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SOLVABILITY OF A BOUNDARY-VALUE PROBLEM FOR DEGENERATE EQUATIONS

РОЗВ'ЯЗНІСТЬ ГРАНИЧНОЇ ЗАДАЧІ ДЛЯ ВИРОДЖЕНИХ РІВНЯНЬ

We consider a boundary-value problem for degenerate equations with discontinuous coefficients and establish the unique strong solvability (almost everywhere) of this problem in the corresponding weighted Sobolev space.

Розглянуто граничну задачу для вироджених рівнянь з розривними коефіцієнтами. Встановлено однозначну сильну (майже скрізь) розв'язність цієї задачі у відповідному зваженому просторі Соболєва.

1. Introduction. The purpose of this work is to prove a unique strong (almost everywhere) solvability of the first boundary-value problem for equation

$$Zu = \sum_{i,j=1}^{n} a_{ij}(x,t)\, u_{ij} + \psi(x,t)\, u_{tt} - u_t = f(x,t), \tag{1.1}$$

$$u|_{\Gamma(Q_T)} = 0, \tag{1.2}$$

in cylinder $Q_T = \Omega \times (0,T)$, $T \in (0,\infty)$, where Ω is a bounded domain in R^n with a boundary $\partial\Omega \subset C^2$. $\Gamma(Q_T) = (\partial\Omega \times [0,T]) \cup \Omega \times \{(x,t) : t = 0\}$ is a parabolic boundary of the domain Q_T, $\psi(x,t)$ and coefficients $a_{ij}(x,t)$ tend to zero. Here

$$u_{ij} = \frac{\partial^2 u(x,t)}{\partial x_i \partial x_j}, \qquad u_{tt} = \frac{\partial^2 u(x,t)}{\partial t^2}, \qquad u_t = \frac{\partial u}{\partial t}.$$

Initial boundary problems for this type of degenerate equations have been studied by many authors (see, for example, [2–4]). In [1], Fichera considered boundary-value problems for degenerate equations in multidimensional case. He proved existence of generalized solutions to these boundary-value problems. Boundary-value problems for the degenerate equations of such type were studied in the stationary case in [5] and in the nonstationary case in [6]. In [8], coercive estimates for this problem have been obtained. We also mention the works [2–4] where strong solvability of the boundary-value problem (1.1), (1.2) was established for equations with smooth coefficients. Similar results Cordes-type discontinuous coefficients have been established in [4]. In [9, 10], some classes of elliptic parabolic equations are considered. In [9], well-posedness of the initial boundary-value problem for pseudoparabolic equations is studied and estimates of the generalized solution are obtained. In [10], solvability results have been obtained in case of Cordes-type discontinuous coefficients. In [11], some general problem for linear and quasilinear equations of parabolic type is considered.

In our paper we consider wide classes of elliptic parabolic equations.

Assume that the coefficients satisfy the conditions: $|a_{ij}(x,t)|$ is a symmetrical matrix with real measurable elements in Q_T and, for any $(x,t) \in Q_T$, $\xi \in R^n$, following inequalities are true:

$$\gamma\omega(x)|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(x,t)\,\xi_i\xi_j \le \gamma^{-1}\omega(x)|\xi|^2, \tag{1.3}$$

where $\gamma \in (0,1]$, $\omega(x) \in A_p$ satisfies the Muckenhoupt condition (see [7]) and

$$\psi(x,t) = \omega(x)\,\lambda(t)\,\varphi(T-t), \tag{1.4}$$

where

$$\lambda(t) \ge 0, \quad \lambda(t) \in C^1[0,T],$$

$$\varphi(z) \ge 0, \qquad \varphi'(z) \ge 0, \qquad \varphi(z) \in C^1[0,T],$$

$$\varphi(0) = \varphi'(0) = 0, \qquad \varphi(z) \ge \beta z\varphi'(z),$$

β is a positive constant.

2. Auxiliary results. Our goal is to establish a unique strong solvability of the boundary-value problem (1.1), (1.2) by means of coercive estimate obtained in [8], using coercive continuation method by parameter. For this purpose, let us prove the solvability for some model equation from the class under consideration. As a model operator we considering operator

$$Z_0 = \omega(x)\,\Delta + \varphi(T-t)\,\frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial t},$$

where $\Delta = \sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}$ is a Laplace operator and function $\varphi(z)$ satisfies the conditions (1.4).

Throughout this paper we consider the most interesting case, where $\varphi(z) > 0$ for $z > 0$. If $\varphi(z) \equiv 0$, then the equation (1.1) is parabolic and the corresponding results on solvability of the boundary-value problem was obtained in this case in [7]. But if $\varphi(z) = 0$ at $z \in [0, z^0]$, then the solution of the problem (1.1), (1.2) can be obtained by combining the solution $u(x,t)$ of the problem in a cylinder Q_{z_0} with the solution $v(x,t)$ of boundary problem for parabolic equation in a cylinder $\Omega \times (z^0, T)$ with boundary conditions $v(x,z^0) = u(x,z^0)$, $v|_{\partial\Omega\times[z^0,T]} = 0$. Let us fix an arbitrary $\varepsilon \in (0,T)$ and introduce a function $\varphi(z)$ by $\varphi_\varepsilon(z) = \varphi(\varepsilon) - \frac{\varphi'(\varepsilon)\varepsilon}{m} + \frac{\varphi'(\varepsilon)}{m\varepsilon^{m-1}}z^m$ for $z \in [0,\varepsilon)$, $\varphi_\varepsilon(z) = \varphi(z)$ for $z \in [\varepsilon, T]$, where $m = \frac{2}{\beta}$. It is easy to see that $\varphi_\varepsilon \in C^1[0,T]$. Let us show that for $z \in [0,T]$

$$\varphi_\varepsilon(z) \ge \frac{1}{2}\varphi(z). \tag{2.1}$$

It suffices to prove (2.1) for $z \in [0,\varepsilon)$. It is clear that due to monotonicity of $\varphi(z)$ the inequality (2.1) will be fulfilled if

$$\varphi(\varepsilon) - \frac{\varphi'(\varepsilon)\varepsilon}{m} \ge \frac{1}{2}\varphi(\varepsilon)$$

or

$$\varphi(\varepsilon) \geq \frac{2}{m}\varphi'(\varepsilon)\varepsilon.$$

The last estimate is true by condition (1.4). Hence, the inequality (2.1) is proved. Without loss of generality, we consider the case $m > 1$. Then

$$q_\varepsilon(T) = \sup_{[0,T]} \varphi'_\varepsilon(z) \leq q(T) = \sup_{[0,T]} \varphi(z). \tag{2.2}$$

