\omega_{1,2}\left(f,1/M_{k-1},1/M_{l-1}\right)_pM_k^{\alpha}M_l^{\beta}+\sum_{r=0}^{k-2}\frac{M_r}{M_k}\omega_1\left(f,1/M_r\right)_p+\sum_{s=0}^{l-2}\frac{M_s}{M_l}\omega_2(f,1/M_s)_p\Bigg)$$

holds true.

***Corollary* 1.** *Let f belongs to L^p for some $p\in[1,\infty]$. If*

$$M_k^{\alpha}\omega_1\left(f,\frac{1}{M_k}\right)_p\to0\quad as\quad k\to\infty,\quad 0<\alpha<1,$$

$$M_l^{\beta}\omega_1\left(f,\frac{1}{M_l}\right)_p\to0\quad as\quad l\to\infty,\quad 0<\beta<1,$$

$$M_k^{\alpha}M_l^{\beta}\omega_{1,2}\left(f,\frac{1}{M_k},\frac{1}{M_l}\right)_p\to0\quad as\quad k,l\to\infty,$$

then

$$\left\|\sigma_{n,m}^{-\alpha,-\beta}(f)-f\right\|_p\to0\quad as\quad n,m\to\infty.$$

***Corollary* 2.** *Let f belongs to L^p for some $p\in[1,\infty]$ and let $\alpha,\beta\in(0,1),\ \alpha+\beta<1$. If*

$$\omega\left(f,\frac{1}{M_n}\right)_p=o\left(\left(\frac{1}{M_n}\right)^{\alpha+\beta}\right),$$

then

$$\left\|\sigma_{n,m}^{-\alpha,-\beta}(f)-f\right\|_p\to0\quad as\quad n,m\to\infty.$$

The following theorem shows that Corollary 2 cannot be improved.

Theorem 2. *For every $\alpha,\beta\in(0,1),\ \alpha+\beta<1$, there exists a function $f_0\in C\left(G_m^2\right)$ for which*

$$\omega\left(f,\frac{1}{M_n}\right)_C=O\left(\left(\frac{1}{M_n}\right)^{\alpha+\beta}\right),$$

and

$$\limsup_{n\to\infty}\left\|\sigma_{M_n,M_n}^{-\alpha,-\beta}(f)-f\right\|_1>0.$$

In order to prove Theorem 1 we need the following lemmas.

Lemma 1 [1]. *Let $\alpha_1, \ldots, \alpha_n$ be real numbers. Then*

$$\frac{1}{n}\int\limits_{G_m}\left|\sum_{k=1}^{n}\alpha_k D_k(x)\right| d\mu(x) \leq \frac{c}{\sqrt{n}}\left(\sum_{k=1}^{n}\alpha_k^2\right)^{1/2},$$

where c is an absolute constant.

Lemma 2. *Let f belongs to $L^p(G_m^2)$ for some $p \in [1,\infty]$. Then, for every $\alpha,\ \beta \in (0,1)$, the following estimation holds:*

$$I := \frac{1}{A_n^{-\alpha}A_m^{-\beta}}\left\|\int\limits_{G_m^2}\sum_{i=0}^{M_{k-1}-1}\sum_{j=0}^{M_{l-1}-1}A_{n-i}^{-\alpha}A_{m-j}^{-\beta}\psi_i(u)\psi_j(v)\times\right.$$

$$\left.\times\left[f(\cdot - u, \cdot - v) - f(\cdot,\cdot)\right]d\mu(u,v)\right\|_p \leq$$

$$\leq c(\alpha,\beta)\left(\sum_{r=0}^{k-1}\frac{M_r}{M_k}\omega_1(f,1/M_r)_p + \sum_{s=0}^{l-1}\frac{M_s}{M_l}\omega_2(f,1/M_s)_p\right),$$

where $M_k \leq n < M_{k+1},\ M_l \leq m < M_{l+1}$.

