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A CLASSIFICATION OF CONFORMAL VECTOR FIELDS ON THE TANGENT BUNDLE

КЛАСИФІКАЦІЯ КОНФОРМНИХ ВЕКТОРНИХ ПОЛІВ НА ДОТИЧНОМУ РОЗШАРУВАННІ

Let (M, g) be a Riemannian manifold and TM be its tangent bundle equipped with a Riemannian (or pseudo-Riemannian) lift metric derived from g. We give a classification of infinitesimal fibre-preserving conformal transformations on the tangent bundle.

Нехай (M, g) — ріманів многовид, TM — його дотичне розшарування з рімановою (або псевдорімановою) метрикою підняття, яка породжується g. Наведено класифікацію нескінченно малих конформних перетворень, що зберігають шари на дотичному розшаруванні.

1. Introduction. Let M be a Riemannian manifold with a Riemannian metric g and X be a vector field on M. Let us consider the local one-parameter group $\{\phi_t\}$ of local transformations of M generated by X. The vector field X is called an infinitesimal conformal transformation if each ϕ_t is a local conformal transformation of M. As is well-known, the vector field X is an infinitesimal conformal transformation or conformal vector field on M if and only if there exists a scalar function ρ on M satisfying $L_X g = 2\rho g$, where L_X denotes the Lie derivation with respect to X. Especially, the vector field X is called an infinitesimal homothetic one when ρ is constant and it is called an isometry or Killing vector field when ρ vanishes.

Let TM be the tangent bundle over M and Φ be a transformation of TM. If the transformation Φ preserves the fibres, it is called a fibre-preserving transformation. Consider a vector field $\tilde{X}$ on TM and the local one-parameter group $\{\Phi_t\}$ of local transformations of TM generated by $\tilde{X}$. The vector field $\tilde{X}$ is called an infinitesimal fibre-preserving transformation if each Φ_t is a local fibre-preserving transformation of TM. An infinitesimal fibre-preserving transformation $\tilde{X}$ on TM is called an infinitesimal fibre-preserving conformal transformation if each Φ_t is a local fibre-preserving conformal transformation of TM. Let $\tilde{g}$ be a Riemannian or pseudo-Riemannian metric on TM. $\tilde{X}$ is an infinitesimal conformal transformation of TM if and only if there exists a scalar function Ω on TM such that $L_{\tilde{X}}\tilde{g} = 2\Omega\tilde{g}$, where $L_{\tilde{X}}$ denotes the Lie derivation with respect to $\tilde{X}$. An infinitesimal conformal transformation $\tilde{X}$ is called essential if Ω depends only on (y^i) with respect to the induced coordinates (x^i, y^i) on TM, and is called inessential if Ω depends only on (x^i), that is, Ω is a constant on each fibre of TM. In this case, Ω induces a function on M.

Let (M, g) be an n-dimensional Riemannian manifold. There are some lift metrics on $TM = \bigcup_{x\in M} T_xM$ as follows: complete lift metric or g_2, diagonal lift metric or $g_1 + g_3$, lift metric $g_2 + g_3$ and lift metric $g_1 + g_2$, where

$$g_1 := g_{ij}dx^i dx^j, \qquad g_2 := 2g_{ij}dx^i\delta y^j, \qquad g_3 := g_{ij}\delta y^i\delta y^j$$

are all bilinear differential forms defined globally on TM. Yamauchi [21] proved that every infinitesimal fibre-preserving conformal transformation on TM with the metric $g_1 + g_3$ is homothetic and it induces an infinitesimal homothetic transformation on M. Also, in the case when M is a complete, simply connected Riemannian manifold with a Riemannian metric, Hasegawa and Yamauchi [6] showed that the Riemannian manifold M is isometric to the standard sphere when the tangent bundle TM equipped with the metric $g_1 + g_2$ admits an essential infinitesimal conformal transformation. In [3], Gezer has studied a similar problem in [20, 21] with respect to the synectic lift metric on the tangent bundle. Also he represents the classification of infinitesimal fibre-preserving conformal transformations on the tangent bundle, equipped with the Cheeger – Gromoll metric [4].