For $R > 0$, $x^0 \in R^n$, we consider a ball $B_R(x^0) = \{x\colon |x - x_0| < R$ and a cylinder $Q_T^R = B_R(x^0)\times(0,T)$. Let $\overline{B}_R(x^0) \subset \Omega$. We say that $u(x,t) \subset A\left(Q_T^R(x^0)\right)$ if $u(x,t) \in C^\infty\left(\overline{Q}_T^R(x^0)\right)$, $u|_{t=0} = 0$ and $\sup p\ u\left(\overline{Q}_T^\rho(x^0)\right)$ for some $\rho \in (0,R)$. We will also use the Banach spaces $W_2^{1,0}(Q_T)$, $W_2^{2,0}(Q_T)$, $W_2^{2,1}(Q_T)$ and $W_{2,\psi}^{2,2}(Q_T)$ of functions $u(x,t)$ given on Q_T with finite norms (see also [8])

$$\|u\|_{W_{2,\omega}^1(Q_T)} = \left(\int\limits_{Q_T} \omega(x)\left(u^2 + \sum_{i=1}^n u_{x_i}^2\right) dxdt\right)^{\frac{1}{2}},$$

$$\|u\|_{W_{2,\omega}^2(Q_T)} = \left(\int\limits_{Q_T} \omega(x)\left(u^2 + \sum_{i=1}^n u_{x_i}^2 + \sum_{i,j=1}^n u_{x_ix_j}^2\right) dxdt\right)^{\frac{1}{2}},$$

$$\|u\|_{W_{2,\omega}^{2,1}(Q_T)} = \|u\|_{W_{2,\omega}^2(Q_T)} + \|u_t\|_{L_2(Q_T)},$$

$$\|u\|_{W_{2,\psi}^{2,2}(Q_T)} = \left(\int\limits_{Q_T}\left[\omega(x)\left(u^2 + \sum_{i=1}^n u_{x_i}^2 + \sum_{i,j=1}^n u_{x_ix_j}^2\right) + u_t^2 + \right.\right.$$

$$\left.\left. + \psi^2(x,t)u_{tt}^2 + \psi(x,t)\sum_{i=1}^n u_{it}^2\right] dxdt\right)^{\frac{1}{2}}.$$

Let $\overset{\circ}{W}{}_{2,\psi}^{2,2}(Q_T)$ be a subspace of $W_{2,\psi}^{2,2}(Q_T)$ consisting of all functions from $C^\infty\left(\overline{Q}_T\right)$ which vanish on $\Gamma(Q_T)$ and form a dense set in $W_{2,\psi}^{2,2}(Q_T)$.

Let us consider the operator

$$Z_\varepsilon = \omega(x)\Delta + \varphi_\varepsilon(T-t)\frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial t}.$$

Lemma 2.1. *If $\omega(x)$ satisfies the Muckenhoupt condition* [7], *then there exists $T_1(\varphi(z),\omega(x),n)$ such that , for $T \leq T_1$ and any function $u(x,t) \in B(Q_T^R(x^0))$, the following estimate is true:*

$$\int\limits_{Q_T^R(x^0)} \left(\omega(x)\sum_{i,j=1}^n u_{ij}^2 + u_t^2 + \varphi_\varepsilon(T-t)\sum_{i=1}^n u_{it}^2 + \varphi_\varepsilon(T-t)\sum_{i=1}^n u_{it}^2\right) dxdt \leq$$

$$\leq (1 + 2(n+1)q(T)) \int\limits_{Q_T^R(x^0)} Z_\varepsilon(u)^2 dxdt. \tag{2.3}$$

(We say that $u(x,t) \in B(Q_T^R(x^0))$ if $u(x,t) \in A(Q_T^R(x^0))$, $u|_{t=T} = u_t|_{t=T}\, 0$.)

Proof. For simplicity we shall write Q_T instead of $Q_T^R(x^0)$. We have

$$\int\limits_Q (Z_\varepsilon u)^2\, dxdt \geq \int\limits_Q \omega(x) \sum_{i,j=1}^n u_{ij}^2 dxdt + \int\limits_Q \varphi_\varepsilon^2\,(T-t)\, u_{tt}^2 dxdt +$$

$$+ \int\limits_Q u_t^2 dxdt + 2\int\limits_Q \varphi_\varepsilon(T-t)\Delta u u_{tt} dxdt - 2\int\limits_Q \varphi_\varepsilon(T-t) u_t u_{tt} dxdt. \tag{2.4}$$

But, on the other hand,

$$2\int\limits_Q \varphi_\varepsilon(T-t)\Delta u\; u_{tt} dxdt - 2\int\limits_Q \sum_{i,j=1}^n (\varphi_\varepsilon(T-t)u_{ii})_t\; u_t dxdt =$$

$$= 2\int\limits_Q \varphi_\varepsilon'(T-t)\sum_{i=1}^n u_{ii}u_t dxdt - 2\int\limits_Q \varphi_\varepsilon(T-t)\sum_{i=1}^n u_{iit}u_t dxdt \geq$$

$$\geq -q_\varepsilon(T)\int\limits_Q \omega(x)\sum_{i,j=1}^n u_{ij}^2 dxdt - nq_\varepsilon(T)\int\limits_Q u_t^2 dxdt +$$

$$+2\int\limits_Q \varphi_\varepsilon(T-t)\sum_{i=1}^n u_i^2 dxdt, \tag{2.5}$$

because

$$u_{ii}|_{t=0} = u_{ii}|_{t=T} = 0.$$

Also we have

$$-2\int\limits_Q \varphi_\varepsilon(T-t)u_t u_{tt} dxdt = -\int\limits_Q \varphi_\varepsilon'(T-t)\sum_{i=1}^n u_i^2 dxdt +$$

$$+\varphi_\varepsilon(T)\int\limits_B u_t^2(x,0)dxdt \geq -q_\varepsilon(T)\int\limits_Q u_t^2 dxdt, \tag{2.6}$$

because

$$u_{ii}|_{t=T} = 0.$$

By virtue of conditions (1.4) for $T \to 0$, we have $q(T) \to 0$. Choose T_1 such that

$$(n+1)q(T_1) \leq \frac{1}{2}.$$

Then, for $T \le T_1$, we have

$$\frac{1}{1-(n+1)q(T)} \le 1 + 2(n+1)q(T).$$

Now using this inequality (2.2), and Lemma 1 of [8], we get the estimate (2.3).

Lemma 2.1 is proved.

Lemma 2.2. *Let the function* $\varphi(z)$ *satisfies the conditions* (1.4), *and* $\omega(x)$ *satisfies the Muckenhoupt condition. Let the operators* Z_ε *with* $\varepsilon > 0$ *be the same as in Lemma* 2.1. *Then, for* $T \le T_2(\varphi, \omega, n, \Omega)$ *and any function* $u(x,t) \in \overset{\circ}{W}{}^{2,2}_{2,\varphi_\varepsilon}(Q_T)$, *the following estimate is true*:

$$\|u\|_{\overset{\circ}{W}{}^{2,2}_{2,\varphi_\varepsilon}(Q_T)} \le C_1(\varphi, \omega, n, \Omega) \|Z_\varepsilon u - \mu u\|_{L_2(Q_T)}, \tag{2.7}$$

where $\mu = \dfrac{1}{T}$, $W^{2,2}_{2,\varphi_\varepsilon}(Q_T)$ *is a Banach space of functions defined above with function* ψ *replaced by* φ_ε.