Proof. Applying Abel's transformation, from (2) we get

$$I \leq \frac{1}{A_n^{-\alpha}A_m^{-\beta}}\left\|\int\limits_{G_m^2}\sum_{i=1}^{M_{k-1}-1}\sum_{j=1}^{M_{l-1}-1}A_{n-i+1}^{-\alpha-1}A_{m-j+1}^{-\beta-1}D_i(u)D_j(v)\times\right.$$

$$\left.\times\left[f(\cdot - u, \cdot - v) - f(\cdot,\cdot)\right]d\mu(u,v)\right\|_p +$$

$$+\frac{1}{A_n^{-\alpha}A_m^{-\beta}}\left\|\int\limits_{G_m^2}A_{m-M_{l-1}+1}^{-\beta}D_{M_{l-1}}(v)\sum_{i=1}^{M_{k-1}-1}A_{n-i+1}^{-\alpha-1}D_i(u)\times\right.$$

$$\left.\times\left[f(\cdot - u, \cdot - v) - f(\cdot,\cdot)\right]d\mu(u,v)\right\|_p +$$

$$+\frac{1}{A_n^{-\alpha}A_m^{-\beta}}\left\|\int\limits_{G_m^2}A_{n-M_{k-1}+1}^{-\alpha}D_{M_{k-1}}(u)\sum_{j=1}^{M_{l-1}-1}A_{m-j+1}^{-\beta-1}D_j(v)\times\right.$$

$$\left.\times\left[f(\cdot - u, \cdot - v) - f(\cdot,\cdot)\right]d\mu(u,v)\right\|_p +$$

$$+\frac{1}{A_n^{-\alpha}A_m^{-\beta}}\left\|\int\limits_{G_m^2} A_{n-M_{k-1}+1}^{-\alpha}A_{m-M_{l-1}+1}^{-\beta}D_{M_{k-1}}(u)D_{M_{l-1}}(v)\times\right.$$

$$\left.\times\left[f(\cdot-u,\cdot-v)-f(\cdot,\cdot)\right]d\mu(u,v)\right\|_p=$$

$$=I_1+I_2+I_3+I_4. \tag{5}$$

From the generalized Minkowski inequality and by (1) and (4), we obtain

$$I_4\le\frac{1}{A_n^{-\alpha}A_m^{-\beta}}\int\limits_{G_m^2}\left|A_{n-M_{k-1}+1}^{-\alpha}A_{m-M_{l-1}+1}^{-\beta}D_{M_{k-1}}(u)D_{M_{l-1}}(v)\right|\times$$

$$\times\left\|f(\cdot-u,\cdot-v)-f(x,y)\right\|_p d\mu(u,v)\le$$

$$\le c(\alpha,\beta)M_{k-1}M_{l-1}\int\limits_{I_{k-1}\times I_{l-1}}\left\|f(\cdot-u,\cdot-v)-f(\cdot,\cdot)\right\|_p d\mu(u,v)=$$

$$=O\left(\omega_1(f,1/M_{k-1})_p+\omega_2(f,1/M_{l-1})_p\right). \tag{6}$$

It is evident that

$$I_1\le\frac{1}{A_n^{-\alpha}A_m^{-\beta}}\sum_{r=0}^{k-2}\sum_{s=0}^{l-2}\left\|\int\limits_{G_m^2}\sum_{i=M_r}^{M_{r+1}-1}\sum_{j=M_s}^{M_{s+1}-1}A_{n-i+1}^{-\alpha-1}A_{m-j+1}^{-\beta-1}D_i(u)D_j(v)\times\right.$$

$$\left.\times\left[f(\cdot-u,\cdot-v)-f(\cdot,\cdot)\right]d\mu(u,v)\right\|_p\le$$

$$\le\frac{1}{A_n^{-\alpha}A_m^{-\beta}}\sum_{r=0}^{k-2}\sum_{s=0}^{l-2}\left\|\int\limits_{G_m^2}\sum_{i=M_r}^{M_{r+1}-1}\sum_{j=M_s}^{M_{s+1}-1}A_{n-i+1}^{-\alpha-1}A_{m-j+1}^{-\beta-1}D_i(u)D_j(v)\times\right.$$

$$\left.\times\left[f(\cdot-u,\cdot-v)-S_{M_r,M_s}(\cdot-u,\cdot-v,f)\right]d\mu(u,v)\right\|_p+$$

$$+\frac{1}{A_n^{-\alpha}A_m^{-\beta}}\sum_{r=0}^{k-2}\sum_{s=0}^{l-2}\left\|\int\limits_{G_m^2}\sum_{i=M_r}^{M_{r+1}-1}\sum_{j=M_s}^{M_{s+1}-1}A_{n-i+1}^{-\alpha-1}A_{m-j+1}^{-\beta-1}D_i(u)D_j(v)\times\right.$$

$$\times \left[S_{M_r,M_s}(\cdot - u, \cdot - v, f) - S_{M_r,M_s}(\cdot,\cdot,f)\right] d\mu(u,v)\Bigg\|_p +$$

