К О Р О Т К І   П О В І Д О М Л Е Н Н Я

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A NOTE ON THE REMOVABILITY OF TOTALLY DISCONNECTED SETS FOR ANALYTIC FUNCTIONS

ЗАУВАЖЕННЯ ПРО УСУВНІСТЬ СКРІЗЬ РОЗРИВНИХ МНОЖИН ДЛЯ АНАЛІТИЧНИХ ФУНКЦІЙ

We prove that each totally disconnected closed subset E of a domain G in the complex plane is removable for analytic functions $f(z)$ defined in $G \setminus E$ and such that for any point $z_0 \in E$ the real or imaginary part of $f(z)$ vanishes at z_0.

Доведено, що будь-яка скрізь розривна замкнена підмножина E області G на комплексній площині є усувною для аналітичних функцій $f(z)$, визначених у $G \setminus E$ і таких, що для довільної точки $z_0 \in E$ дійсна або уявна частина $f(z)$ зникає в z_0.

Let G be a domain in the complex plane $\mathbb{C}$, E a totally disconnected closed subset of G, and $f(z) = u(z) + iv(z)$ an analytic function in $G \setminus E$ ($u(z) = \operatorname{Re} f(z)$, $v(z) = \operatorname{Im} f(z)$). Fedorov [1] proved that, if $f(z)$ is continuously extended from $G \setminus E$ to G and $u(z)$ vanishes on E, then this extension is an analytic function in G. Ischanov [2] (see also [3, 4]) generalized this result as follows: if $u(z)$ vanishes on E, then $f(z)$ is analytically extended from $G \setminus E$ to G. The aim of this paper is to prove the following generalization of the mentioned results.

Theorem 1. *Let G be a domain in $\mathbb{C}$, E a totally disconnected closed subset of G, and $f(z) = u(z) + iv(z)$ an analytic function in $G \setminus E$ such that for any $z_0 \in E$ we have either $u(z) \to 0$ or $v(z) \to 0$ as $z \to z_0$, $z \in G \setminus E$. Then the function $f(z)$ can be analytically extended from $G \setminus E$ to G.*

Proof. Let the conditions of Theorem 1 be satisfied and let $z_0 \in E$. Then we have one of the following cases:

(a) the function $f(z)$ is bounded in the intersection of $G \setminus E$ with some neighborhood of the point z_0;

(b) $u(z) \to 0$ as $z \to z_0$, $z \in G \setminus E$, and $\limsup\limits_{z \to z_0, z \in G \setminus E} |v(z)| = +\infty$;

(c) $v(z) \to 0$ as $z \to z_0$, $z \in G \setminus E$, and $\limsup\limits_{z \to z_0, z \in G \setminus E} |u(z)| = +\infty$.

Consider the case (a). Then there is an $r > 0$ such that the disk $D(z_0, r) := \{z \in \mathbb{C} : |z - z_0| < r\}$ is contained in G and the function $f(z)$ is bounded in $D(z_0, r) \setminus E$. Define the function

$$f_1(z) := -if^2(z), \qquad z \in D(z_0, r) \setminus E.$$

Then we have

$$u_1(z) := \operatorname{Re} f_1(z) = 2u(z)v(z), \qquad v_1(z) = \operatorname{Im} f_1(z) = v^2(z) - u^2(z).$$

Since the functions $u(z)$ and $v(z)$ are bounded in $D(z_0,r)\setminus E$, then for any $\zeta\in D(z_0,r)\cap E$ we have $u_1(z)\to 0$ as $z\to\zeta$, $z\in D(z_0,r)\setminus E$. The Ischanov theorem implies the existence of an analytic extension $F(z)$ of the function $f_1(z)$ from $D(z_0,r)\setminus E$ to $D(z_0,r)$. Let $\zeta\in E\cap D(z_0,r)$. Suppose that $F(\zeta)\neq 0$ and take an $\varepsilon\in(0,r)$ such that

$$D(\zeta,\varepsilon)\subset D(z_0,r)\quad\text{and}\quad |F(z)-F(\zeta)|<|F(\zeta)|/2\quad\text{for all}\quad z\in D(\zeta,\varepsilon).$$

Then $\sqrt{iF(z)}$ is a univalent analytic function in $D(\zeta,\varepsilon)$, where the branch of the square root in $D(iF(\zeta),|F(\zeta)|/2)$ is fixed by the condition $\sqrt{iF(z)}=f(z)$ for all $z\in D(\zeta,\varepsilon)\setminus E$.

Thus we justified the existence of an analytic continuation $\bar f(z)$ of the function $f(z)$ from $D(z_0,r)\setminus E$ to $D(z_0,r)\setminus(F^{-1}(0)\cap E)$, where the set $F^{-1}(0):=\{z\in D(z_0,r)\colon F(z)=0\}$ contains only isolated points. Since the function $\bar f(z)$ is bounded in $D(z_0,r)\setminus(F^{-1}(0)\cap E)$, then each point of the set $F^{-1}(0)\cap E$ is a removable singular point for the function $\bar f(z)$.

The above arguments show that we can assume without loss of generality in the proof of Theorem 1 that for any $z_0\in E$ we have either the case (b) or the case (c). Fix an arbitrary domain $G_0\Subset G$, define E_1 and E_2 as the sets consisting of all points $z_0\in E\cap G_0$ satisfying the conditions (b) and (c), respectively, and denote

$$\operatorname{dist}(E_1,E_2):=\inf\{|z_1-z_2|\colon z_1\in E_1, z_2\in E_2\}.$$

Suppose that $\operatorname{dist}(E_1,E_2)=0$. Then there are sequences $\{z_{1n}\}_{n=1}^{\infty}\subset E_1$ and $\{z_{2n}\}_{n=1}^{\infty}\subset E_2$ such that $|z_{1n}-z_{2n}|\to 0$ as $n\to\infty$ whence the compactness of the set $E\cap\overline{G}_0$, where $\overline{G}_0$ is the closure of G_0, implies the existence of a point $z_0\in E\cap\overline{G}_0$ such that

$$\limsup_{\zeta\to z_0,\zeta\in G\setminus E}|u(\zeta)|=\limsup_{\zeta\to z_0,\zeta\in G\setminus E}|v(z)|=+\infty.$$

Therefore, the case (b) or (c) is impossible. Hence, $\operatorname{dist}(E_1,E_2)>0$ and consequently E_1 and E_2 are totally disconnected closed subsets of G_0 such that for any $z_0\in E_1$ we have $u(z)\to 0$ as $z\to z_0$, $z\in G_0\setminus(E_1\cup E_2)$, and for any $z_0\in E_2$ we have $v(z)\to 0$ as $z\to z_0$, $z\in G_0\setminus(E_1\cup E_2)$. Since $\operatorname{dist}(E_1,E_2)>0$, then applying Ischanov's theorem once again we conclude that the function $f(z)$ has an analytic continuation from $G_0\setminus E$ to G_0. Taking into account the arbitrariness in the selection of the domain G_0 we complete the proof of Theorem 1.

References

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