Proof. It suffices to prove the lemma for functions $u(x,t) \in C^\infty\left(\overline{Q}_T\right)$, $u|_{\partial Q_T} = 0$. Note that, according to the above mentioned, $q\left(\dfrac{T}{T_1}\right) \le 1$. Then, as in the proof of coercive estimate [8], we derive from (1.1) the existence of $T_3(\varphi, \omega, n, \Omega) \le T_1$ such that if $T \le T_3$, then for any function $v(x,t) \in C^\infty\left(\overline{Q}_T\right)$, $v|_{\Gamma(Q_T)} = 0$, $v|_{t=T} = v_t|_{t=T} = 0$ the following estimate is true:

$$\|v\|_{W^{2,2}_{2,\varphi_\varepsilon}(Q_T)} \le C_2(\varphi, \omega, n, \Omega)\left(\|Z_\varepsilon v\|_{L_2(Q_T)} + \|v\|_{L_2(Q_T)}\right). \tag{2.8}$$

Let $T \le T_3/2$. We take $R = T/4$ and let $u(x,t) \in C^\infty\left(\overline{Q}_T\right)$, $u|_{\partial Q_T} = 0$. We consider a function $g(t) \in C^\infty[0,T]$ such that $g(t) = 1$ for $t \in [0, T-R]$, $g(t) = 0$ at $t \in \left[T - \dfrac{R}{2}, T\right]$, $0 \le g(t) \le 1$ and

$$\left|g'(t)\right| \le C_3/R, \qquad \left|g'(t)\right| \le C_3/R^2. \tag{2.9}$$

Putting in (2.8) $v(x,t) = u(x,t)g(t)$ and taking into account (2.9), we get

$$\|u\|_{W^{2,2}_{2,\varphi_\varepsilon}(Q_{T-R})} \le C_4(\varphi, \omega, n, \Omega)\left(\|Z_\varepsilon(ug)\|_{L_2(Q_T)} + \|u\omega(x)\|_{L_2(Q_T)}\right) \le$$
$$\le C_5(\varphi, \omega, n, \Omega)\left(\|Z_\varepsilon u\|_{L_2(Q_T)} + \left(\frac{C_6}{R} + 1\right)\|u\omega(x)\|_{L_2(Q_T)}\right) +$$
$$+ \frac{2C_6}{R}\|\varphi_\varepsilon u_t\|_{L_2(Q_T)} + \frac{2C_6}{R^2}\|\varphi_\varepsilon u\|_{L_2(Q_T)}. \tag{2.10}$$

From the conditions (1.4) it follows that $\sup_{[0,T]} \varphi(z) \le C_7(\varphi)T$. So, taking into account that $\sup_{[0,T]} \varphi_\varepsilon(z) \le \sup_{[0,T]} \varphi(z)$, we conclude

$$\|\varphi_\varepsilon u\|_{L_2(Q_T)} \le C_7 T \|u\omega(x)\|_{L_2(Q_T)}. \tag{2.11}$$

On the other hand, for any $\alpha' > 0$ the interpolation inequality

$$\|\varphi_\varepsilon u_t\|_{L_2(Q_T)} \le C_8 T\alpha' \|\varphi_\varepsilon u_{tt}\|_{L_2(Q_T)} + \frac{1}{\alpha'}\|u\|_{L_2(Q_T)} \tag{2.12}$$

holds. Indeed, let us fix an arbitrary α' and consider for $\nu > 0$ the integral

$$k = \int\limits_{Q_T} \left[\nu\varphi_\varepsilon^2(T-t)u_{tt} + \frac{1}{\nu}u\omega(x)\right]^2 dxdt.$$

It is clear that $k \ge 0$. At the same time

$$k = \nu^2 \int\limits_{Q_T} \varphi_\varepsilon^2(T-t)u_{tt}^2 dxdt + \frac{1}{\nu^2}\int\limits_{Q_T}\omega(x)u^2 dxdt + 2\int\limits_{Q_T}\varphi_\varepsilon^2(T-t)u_{tt}u dxdt \le$$

$$\le C_8^2 T^2\nu^2 \int\limits_{Q_T}\varphi_\varepsilon^2(T-t)u_{tt}^2 dxdt + \frac{1}{\nu^2}\int\limits_{Q_T}\omega(x)u^2 dxdt - 2\int\limits_{Q_T}\varphi_\varepsilon^2(T-t)u_{tt}^2 dxdt +$$

$$+4\int\limits_{Q_T}\varphi_\varepsilon(T-t)\varphi_\varepsilon'(T-t)uu_t dxdt.$$

Besides, by using the fact that $q(T) \le 1$ and the inequality (2.2), we get

$$4\int\limits_{Q_T}\varphi_\varepsilon(T-t)\varphi_\varepsilon'(T-t)uu_t dxdt \le \int\limits_{Q_T}\varphi_\varepsilon^2(T-t)u_t^2 dxdt +$$

$$+4\int\limits_{Q_T}(\varphi_\varepsilon'(T-t))u^2 dxdt \le \int\limits_{Q_T}\varphi_\varepsilon^2(T-t)u_t^2 dxdt + 4\int\limits_{Q_T}\omega(x)u^2 dxdt. \tag{2.13}$$

From (2.12), (2.13) it follows that

$$\int\limits_{Q_T}\varphi_\varepsilon^2(T-t)u_t^2 dxdt \le C_8^2T^2\nu^2\int\limits_{Q_T}\varphi_\varepsilon^2(T-t)u_{tt}^2 dxdt +$$

$$+\left(\frac{1}{\nu^2}+4\right)\int\limits_{Q_T}\omega(x)u^2 dxdt.$$

Now putting $\nu = \min\{\alpha', 1\}$ we prove the inequality (2.12).

By using (2.11) and (2.12) in (2.10), we conclude that, for any $\alpha' > 0$, the inequality

$$\|u\|_{W^{2,2}_{2,\varphi_\varepsilon}(Q_{T-R})} \le C_4\|Z_\varepsilon(ug)\|_{L_2(Q_T)} + 8\alpha' C_4C_5C_6\|u\|_{W^{2,2}_{2,\varphi_\varepsilon}(Q_T)} +$$

$$+\frac{C_7(\varphi,\omega,n,\Omega)}{\alpha' R}\|u\omega(x)\|_{L_2(Q_T)} \tag{2.14}$$

holds.