$$+\frac{1}{A_n^{-\alpha}A_m^{-\beta}}\sum_{r=0}^{k-2}\sum_{s=0}^{l-2}\Bigg\|\int\limits_{G_m^2}\sum_{i=M_r}^{M_{r+1}-1}\sum_{j=M_s}^{M_{s+1}-1}A_{n-i+1}^{-\alpha-1}A_{m-j+1}^{-\beta-1}D_i(u)D_j(v)\times$$

$$\times \left[S_{M_r,M_s}(\cdot,\cdot,f) - f(\cdot,\cdot)\right] d\mu(u,v)\Bigg\|_p =$$

$$= I_{11} + I_{12} + I_{13}. \tag{7}$$

It is easy to show that

$$I_{12} = 0. \tag{8}$$

By using Lemma 1, for I_{11}, we can write

$$I_{11} \le \frac{1}{A_n^{-\alpha}A_m^{-\beta}}\sum_{r=0}^{k-2}\sum_{s=0}^{l-2}\int\limits_{G_m^2}\left|\sum_{i=M_r}^{M_{r+1}-1}\sum_{j=M_s}^{M_{s+1}-1}A_{n-i+1}^{-\alpha-1}A_{m-j+1}^{-\beta-1}D_i(u)D_j(v)\right|\times$$

$$\times \left\|f(\cdot - u, \cdot - v) - S_{M_r,M_s}(\cdot - u, \cdot - v, f)\right\|_p d\mu(u,v) \le$$

$$\le c(\alpha,\beta)n^\alpha m^\beta \sum_{r=0}^{k-2}\sum_{s=0}^{l-2}\left(\omega_1(f,1/M_r)_p + \omega_2(f,1/M_s)_p\right)\times$$

$$\times\left(\int\limits_{G_m}\left|\sum_{i=M_r}^{M_{r+1}-1}A_{n-i+1}^{-\alpha-1}D_i(u)\right|d\mu(u)\right)\left(\int\limits_{G_m}\left|\sum_{j=M_s}^{M_{s+1}-1}A_{m-j+1}^{-\beta-1}D_j(v)\right|d\mu(v)\right) \le$$

$$\le c(\alpha,\beta)n^\alpha m^\beta \sum_{r=0}^{k-2}\sum_{s=0}^{l-2}\left(\omega_1(f,1/M_r)_p + \omega_2(f,1/M_s)_p\right)\times$$

$$\times\left(\sqrt{M_{r+1}}\left(\sum_{i=M_r}^{M_{r+1}-1}(n-i+1)^{-2\alpha-2}\right)^{1/2}\right)\times$$

$$\times\left(\sqrt{M_{s+1}}\left(\sum_{j=M_s}^{M_{s+1}-1}(m-j+1)^{-2\beta-2}\right)^{1/2}\right)\le$$

$$\le c(\alpha,\beta)n^\alpha m^\beta \sum_{r=0}^{k-2}\sum_{s=0}^{l-2}\left(\omega_1(f,1/M_r)_p + \omega_2(f,1/M_s)_p\right)\times$$

$$\times \left(\sqrt{M_{r+1}}(n-M_{r+1})^{-\alpha-1}\sqrt{M_{r+1}}\right)\left(\sqrt{M_{s+1}}(n-M_{s+1})^{-\beta-1}\sqrt{M_{s+1}}\right) \leq$$

$$\leq c(\alpha,\beta)n^{\alpha}m^{\beta}\sum_{r=0}^{k-2}\sum_{s=0}^{l-2}\frac{M_{r+1}}{M_k^{\alpha+1}}\frac{M_{s+1}}{M_l^{\beta+1}}\left(\omega_1(f,1/M_r)_p+\omega_2(f,1/M_s)_p\right) \leq$$

$$\leq c(\alpha,\beta)\left(\sum_{r=0}^{k-2}\frac{M_r}{M_k}\omega_1(f,1/M_r)_p+\sum_{s=0}^{l-2}\frac{M_s}{M_l}\omega_2(f,1/M_s)_p\right). \tag{9}$$

Analogously, we can prove that

$$I_{13} \leq c(\alpha,\beta)\left(\sum_{r=0}^{k-2}\frac{M_r}{M_k}\omega_1(f,1/M_r)_p+\sum_{s=0}^{l-2}\frac{M_s}{M_l}\omega_2(f,1/M_s)_p\right). \tag{10}$$

Combining (7)–(10), for I_1, we receive

$$I_1 \leq c(\alpha,\beta)\left(\sum_{r=0}^{k-2}\frac{M_r}{M_k}\omega_1(f,1/M_r)_p+\sum_{s=0}^{l-2}\frac{M_s}{M_l}\omega_2(f,1/M_s)_p\right). \tag{11}$$

For I_2 we can write

$$I_2 \leq \frac{1}{A_n^{-\alpha}A_m^{-\beta}}\Bigg\| \int_{G_m^2} A_{m-M_{l-1}+1}^{-\beta}D_{M_{l-1}}(v)\sum_{i=1}^{M_{k-1}-1}A_{n-i+1}^{-\alpha-1}D_i(u)\times$$

$$\times\left[f(\cdot-u,\cdot-v)-f(\cdot-u,\cdot)\right]d\mu(u,v)\Bigg\|_p +$$

$$+\frac{1}{A_n^{-\alpha}A_m^{-\beta}}\Bigg\| \int_{G_m^2} A_{m-M_{l-1}+1}^{-\beta}D_{M_{l-1}}(v)\sum_{i=1}^{M_{k-1}-1}A_{n-i+1}^{-\alpha-1}D_i(u)\times$$

$$\times\left[f(\cdot-u,\cdot)-f(\cdot,\cdot)\right]d\mu(u,v)\Bigg\|_p =$$

$$= I_{21}+I_{22}. \tag{12}$$

From the generalized Minkowski inequality and by (1) and (4), we obtain

$$I_{21} \leq c(\alpha,\beta)\frac{M_{l-1}}{A_n^{-\alpha}}\int_{I_{l-1}}\left(\int_{G_m}\left|\sum_{i=1}^{M_{k-1}-1}A_{n-i+1}^{-\alpha-1}D_i(u)\right|\times\right.$$

$$\left.\times\left\|f(\cdot-u,\cdot-v)-f(\cdot-u,\cdot)\right\|_p d\mu(u)\right)d\mu(v) \leq$$

$$\leq c(\alpha,\beta)n^{\alpha}\omega_2(f,1/M_{l-1})\left(\int\limits_{G_m}\left|\sum_{i=1}^{M_{k-1}-1}A_{n-i+1}^{-\alpha-1}D_i(u)\right|d\mu(u)\right)\leq$$

$$\leq c(\alpha,\beta)n^{\alpha}\omega_2(f,1/M_{l-1})\left(\sqrt{M_{k-1}}\left(\sum_{i=1}^{M_{k-1}-1}(n-i+1)^{-2\alpha-2}\right)^{1/2}\right)\leq$$

$$\leq c(\alpha,\beta)n^{\alpha}\omega_2(f,1/M_{l-1})\left(\sqrt{M_{k-1}}(n-M_{k-1})^{-\alpha-1}\sqrt{M_{k-1}}\right)\leq$$

$$\leq c(\alpha,\beta)\omega_2(f,1/M_{l-1}). \tag{13}$$

The estimation of I_{22} is analogous to the estimation of I_1 and we have

$$I_{22}\leq c(\alpha,\beta)\sum_{r=0}^{k-2}\frac{M_r}{M_k}\omega_1(f,1/M_r)_p. \tag{14}$$

So, combining (12)–(14), for I_2, we get

$$I_2\leq c(\alpha,\beta)\left(\sum_{r=0}^{k-2}\frac{M_r}{M_k}\omega_1(f,1/M_r)_p+\omega_2(f,1/M_{l-1})\right). \tag{15}$$

The estimation I_3 is analogous to the estimation of I_2, and we obtain

$$I_3\leq c(\alpha,\beta)\left(\sum_{s=0}^{l-2}\frac{M_s}{M_l}\omega_2(f,1/M_s)_p+\omega_1(f,1/M_{k-1})\right). \tag{16}$$

Combining (5), (6), (10), (15), (16), we receive the proof of Lemma 2.