Let us fix an arbitrary $\alpha > 0$ and choose $\alpha' = \dfrac{\alpha}{C_4C_5C_6}$. Then from (2.14) follows that

$$\|u\|_{W^{2,2}_{2,\varphi_\varepsilon}(Q_{T-R})} \leq C_4 \|Z_\varepsilon u)\|_{L_2(Q_T)} + \alpha \|u\|_{W^{2,2}_{2,\varphi_\varepsilon}(Q_T)} +$$

$$+\frac{C_8(\varphi,\omega,n,\Omega)}{\alpha T^2} \|u\omega(x)\|_{L_2(Q_T)} . \tag{2.15}$$

Similarly, we can show that if $Q' = \Omega \times (T-2R, T+2R)$, $Q'' = \Omega \times (T-R, T+R)$, $S(Q') = \partial\Omega \times [T-2R, T+2R]$, then for any function $W(x,t) \in C^\infty\left(\overline{Q}_T\right)$, $W\omega(x)|_{S(Q_T)} = 0$ and for any $\alpha > 0$ the following estimate is true:

$$\|W\|_{W^{2,2}_{2,\varphi_\varepsilon}(Q'')} \leq C_4 \|Z_\varepsilon W\|_{L_2(Q')} + \alpha \|W\|_{W^{2,2}_{2,\varphi_\varepsilon}(Q')} +$$

$$+\frac{C_9(\varphi,\omega,n,\Omega)}{\alpha' R} \|W\omega(x)\|_{L_2(Q_T)} . \tag{2.16}$$

Let $Q'_t = \Omega \times (T-2R, T)$, $Q'_- = \Omega \times (T, T+2R)$, $Q''_+ = \Omega \times (T-R, T)$. Let us extend the function $\varphi_\varepsilon(T-t) - ih$ through the hyperplane $t = T$ from Q'_+ onto Q'_- in an even and odd ways. Denote the extended functions also by $u(x,t)$ and $\varphi_\varepsilon(T-t)$, respectively.

Putting $w = u$ in (2.16) and taking into account the inequality

$$\|u\|_{W^{2,2}_{2,\varphi_\varepsilon}(Q''_+)} \leq \sqrt{2}\, \|u\|_{W^{2,2}_{2,\varphi_\varepsilon}(Q'_+)}$$

and similar inequalities for the norms $\|u\|_{W^{2,2}_{2,\varphi_\varepsilon}(Q')}$, $\|u\|_{L_2(Q')}$, $\|Z_\varepsilon u\|_{L_2(Q')}$, we get

$$\|u\|_{W^{2,2}_{2,\varphi_\varepsilon}(Q''_+)} \leq C_{10} \|Z_\varepsilon u\|_{L_2(Q'_+)} + \alpha \|u\|_{W^{2,2}_{2,\varphi_\varepsilon}(Q'_+)} +$$

$$+\frac{C_{11}(\varphi,\omega,n,\Omega)}{\alpha T} \|u\|_{L_2(Q'_+)} . \tag{2.17}$$

Combining (2.15), (2.17) and choosing the corresponding α, we conclude

$$\|u\|^2_{W^{2,2}_{2,\varphi_\varepsilon}(Q_+)} \leq C_{12}(\varphi,\omega,n,\Omega) \|Z_\varepsilon u\|^2_{L_2(Q_T)} + \frac{1}{T^2} \|u\omega(x)\|^2_{L_2(Q_T)} . \tag{2.18}$$

On the other hand, recalling that $\mu = \dfrac{1}{T}$, we have

$$\int\limits_{Q_T} (Z_\varepsilon u - \mu u)^2 \, dxdt = \|Z_\varepsilon u\|^2_{L_2(Q_T)} + \mu^2 \|u\omega(x)\|^2_{L_2(Q_T)} - 2\mu \int\limits_{Q_T} u Z_\varepsilon u dxdt =$$

$$= \|Z_\varepsilon u\|^2_{L_2(Q_T)} + \mu^2 \|u\omega(x)\|^2_{L_2(Q_T)} + K_1. \tag{2.19}$$

Moreover,

$$K_1 = -2\mu \int\limits_{Q_T} u\omega(x)\, (\Delta u + \varphi_\varepsilon(T-t)\, u_{tt} - u_t) dxdt =$$

$$= 2\mu \int\limits_{Q_T} \sum_{i=1}^n \omega(x) u_i^2 dxdt - 2\mu \int\limits_{Q_T} \varphi_\varepsilon(T-t) u u_{tt} dxdt + \mu \int\limits_{Q_T} (u^2)_t dxdt \geq$$

$$\geq 2\mu \int\limits_{Q_T} \varphi_\varepsilon(T-t)uu_{tt}dxdt - 2\mu \int\limits_{Q_T} \varphi'_\varepsilon(T-t)uu_t dxdt. \tag{2.20}$$

Let us show that for $z \in (0,T)$

$$\varphi_\varepsilon(z) \geq \beta z \varphi'_\varepsilon(z). \tag{2.21}$$

Due to (1.4) it suffices to prove (2.21) only for $z \in (0,\varepsilon)$. But for such z the inequality (2.21) is equivalent to the inequality

$$\varphi(\varepsilon) - \frac{\varphi'(\varepsilon)\varepsilon}{m} \geq \frac{\varphi'(\varepsilon)z^m}{m\varepsilon^{m-1}}, \text{ where } m = \frac{2}{\beta}.$$

The last inequality is true if the following estimate holds:

$$\varphi(\varepsilon) \geq \frac{2}{m}\varphi'(\varepsilon)\varepsilon. \tag{2.22}$$

Note that (2.22) is follows from the conditions (1.4). Hence, from (2.20), (2.21) and (2.22), we obtain

$$k_1 \geq -\frac{\mu}{2}\int\limits_{Q_T} \frac{[\varphi'_\varepsilon(T-t)]^2}{\varphi_\varepsilon(T-t)}u^2 dxdt \geq \frac{\mu}{2\beta}\int\limits_{Q_T} \frac{\varphi'_\varepsilon(T-t)}{T-t}u^2 dxdt \geq$$

$$\geq -\frac{\mu q(T)T}{2\beta}\int\limits_{Q_T} \frac{\omega(x)u^2}{(T-t)^2}dxdt. \tag{2.23}$$

We apply the Hardy inequality according to which

$$\int\limits_{Q_T} \frac{\omega(x)u^2}{(T-t)^2}dxdt \ \leq 4\int\limits_{Q_T} u_t^2 dxdt. \tag{2.24}$$

Then, from (2.19), (2.23) and (2.24), we conclude

$$\|Z_\varepsilon u\|^2_{L_2(Q_T)} + \mu^2 \|\omega(x)u\|^2_{L_2(Q_T)} \leq \|Z_\varepsilon u - \mu u\|^2_{L_2(Q_T)} +$$

$$+\frac{2q(T)}{\beta}\|u\|_{W^{2,2}_{2,\varphi_\varepsilon}(Q_T)}. \tag{2.25}$$

Now let us choose $T_4(\varphi,\omega,n,\Omega)$ small enough to satisfy

$$q(T_4) \leq \frac{\beta}{4C_{12}} \text{ and fix } T_2 = \min\left\{\frac{T_3}{2}, T_4\right\}.$$

Then, from (2.18) and (2.25), we obtain the estimate (2.7).

Lemma 2.2 is proved.

Now let us establish the solvability of our problem for a model equation. Let us consider the operator

$$Z'_0 = \omega(x)\Delta + \psi(x,t)\frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial t},$$

where $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ is a Laplace operator.

Lemma 2.3. *If $\omega(x)$ satisfies the Muckenhoupt condition and $\psi(x,t)$ satisfies the conditions* (1.4), *then, for $T \le T_S(\psi)$, $\tau \in [0,1]$ and any function $u(x,t) \in A\left(Q_T^R(x^0)\right)$, the following estimate is true:*

$$\int\limits_{Q_T^R(x^0)} \left(\omega^2(x)\sum_{i,j=1}^n u_{ij}^2 + u_t^2 + \psi^2(x,t)u_{tt}^2 + \psi(x,t)\sum_{i=1}^n u_{it}^2\right) dxdt \le$$

$$\le (1 + D(T)S_2) \int\limits_{Q_T^R(x^0)} \left(Z_0'u - \frac{\tau}{T}\omega(x)u\right)^2 dxdt, \qquad (2.26)$$

where $S_2 = S_2(\psi, n)$ is some constant, $D(T) = q(T) + q_1(T)$, $q_1(T) = \sup_{t\in[0,T]} \varphi(t)$, $q(T) = \sup_{t\in[0,T]} \varphi'(t)$.

Proof. It suffices to consider the case $\tau > 0$. We denote $\dfrac{\tau}{T}$ by μ'. Then we have

$$I_1 = \int\limits_{Q_T^R(x^0)} \left(Z_0'u - \mu'\omega(x)u\right)^2 dxdt = \int\limits_{Q_T^R(x^0)} \left(Z_0'u\right)^2 dxdt +$$

$$+ \left(\mu'\right)^2 \int\limits_{Q_T^R(x^0)} \omega(x)u^2 dxdt - 2\mu' \int\limits_{Q_T^R(x^0)} \omega(x)u\Delta u dxdt +$$

$$+ 2\mu' \int\limits_{Q_T^R(x^0)} uu_t dxdt - 2\mu' \int\limits_{Q_T^R(x^0)} \psi(x,t)u_{tt}u dxdt. \qquad (2.27)$$

In Lemma 2.1 of [8], the following estimate has been obtained:

$$\int\limits_{Q_T^R(x^0)} \left(\omega^2(x)\sum_{i,j=1}^n u_{ij}^2 + u_t^2 + \psi^2(x,t)u_{tt}^2 + \psi(x,t)\sum_{i=1}^n u_{it}^2\right) dxdt \le$$

$$\le (1 + DS) \int\limits_{Q_T^R(x^0)} \left(Z_0'u\right)^2 dxdt,$$

where $S = S(\psi, n)$ is some constant.

We can rewrite it as follows:

$$\int\limits_{Q_T^R(x^0)} \left(Z_0'u\right)^2 dxdt \ge \frac{1}{1 + SD(T)} \int\limits_{Q_T^R(x^0)} \left(\omega^2(x)\sum_{i,j=1}^n u_{ij}^2 + u_t^2 +\right.$$

$$\left. + \psi^2(x,t)u_{tt}^2 + \psi(x,t)\sum_{i=1}^n u_{it}^2\right) dxdt.$$

But $\dfrac{1}{1 + SD(T)} = 1 - \dfrac{SD(T)}{1 + SD(T)} \ge 1 - SD(T)$ and

$$\int\limits_{Q_T^R(x^0)} \left(Z_0'u\right)^2 dxdt \geq (1 - SD(T)) \int\limits_{Q_T^R(x^0)} \left(\omega^2(x) \sum_{i,j=1}^n u_{ij}^2 + u_t^2 + \right.$$

$$\left. + \psi^2(x,t)u_{tt}^2 + \psi(x,t) \sum_{i=1}^n u_{it}^2 \right) dxdt.$$

We have used the last inequality to estimate the first term in (2.27). For the third term in (2.27) we have

$$-2\mu' \int\limits_{Q_T^R(x^0)} \omega^2(x) u\Delta u dxdt = 2\mu'\mu \int\limits_{Q_T^R(x^0)} \omega_{x_i}(x) \sum_{i=1}^n u_i^2 dxdt \geq$$

$$\geq 2\mu' M \int\limits_{Q_T^R(x^0)} \omega(x) \sum_{i=1}^n u_i^2 dxdt \geq 0,$$

where $M = \sup_{Q_T^R(x^0)} |u(x)|$.

For the fourth term we get

$$2\mu' \int\limits_{Q_T^R(x^0)} \omega^2(x) u u_t dxdt = \mu' \int\limits_{Q_T^R(x^0)} \omega^2(x) u^2(x,T) dx \geq 0.$$

Let us consider the fifth term in (2.27) in detail:

$$-2\mu' \int\limits_{Q_T^R(x^0)} \psi(x,t) u_{tt} u dxdt = -2\mu' \int\limits_{Q_T^R(x^0)} \varphi(T-t)\lambda(t)\omega(x) u_{tt} u dxdt =$$

$$= -2\mu' \int\limits_{Q_T^R(x^0)} \psi(x,t) u_t^2 dxdt - 2\mu' \int\limits_{Q_T^R(x^0)} \varphi'(t-T)\lambda(t)\omega(x) u_t u dxdt -$$

$$-2\mu' \int\limits_{Q_T^R(x^0)} \varphi(T-t)\lambda'(t)\omega(x) u_t u dxdt \geq$$

$$\geq -2\mu' \int\limits_{Q_T^R(x^0)} \varphi'(T-t)\lambda(t)\omega(x) |u| |u_t| dxdt -$$

$$-2\mu' \int\limits_{Q_T^R(x^0)} \varphi(T-t) |\lambda(t)| \omega(x) |u| |u_t| dxdt \geq -\mu' C_{13}(\omega) C_{14}(\lambda) 2q(T) \times$$

$$\times \int\limits_{Q_T^R(x^0)} u_t^2 dxdt - \frac{\mu'}{\alpha} C_{13}(\omega) C_{14}(\lambda) q(T) \int\limits_{Q_T^R(x^0)} \omega^2(x) u^2 dxdt -$$

$$-\mu' C_{13}(\omega)C_{14}(\lambda)\alpha q_1(T)\int\limits_{Q_T^R(x^0)} u_t^2 dxdt-$$

$$-\frac{\mu'}{\alpha}C_{13}(\omega)C_{14}(\lambda)q_1(T)\int\limits_{Q_T^R(x^0)}\omega^2(x)u^2dxdt. \tag{2.28}$$