Lemma 3. *Let f belongs to $L^p(G_m^2)$ for some $p\in[1,\infty]$. Then, for every $\alpha,\beta\in(0,1)$, the following estimations hold*:

$$II:=\frac{1}{A_n^{-\alpha}A_m^{-\beta}}\left\|\int\limits_{G_m^2}\sum_{i=M_{k-1}}^{n}\sum_{j=0}^{M_{l-1}-1}A_{n-i}^{-\alpha}A_{m-j}^{-\beta}\psi_i(u)\psi_j(v)\times\right.$$

$$\left.\times\left[f(\cdot-u,\cdot-v)-f(\cdot,\cdot)\right]d\mu(u)d\mu(v)\right\|_p\leq c(\alpha,\beta)\omega_1\left(f,1/M_{k-1}\right)_p M_k^{\alpha},$$

$$III:=\frac{1}{A_n^{-\alpha}A_m^{-\beta}}\left\|\int\limits_{G_m^2}\sum_{i=0}^{M_{k-1}-1}\sum_{j=M_{l-1}}^{m}A_{n-i}^{-\alpha}A_{m-j}^{-\beta}\psi_i(u)\psi_j(v)\times\right.$$

$$\left.\times\left[f(\cdot-u,\cdot-v)-f(\cdot,\cdot)\right]d\mu(u)d\mu(v)\right\|_p\leq c(\alpha,\beta)\omega_2(f,1/M_{l-1})_pM_l^{\beta},$$

where $M_k\leq n<M_{k+1},\ M_l\leq m<M_{l+1}$.

Proof. From the generalized Minkowski inequality, we have

$$II = \frac{1}{A_n^{-\alpha} A_m^{-\beta}} \left\| \int_{G_m^2} \sum_{i=M_{k-1}}^{n} \sum_{j=0}^{M_{l-1}-1} A_{n-i}^{-\alpha} A_{m-j}^{-\beta} \psi_i(u) \psi_j(v) \times \right.$$

$$\left. \times f(\cdot - u, \cdot - v) d\mu(u,v) \right\|_p =$$

$$= \frac{1}{A_n^{-\alpha} A_m^{-\beta}} \left\| \int_{G_m^2} \sum_{i=M_{k-1}}^{n} \sum_{j=0}^{M_{l-1}-1} A_{n-i}^{-\alpha} A_{m-j}^{-\beta} \psi_i(u) \psi_j(v) \times \right.$$

$$\left. \times \left[f(\cdot - u, \cdot - v) - S_{M_{k-1}}^{(1)}(\cdot - u, \cdot - v, f) \right] d\mu(u,v) \right\|_p \le$$

$$\le \frac{1}{A_n^{-\alpha} A_m^{-\beta}} \int_{G_m^2} \left| \sum_{i=M_{k-1}}^{M_k-1} \sum_{j=0}^{M_{l-1}-1} A_{n-i}^{-\alpha} A_{m-j}^{-\beta} \psi_i(u) \psi_j(v) \right| \times$$

$$\times \left\| f(\cdot - u, \cdot - v) - S_{M_{k-1}}^{(1)}(\cdot - u, \cdot - v, f) \right\|_p d\mu\,(u,v) +$$

$$+ \frac{1}{A_n^{-\alpha} A_m^{-\beta}} \int_{G_m^2} \left| \sum_{i=M_k}^{n} \sum_{j=0}^{M_{l-1}-1} A_{n-i}^{-\alpha} A_{m-j}^{-\beta} \psi_i(u) \psi_j(v) \right| \times$$

$$\times \left\| f(\cdot - u, \cdot - v) - S_{M_{k-1}}^{(1)}(\cdot - u, \cdot - v, f) \right\|_p d\mu(u)\, d\mu(v) = II_1 + II_2. \tag{17}$$

In [15], present author showed that the inequality

$$\int_{G_m} \left| \sum_{v=M_{k-1}}^{M_k-1} A_{n-v}^{-\alpha} \psi_v(u) \right| d\mu(u) \le c\,(\alpha)\,, \qquad k = 1, 2, \ldots, \tag{18}$$

holds true.

Using Lemma 1, by (4) and (18), for II_1, we can write

$$II_1 \le c(\alpha,\beta) n^\alpha m^\beta \omega_1\,(f, 1/M_{k-1})_p \left(\int_{G_m} \left| \sum_{i=M_{k-1}}^{M_k-1} A_{n-i}^{-\alpha} \psi_i(u) \right| d\mu(u) \right) \times$$

$$\times \left(\int_{G_m} \left| \sum_{j=1}^{M_{l-1}} A_{m-j+1}^{-\beta} \psi_{j-1}(v) \right| d\mu(v) \right) \le$$

$$\leq c(\alpha,\beta)n^{\alpha}m^{\beta}\omega_1(f,1/M_{k-1})_p\left(\sqrt{M_{l-1}}\left(\sum_{i=1}^{M_{l-1}}(m-j+1)^{-2\beta-2}\right)^{1/2}\right)\leq$$

$$\leq c(\alpha,\beta)n^{\alpha}m^{\beta}\omega_2(f,1/M_{k-1})_p\left(\sqrt{M_{l-1}}\,(n-M_{l-1})^{-\beta-1}\sqrt{M_{l-1}}\right)\leq$$

$$\leq c(\alpha,\beta)\omega_1\left(f,1/M_{k-1}\right)_p M_k^{\alpha}. \tag{19}$$