Let $C_{15}=\max\{C_{13},C_{14}\},\ C_{16}=C_{13}C_{15}$. Then, from (2.28), we obtain

$$-2\mu'\int\limits_{Q_T^R(x^0)}\psi(x,t)u_{tt}udxdt\geq -C_{15}\alpha D(T)\int\limits_{Q_T^R(x^0)}u_t^2dxdt-$$

$$-\frac{\mu'}{\alpha}C_{15}D(T)\int\limits_{Q_T^R(x^0)}\omega^2(x)u^2dxdt. \tag{2.29}$$

Let $T\leq T_S(\psi)$ be so small that $C_{15}D(T)\leq 1$. Then, taking into account the above inequalities, from (2.27) we get

$$I_1\geq(1-SD(T))\int\limits_{Q_T^R(x^0)}\left(\omega^2\sum_{i,j=1}^n u_{ij}^2+u_t^2+\right.$$

$$\left.+\psi^2(x,t)u_{tt}^2+\psi(x,t)\sum_{i=1}^n u_{it}^2\right)dxdt+\left(\mu'\right)^2\int\limits_{Q_T^R(x^0)}\omega^2u^2dxdt-$$

$$-\mu' C_{15}\alpha D(T)\int\limits_{Q_T^R(x^0)}u_t^2dxdt-\frac{\mu'}{\alpha}\int\limits_{Q_T^R(x^0)}\omega^2(x)u^2dxdt.$$

If we put $\alpha=\dfrac{1}{\mu'}$, then we have

$$I_1\geq(1-SD(T))\int\limits_{Q_T^R(x^0)}\left(\omega^2(x)\sum_{i,j=1}^n u_{ij}^2+u_t^2+\right.$$

$$\left.+\psi^2(x,t)u_{tt}^2+\psi(x,t)\sum_{i=1}^n u_{it}^2\right)dxdt-$$

$$-C_{15}D(T)\int\limits_{Q_T^R(x^0)}u_t^2dxdt=(1-S_3D(T))\int\limits_{Q_T^R(x^0)}\left(\omega^2(x)\sum_{i,j=1}^n u_{ij}^2+\right.$$

$$\left.+u_t^2+\psi^2(x,t)u_{tt}^2+\psi(x,t)\sum_{i=1}^n u_{it}^2\right)dxdt,$$

where $S_3 = S + C_{15}$. Hence,

$$\int\limits_{Q_T^R(x^0)} \left(\omega^2(x) \sum_{i,j=1}^{n} u_{ij}^2 + u_t^2 + \psi^2(x,t) u_{tt}^2 + \psi(x,t) \sum_{i=1}^{n} u_{it}^2 \right) dxdt \leq$$

$$\leq \frac{1}{1 - S_3 D(T)} I_1 + \frac{S_3 D(T)}{1 - S_3 D(T)} I_1.$$

Let T_5 be so small that $S_3 D(T) \leq \frac{1}{2}$. Then

$$\int\limits_{Q_T^R(x^0)} \left(\omega^2(x) \sum_{i,j=1}^{n} u_{ij}^2 + u_t^2 + \psi^2(x,t) u_{tt}^2 + \psi(x,t) \sum_{i=1}^{n} u_{it}^2 \right) dxdt \leq$$

$$\leq \left(1 + 2 S_3 D(T)\right) I_1 = \left(1 + S_4 D(T)\right) I_1.$$

So, we get the needed estimate (2.26).

Lemma 2.3 is proved.

Lemma 2.4. *Let the coefficients of the operator Z satisfy the conditions* (1.3), (1.4). *Then, for any function $u(x,t) C^\infty\left(\overline{Q}_T\right)$, $u|_{\Gamma(Q_T)} = 0$, for $T \leq T_6(\gamma, \psi, n, \Omega)$ and any $\tau \in [0,1]$, the following estimate is true:*

$$\|u\|_{W_{2,\varphi_\varepsilon}^{2,2}(Q''_+)} \leq C_{16}(\gamma, \psi, n) \left\| Zu - \frac{\tau}{T} \omega^2(x) u \right\|_{L_2(Q_T)}.$$

Proof is similar to the proof of coercive estimate for the operator Z in [8].

In what follows, we will denote the operators $Z_0 - \mu$ and $Z_\varepsilon - \mu$ by M_0 and M, respectively. We will also denote $T_0 = \min\{T_9, T_6\}$.

3. Strong solvability of boundary-value problem. Main results.

Theorem 3.1. *Let the function $\varphi(z)$ satisfies the conditions* (1.4). *Then, for $T \leq T^0$, the boundary-value problem*

$$M_0 u = f(x,t)(x,t) \in Q_T, \tag{3.1}$$

$$u|_{\Gamma(Q_T)} = 0, \tag{3.2}$$

has a unique strong solution in the space $W_{2,\varphi}^{2,2}(Q_T)$ for any function $f(x,t) \in L_2(Q_T)$.

Proof. First assume that $f(x,t) \in C^\infty(Q_T)$. Let $v(x,t)$ be a classical solution of the boundary-value problem

$$\omega(x)\Delta v - v_t = f(x,t), \qquad (x,t) \in Q_T,$$

$$v|_{\Gamma(Q_T)} = 0.$$

It is clear that this solution exists and due to [7, 9]

$$v(x,t) \in W_{2,\omega}^{2,2}(Q_T),$$

and

$$\|v\|_{W^{2,2}_{2,\omega}(Q_T)} \le C_{17}(n,\Omega,f), \tag{3.3}$$

where $W^{2,2}_{2,\omega}(Q_T)$ is a Banach space of functions given on Q_T with finite norms of $W^{2,2}_{2,\psi}(Q_T)$ type. For $\varepsilon \in (0,T)$ we have $\varphi_\varepsilon(z) \le 1$. Then, we conclude from (3.3) that

$$\|v\|_{W^{2,2}_{2,\varphi_\varepsilon}(Q_T)} \le C_{17}. \tag{3.4}$$

We denote by $\overset{\circ}{W}{}^{2,2}_{2,\omega}(Q_T)$ the complement of a set of all functions from $C^\infty(Q_T)$ vanishing with respect to the norm of the space $W^{2,2}_{2,\omega}(Q_T)$, and by $u^\varepsilon(x,t)$ the strong (almost everywhere) solution of the problem

$$M_\varepsilon u^\varepsilon = f(x,t), \qquad (x,t)\in Q_T,$$

$$(u^\varepsilon(x,t) - v(x,t)) \in \overset{\circ}{W}{}^{2,2}_{2,\omega}(Q_T).$$

This solution exists for every $\varepsilon > 0$ due to [7]. It is clear that $(u^\varepsilon(x,t) - v(x,t)) \in W^{2,2}_{2,\varphi_\varepsilon}(Q_T)$. Taking into account $v|_{\Gamma(Q_T)} = 0$ and the inequality (2.1), we get

$$u^\varepsilon(x,t) \in \overset{\circ}{W}{}^{2,2}_{2,\varphi_\varepsilon}(Q_T).$$

Moreover, for $F_\varepsilon(x,t) = M_\varepsilon v$, taking into account (3.3), we have

$$\|F_\varepsilon\|_{L_2(Q_T)} \le C_{18}(n,\Omega,T,f). \tag{3.5}$$

From Lemma 2.2 it follows that

$$\|u^\varepsilon - v\|_{W^{2,2}_{2,\varphi_\varepsilon}(Q_T)} \le C_1\left(\|f\|_{L_2(Q_T)} + \|F_\varepsilon\|_{L_2(Q_T)}\right).$$

Then, from (3.3), (3.4) and (2.1) we conclude

$$\|u^\varepsilon\|_{W^{2,2}_{2,\varphi}(Q_T)} \le C_{15}\,\|u\|_{W^{2,2}_{2,\varphi_\varepsilon}(Q_T)} \le C_{20}(n,\Omega,T,f).$$

Thus, a family of functions $\{u^\varepsilon(x,t)\}$ is bounded by the norm of the space $W^{2,2}_{2,\varphi}(Q_T)$ uniformly with respect to ε. So, this family is weakly compact in $\overset{\circ}{W}{}^{2,2}_{2,\varphi}(Q_T)$. This means, in particular, that there exist the sequences of positive numbers $\{\varepsilon_k\}, \lim_{k\to\infty}\varepsilon_k = 0$ and a function $u_0(x,t) \in W^{2,2}_{2,\varphi}(Q_T)$ such that for any $h(x,t) \in C^\infty\left(\overline{Q}_T\right)$

$$\lim_{k\to\infty}\left(\mu_0 u^{\varepsilon_k}, h\right) = \left(\mu_0 u_0, h\right), \tag{3.6}$$

where $(a,b) = \displaystyle\int_{Q_T} ab\,dx\,dt.$

But

$$(\mu_0 u^{\varepsilon_k}, h) = ((\mu_0 - \mu_{\varepsilon_k})u^{\varepsilon_k}, h) + \mu_{\varepsilon_k} u^{\varepsilon_k}, h) = ((\mu_0 - \mu_{\varepsilon_k})u^{\varepsilon_k}, h) + (f,h). \tag{3.7}$$

Besides, taking into account (2.1) and (3.5), we have

$$J(k) = |(\mu_0 - \mu_{\varepsilon_k})u^{\varepsilon_k}, h)| \leq \|(\varphi - \varphi_{\varepsilon_k})u_{tt}^{\varepsilon_k}\|_{L_2(Q(\varepsilon_k))} \|h\|_{L_2(Q(\varepsilon_k))} \leq$$

$$\leq 3\|u^{\varepsilon}\|_{W^{2,2}_{2,\varphi_{\varepsilon_k}}(Q_T)} \|h\|_{L_2(Q(\varepsilon_k))} \leq 3C_{20}\|h\|_{L_2(Q(\varepsilon_k))}, \tag{3.8}$$

where $Q(\varepsilon) = \Omega \times (T - \varepsilon, T)$. Thus, we have $J(k) \to 0$ as $k \to \infty$. From (3.6)–(3.8) it follows that $(\mu_0 u_0, h) = (f, h)$ and $\mu_0 u_0 = f(x,t)$ almost everywhere in Q_T. Now let $f(x,t) \in L_2(Q_T)$. In this case there exists a sequence $\{f_m(x,t)\}$, $m = 1, 2, \ldots$, such that $f_m(x,t) \in C^{\infty}\left(\overline{Q}_T\right)$ and $\lim_{m\to\infty} \|f_m - f\|_{L_2(Q_T)} = 0$. For any positive integer m, consider a sequence $\{u_m(x,t)\}$ of strong solutions of the boundary-value problems

$$M_0 u_m = f_m(x,t), \qquad (x,t) \in Q_T,$$

$$u_m|_{\Gamma(Q_T)} = 0.$$

Based on the above, we can say that for any m there exist the function $u_m(x,t)$ such that using the estimate obtained in the previous lemma, for the operator Z_0' and $\tau = 1$, we get

$$\|u_m\|_{W^{2,2}_{2,\varphi}(Q_T)} \leq C_{21}\|f_m\|_{L_2(Q_T)} \leq C_{20}(\varphi, \omega, n, \Omega, f). \tag{3.9}$$

Thus, the sequence $\{u_m(x,t)\}$ is weakly compact in $\overset{\circ}{W}{}^{2,2}_{2,\varphi}(Q_T)$, i.e., there exists a subsequence $\{m_k\} \in N$, $\lim_{k\to\infty} m_k = \infty$ and a function $u(x,t) \in \overset{\circ}{W}{}^{2,2}_{2,\varphi}(Q_T)$, such that for any $h(x,t) \in C^{\infty}\left(\overline{Q}_T\right)$ $\lim_{k\to\infty}(\mu_0 u_{m_k}, h) = (\mu_0 u, h)$. But

$$\lim_{k\to\infty}(M_0 u_{m_k}, h) = \lim_{k\to\infty}(f_{m_k}, h) = (f, h).$$

Therefore, $(\mu_0 u_{m_k}, h) = (f, h)$ and $\mu_0 u = f(x,t)$ almost everywhere in Q_T. Thus, the existence of strong solution of the problem (3.1), (3.2) is proved. The uniqueness of the solution follows from Lemma 2.4.

Theorem 3.1 is proved.

Theorem 3.2. *Let the coefficients of the operator* Z *satisfy the conditions* (1.3), (1.4). *Then, for* $T \leq T^0$, *the boundary-value problem* (1.1), (1.2) *has a unique strong solution for* $f(x,t) \in L_2(Q_T)$ *and the following estimate is true:*

$$\|u_m\|_{W^{2,2}_{2,\psi}(Q_T)} \leq C_{21}\|f\|_{L_2(Q_T)}. \tag{3.10}$$

Proof. The estimate (3.10) and the uniqueness of the solution follow from the coercive estimate in [8]. Therefore, we only need to prove the existence of the solution. Consider a family of operators

$$Z^{(\tau)} = (1 - \tau)\mu_0 + \tau Z \quad \text{for} \quad \tau \in [0, 1].$$

Let us show that the set E of points τ for which the problem

$$Z^{(\tau)}u = f(x,t), \qquad (x,t) \in Q_T, \tag{3.11}$$

$$u|_{\Gamma(Q_T)} = 0, \tag{3.12}$$

has a unique strong solution in $\overset{\circ}{W}{}^{2,2}_{2,\psi}(Q_T)$ for any function $f(x,t) \in L_2(Q_T)$, is nonempty and simultaneously open and closed with respect to $[0,1]$. Hence, we get $E=[0,1]$ and, in particular, the problem (3.11), (3.12) is solvable at $\tau=1$, i.e., when $Z^{(1)}=Z$.