The estimation of II_2 is analogous to the estimation of II_1 and we have

$$II_2\leq c(\alpha,\beta)\omega_1(f,1/M_{k-1})_pM_k^{\alpha}. \tag{20}$$

Combining (17) – (20), we obtain

$$II\leq c(\alpha,\beta)\omega_1\left(f,1/M_{k-1}\right)_p M_k^{\alpha}. \tag{21}$$

Analogously, we can prove that

$$III\leq c(\alpha,\beta)\omega_2(f,1/M_{l-1})_pM_l^{\beta}. \tag{22}$$

Combining (21), (22), we receive the proof of Lemma 3.

Lemma 4. *Let f belongs to $L^p(G_m^2)$ for some $p\in[1,\infty]$. Then, for every $\alpha,\beta\in(0,1)$, the following estimation holds*:

$$IV:=\frac{1}{A_n^{-\alpha}A_m^{-\beta}}\left\|\int_{G_m^2}\sum_{i=M_{k-1}}^{n}\sum_{j=M_{l-1}}^{m}A_{n-i}^{-\alpha}A_{m-j}^{-\beta}\psi_i(u)\psi_j(v)\times\right.$$

$$\left.\times\left[f(\cdot-u,\cdot-v)-f(\cdot,\cdot)\right]d\mu(u)d\mu(v)\right\|_p\leq c(\alpha,\beta)\omega_{1,2}\left(f,1/M_k,1/M_l\right)_p M_k^{\alpha}M_l^{\beta},$$

where $M_k\leq n<M_{k+1}, M_l\leq m<M_{l+1}$.

Proof. From the generalized Minkowski inequality, and by (1) and (4), we have

$$IV=\frac{1}{A_n^{-\alpha}A_m^{-\beta}}\left\|\int_{G_m^2}\sum_{i=M_{k-1}}^{n}\sum_{j=M_{l-1}}^{m}A_{n-i}^{-\alpha}A_{m-j}^{-\beta}\psi_i(u)\psi_j(v)\times\right.$$

$$\left.\times f(\cdot-u,\cdot-v)d\mu(u,v)\right\|_p\leq$$

$$\leq\frac{1}{A_n^{-\alpha}A_m^{-\beta}}\left\|\int_{G_m^2}\sum_{i=M_{k-1}}^{n}\sum_{j=M_{l-1}}^{m}A_{n-i}^{-\alpha}A_{m-j}^{-\beta}\psi_i(u)\psi_j(v)\times\right.$$

$$\times\Big[S_{M_{k-1},M_{l-1}}(\cdot-u,\cdot-v,f)-S_{M_{k-1}}^{(1)}(\cdot-u,\cdot-v,f)-$$

$$-S_{M_{l-1}}^{(2)}(\cdot - u, \cdot - v, f) + f(\cdot - u, \cdot - v)\Big] d\mu(u,v)\Bigg\|_p \leq$$

$$\leq \frac{1}{A_n^{-\alpha} A_m^{-\beta}} \int\limits_{G_m^2} \left| \sum_{i=M_{k-1}}^{n} \sum_{j=M_{l-1}}^{m} A_{n-i}^{-\alpha} A_{m-j}^{-\beta} \psi_i(u)\psi_j(v) \right| \times$$

$$\times \left\| S_{M_{k-1},M_{l-1}}(\cdot - u, \cdot - v, f) - S_{M_{k-1}}^{(1)}(\cdot - u, \cdot - v, f) - \right.$$

$$\left. -S_{M_{l-1}}^{(2)}(\cdot - u, \cdot - v, f) + f(\cdot - u, \cdot - v) \right\|_p d\mu(u,v) \leq$$

$$\leq c(\alpha,\beta) n^{\alpha} m^{\beta} \omega_{1,2}\left(f, 1/M_{k-1}, 1/M_{l-1}\right)_p \times$$

$$\times \int\limits_{G_m^2} \left| \sum_{i=M_{k-1}}^{n} \sum_{j=M_{l-1}}^{m} A_{n-i}^{-\alpha} A_{m-j}^{-\beta} \psi_i(u)\psi_j(v) \right| d\mu(u,v) \leq$$

$$\leq c(\alpha,\beta) n^{\alpha} m^{\beta} \omega_{1,2}\left(f, 1/M_{k-1}, 1/M_{l-1}\right)_p \leq$$

$$\leq c(\alpha,\beta) M_k^{\alpha} M_l^{\beta} \omega_{1,2}\left(f, 1/M_{k-1}, 1/M_{l-1}\right)_p. \tag{23}$$

Lemma 4 is proved.

***Proof of Theorem* 1.** It is evident that

$$\sigma_{n,m}^{-\alpha,-\beta}(f,x,y) - f(x,y) = \frac{1}{A_n^{-\alpha} A_m^{-\beta}} \int\limits_{G_m^2} \sum_{i=0}^{M_{k-1}-1} \sum_{j=0}^{M_{l-1}-1} A_{n-i}^{-\alpha} A_{m-j}^{-\beta} \psi_i(u)\psi_j(v) \times$$

$$\times \left[f(\cdot - u, \cdot - v) - f(\cdot,\cdot)\right] d\mu(u,v) +$$

$$+\frac{1}{A_n^{-\alpha} A_m^{-\beta}} \int\limits_{G_m^2} \sum_{i=M_{k-1}}^{n} \sum_{j=0}^{M_{l-1}-1} A_{n-i}^{-\alpha} A_{m-j}^{-\beta} \psi_i(u)\psi_j(v) \times$$

$$\times \left[f(\cdot - u, \cdot - v) - f(\cdot,\cdot)\right] d\mu(u,v) +$$

$$+\frac{1}{A_n^{-\alpha} A_m^{-\beta}} \int\limits_{G_m^2} \sum_{i=0}^{M_{k-1}-1} \sum_{j=M_{l-1}}^{m} A_{n-i}^{-\alpha} A_{m-j}^{-\beta} \psi_i(u)\psi_j(v) \times$$

$$\times \left[f(\cdot - u, \cdot - v) - f(\cdot,\cdot)\right] d\mu(u,v) +$$

$$+\frac{1}{A_n^{-\alpha} A_m^{-\beta}} \int\limits_{G_m^2} \sum_{i=M_{k-1}}^{n} \sum_{j=M_{l-1}}^{m} A_{n-i}^{-\alpha} A_{m-j}^{-\beta} \psi_i(u)\psi_j(v) \times$$

$$\times \left[f(\cdot - u, \cdot - v) - f(\cdot,\cdot)\right] d\mu(u) d\mu(v) = I + II + III + IV.$$

Since

$$\left\|\sigma_{n,m}^{-\alpha,-\beta}(f,x)-f(x)\right\|_p \le \|I\|_p+\|II\|_p+\|III\|_p+\|IV\|_p\,.$$

From Lemmas 2 – 4 the proof of theorem is complete.

***Proof of Corollary* 2.** Since

$$\omega_i\left(f,\frac{1}{M_n}\right)\le\omega\left(f,\frac{1}{M_n}\right),\quad i=1,2,$$

$$\omega_{1,2}\left(f,\frac{1}{M_n},\frac{1}{M_m}\right)\le 2\omega_1\left(f,\frac{1}{M_n}\right)$$

and

$$\omega_{1,2}\left(f,\frac{1}{M_n},\frac{1}{M_m}\right)\le 2\omega_2\left(f,\frac{1}{M_m}\right),$$

we obtain

$$\omega_{1,2}\left(f,\frac{1}{M_n},\frac{1}{M_m}\right)=\left(\omega_{1,2}\left(f,\frac{1}{M_n},\frac{1}{M_m}\right)\right)^{\frac{\alpha}{\alpha+\beta}}\left(\omega_{1,2}\left(f,\frac{1}{M_n},\frac{1}{M_m}\right)\right)^{\frac{\beta}{\alpha+\beta}}\le$$

$$\le 2\left(\omega_1\left(f,\frac{1}{M_n}\right)\right)^{\frac{\alpha}{\alpha+\beta}}\left(\omega_2\left(f,\frac{1}{M_m}\right)\right)^{\frac{\beta}{\alpha+\beta}}\le$$

$$\le 2\left(\omega\left(f,\frac{1}{M_n}\right)\right)^{\frac{\alpha}{\alpha+\beta}}\left(\omega\left(f,\frac{1}{M_m}\right)\right)^{\frac{\beta}{\alpha+\beta}}.$$

The validity of Corollary 2 follows immediately from Corollary 1.