The nonemptiness of the set E follows directly from Theorem 3.1. Let us prove its openness. Let $\tau_0 \in E$. $\varepsilon > 0$ will be specified later. Let us show that the problem (3.11), (3.12) is solvable. Then, we can rewrite this problem in the following equivalent form:

$$Z^{(\tau)} u = f(x,t) - \left(Z^{(\tau)} - Z^{(\tau_0)}\right) u, \quad (x,t) \in Q_T, \tag{3.13}$$

where $u(x,t) \in \overset{\circ}{W}{}^{2,2}_{2,\psi}(Q_T)$. It is clear that $\left(Z^{(\tau)} - Z^{(\tau_0)}\right) \;\; v(x,t) \in L_2(Q_T)$. Note that for all operators $Z^{(\tau)}$ the conditions (1.3) and (1.4) with constants $\gamma'_{(\tau)} \geq \min\{\gamma', n\}$ are fulfilled. Now let us note that from the above mentioned considerations and Lemma 2.4 it follows that for $T \leq T^0$, any $\tau = [0,1]$ and any function $u(x,t) \in W^{2,2}_{2,\psi}(Q_T)$ the following estimate is true:

$$\|u_m\|_{W^{2,2}_{2,\psi}(Q_T)} \leq C_{22} \left\| Z^{(\tau)} u \right\|_{L_2(Q_T)}. \tag{3.14}$$

By the assumption, the boundary-value problem (3.13) has a strong solution $u(x,t)$ for any $v(x,t) \in$ $\in W^{2,2}_{2,\psi}(Q_T)$. Thus, the operator F from $\overset{\circ}{W}{}^{2,2}_{2,\psi}(Q_T)$ into $\overset{\circ}{W}{}^{2,2}_{2,\psi}(Q_T)$ is defined and

$$u = Fv.$$

Operator F is a contration operator for properly chosen ε. Indeed, let

$$v^{(i)}(x,t) \in W^{2,2}_{2,\psi}(Q_T), \qquad u^{(i)} = Fv^{(i)}, \qquad i = 1,2.$$

Then, taking into account the equality

$$\left(Z^{(\tau)} - Z^{(\tau_i)}\right) = (\tau - \tau_0)\,(Z - \mu_0)\,,$$

we conclude that $u^{(1)}(x,t)-$ $u^{(2)}(x,t)$ is a strong solution of the boundary-value problem

$$Z^{(\tau_0)} \left(u^{(1)}(x,t) - u^{(2)}(x,t)\right) = (\tau - \tau_0)\,(Z - \mu_0) \left(v^{(1)}(x,t) - v^{(2)}(x,t)\right),$$

$$\left(u^{(1)}(x,t) - u^{(2)}(x,t)\right) \in W^{2,2}_{2,\psi}(Q_T).$$

By using (3.14), we get

$$\left\| u^{(1)}(x,t) - u^{(2)}(x,t) \right\|_{W^{2,2}_{2,\psi}(Q_T)} \leq$$

$$\leq C_{23}\,|\tau - \tau_0| \left\| (Z - \mu_0) \left(v^{(1)} - v^{(2)}\right) \right\|_{L_2(Q_T)}. \tag{3.15}$$

On the other hand,

$$\left\| (Z - M_0) \left(v^{(1)}(x,t) - v^{(2)}(x,t)\right) \right\|_{L_2(Q_T)} \leq$$

$$\leq C_{24}\left(Z,n,\Omega,T\right)\left\|v^{(1)}(x,t)-v^{(2)}(x,t)\right\|_{W^{2,2}_{2,\psi}(Q_T)}. \qquad (3.16)$$

Thus,

$$\left\|u^{(1)}(x,t)-u^{(2)}(x,t)\right\|_{W^{2,2}_{2,\psi}(Q_T)}\leq C_{23}C_{24}\left\|v^{(1)}(x,t)-v^{(2)}(x,t)\right\|_{W^{2,2}_{2,\psi}(Q_T)}.$$

Now choosing $\varepsilon=\dfrac{1}{2}C_{23}C_{24}$ we prove that the operator F has a fixed point $u=Fu$, which is a strong solution of the boundary-value problem (3.13) and, consequently, (3.11), (3.12). Therefore, the openness of the set E is proved.

Now let the set E be closed. Let $\tau_k\in E, k=1,2,\ldots,$ and $\lim_{k\to\infty}\tau_k=\tau$. For positive integer k, denote by $u_{[k]}(x,t)$ a strong solution of the boundary-value problem

$$Z^{(\tau_k)}u_{[k]}(x,t)=f(x,t),\qquad (x,t)\in Q_T,$$

$$\frac{u_{[k]}(x,t)}{\Gamma(Q_T)}=0.$$

According to (3.14), we have

$$\left\|u_{[k]}(x,t)\right\|_{W^{2,2}_{2,\psi}(Q_T)}\leq C_{25}\left\|f\right\|_{L_2(Q_T)}. \qquad (3.17)$$

So, the family of functions $\{u_{[k]}(x,t)\}$ is weakly compact in $\overset{\circ}{W}{}^{2,2}_{2,\psi}(Q_T)$, i.e., there exists a subsequence of positive integers $\{k_l\}\lim_{l\to\infty}k_i=\infty$ and a function $u(x,t)\in\overset{\circ}{W}{}^{2,2}_{2,\psi}(Q_T)$, such that for any $\psi(x,t)\in C^{\infty}\left(\overline{Q}_T\right)$

$$\lim_{l\to\infty}\left(Z^{(\tau_{k_l})}u_{[k]},\psi\right)=\left(Z^{(\tau)}u,\psi\right). \qquad (3.18)$$

But

$$\left(Z^{(\tau_{k_l})}u_{[k_l]},\psi\right)=\left(Z^{(\tau)}-Z^{(\tau_{k_l})}u_{[k]},\psi\right)+(f,\psi)=J_1(l)+(f,\psi). \qquad (3.19)$$

Moreover, taking into account (3.15) and (3.16), we have

$$|J_1(l)|\leq|\tau-\tau_{k_l}|\left|\left(\left(Z-\mu_0\right)u_{[k]},\psi\right)\right|\leq|\tau-\tau_{k_l}|\,C_{26}\left\|u_{[k_l]}\right\|_{W^{2,2}_{2,\psi}(Q_T)},$$

$$\|\psi\|_{L_2(Q_T)}\leq C_{25}C_{26}\,|\tau-\tau_{k_l}|\,\|f\|_{L_2(Q_T)}\,\|\psi\|_{L_2(Q_T)}. \qquad (3.20)$$

From (3.20) it follows that $\lim_{l\to\infty}J_1(l)=0$.

Further, from (3.18) and (3.19) we conclude that

$$\left(Z^{(\tau)}u,\psi\right)=(f,\psi),$$

i.e.,

$$Z^{(\tau)}u=f(x,t)$$

almost everywhere in Q_T. So, we have showen that $\tau\in E$, i.e., the set E is closed.

Theorem 3.2 is proved.

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