***Proof of Theorem* 2.** First, we set

$$f_j(x)=\rho_j(x)=\exp\frac{2\pi i x_j}{m_j}.$$

Then we define the function

$$f(x,y)=\sum_{j=1}^{\infty}\frac{1}{M_j^{(\alpha+\beta)}}f_j(x)f_j(y)\,.$$

First, we prove that

$$\omega\left(f,\frac{1}{M_n}\right)_C=O\left(\left(\frac{1}{M_n}\right)^{\alpha+\beta}\right). \tag{24}$$

Since

$$|f_j(x-t)-f_j(x)|=0,\qquad j=0,1,\ldots,n-1,\quad t\in I_n,$$

we find

$$|f(x-t,y)-f(x,y)| \le \sum_{j=1}^{n-1} \frac{1}{M_j^{(\alpha+\beta)}} |f_j(x-t)-f_j(x)| + \sum_{j=n}^{\infty} \frac{2}{M_j^{(\alpha+\beta)}} \le$$

$$\le \frac{c}{M_n^{(\alpha+\beta)}}.$$

Hence,

$$\omega_1\left(f, \frac{1}{M_n}\right) = O\left(\left(\frac{1}{M_n}\right)^{\alpha+\beta}\right). \qquad (25)$$

Analogously, we have

$$\omega_2\left(f, \frac{1}{M_m}\right) = O\left(\left(\frac{1}{M_m}\right)^{\alpha+\beta}\right). \qquad (26)$$

Now, by (25) and (26), we obtain (24).

Next, we shall prove that $\sigma_{M_n,M_n}^{-\alpha,-\beta}(f)$ diverge in the metric of L^1. It is clear that

$$\left\|\sigma_{M_n,M_n}^{-\alpha,-\beta}(f) - f\right\|_1 \ge \left| \int_{G_m^2} \left[\sigma_{M_n,M_n}^{-\alpha,-\beta}(f;x,y) - f(x,y)\right] \psi_{M_k}(x)\psi_{M_k}(y)\, d\mu(x,y) \right| \ge$$

$$\ge \left| \int_{G_m^2} \sigma_{M_n,M_n}^{-\alpha,-\beta}(f;x,y)\, \psi_{M_k}(x)\psi_{M_k}(y)\, dxdy \right| - \left|\widehat{f}(M_k,M_k)\right| =$$

$$= \left| \frac{1}{A_{M_k}^{-\alpha} A_{M_k}^{-\beta}} \sum_{i=0}^{M_{l_k}} \sum_{j=0}^{M_{l_k}} A_{M_k-i}^{-\alpha} A_{M_k-j}^{-\beta} \hat{f}(i,j) \int_{G_m^2} \psi_i(x)\psi_j(y)\, \psi_{M_k}(x)\psi_{M_k}(y)\, d\mu(x,y) \right| -$$

$$- \left|\widehat{f}(M_k,M_k)\right| = \frac{1}{A_{M_{l_k}}^{-\alpha} A_{M_{l_k}}^{-\beta}} \left|\widehat{f}(M_k,M_k)\right| - \left|\widehat{f}(M_k,M_k)\right|.$$

We have

$$\widehat{f}(M_k,M_k) = \int_{G_m^2} f(x,y)\psi_{M_k}(x)\psi_{M_k}(y)\, d\mu(x,y) =$$

$$= \sum_{j=1}^{\infty} \frac{1}{M_j^{(\alpha+\beta)}} \int_{G_m^2} \rho_j(x)\rho_j(y)\, \psi_{M_k}(x)\psi_{M_k}(y)\, d\mu(x,y) =$$

$$= \sum_{j=1}^{\infty} \frac{1}{M_j^{(\alpha+\beta)}} \int_{G_m} \rho_j(x)\psi_{M_k}(x) d\mu(x) \int_{G_m} \rho_j(y)\, \psi_{M_k}(y)\, d\mu(y) = \frac{1}{M_k^{(\alpha+\beta)}}.$$

So, we can write

$$\left\|\sigma_{M_n,M_n}^{-\alpha,-\beta}(f) - f\right\|_1 \ge c(\alpha,\beta).$$

Theorem 2 is proved.

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