In [1], Abbassi and Sarih classified Killing vector fields on (TM, g_{CG}); that is, they found general forms of all Killing vector fields on (TM, g_{CG}). Also, they showed that if (TM, g_{CG}) is the tangent bundle with the Cheeger – Gromoll metric g_{CG} of a Riemannian, compact and orientable manifold (M, g) with vanishing first and second Betti numbers, then the Lie algebras of Killing vector fields on (M, g) and on (TM, g_{CG}) are isomorphic. Finally, they showed that the sectional curvature of the tangent bundle (TM, g_{CG}) with the Cheeger – Gromoll metric g_{CG} of a Riemannian manifold (M, g) is never constant.

Peyghan, Tayebi and Zhong introduced a class of g-natural metrics $G_{a,b}$ on the tangent bundle of a Finsler manifold (M, F) which generalizes the associated Sasaki$-$Matsumoto metric and Miron metric and They investigated Killing vector fields associated to $G_{a,b}$ in [11]. Two first authors introduced two vector fields of horizontal Liouville type on a slit tangent bundle endowed with a Riemannian metric of Sasaki – Finsler type and proved that these vector fields are Killing if and only if the base Finsler manifold is of positive constant curvature. In the special case of one of them, they showed that if it is Killing vector field then the base manifold is the Einstein$-$Finsler manifold [14]. For the other progress, see [5, 7 – 10, 12, 13, 15 – 18].

In [2], Bidabad introduced a new Riemannian (or pseudo-Riemannian) lift metrics on TM, $\tilde{g} = \alpha g_1 + \beta g_2 + \mu g_3$, where α, β and μ are certain constant real numbers. That is a combination of diagonal lift, and complete lift metrics. He had proved that if (M, g) is an n-dimensional Riemannian manifold and TM is its tangent bundle with metric $\tilde{g}$, Then every complete lift conformal vector field on TM is homothetic.

The purpose of the present paper is to characterize infinitesimal fibre-preserving conformal transformations with respect to the lift metric $\tilde{g}$.

2. Preliminaries. Let M be a real n-dimensional manifold of class C^∞. We denote by $TM \to M$ the bundle of tangent vectors and by $\pi \colon TM\backslash\{0\} \to M$ the fiber bundle of non-zero tangent vectors. Let $\mathcal{V}_v TM = \ker \pi_*^v$ be the set of the vectors tangent to the fiber through $v \in TM\backslash\{0\}$. Then a vertical vector bundle on M is defined by $\mathcal{V}TM := \bigcup_{v \in TM\backslash\{0\}} \mathcal{V}_v TM$. A nonlinear connection or a horizontal distribution on $TM\backslash\{0\}$ is a complementary distribution $\mathcal{H}TM$ for $\mathcal{V}TM$ on $T(TM\backslash\{0\})$. Therefore, we have the decomposition

$$T(TM\backslash\{0\}) = \mathcal{V}TM \oplus \mathcal{H}TM.$$

Using the local coordinates (x^i, y^i) on TM we have the local field of frames $\left\{\dfrac{\partial}{\partial x^i}, \dfrac{\partial}{\partial y^i}\right\}$ on TTM. It is well-known that we can choose a local field of frames $\left\{\dfrac{\delta}{\delta x^i}, \dfrac{\partial}{\partial y^i}\right\}$ adapted to the above decomposition, i.e., $\dfrac{\delta}{\delta x^i} \in \Gamma(\mathcal{H}TM)$ and $\dfrac{\partial}{\partial y^i} \in \Gamma(\mathcal{V}TM)$ set of vector fields on $\mathcal{H}TM$ and $\mathcal{V}TM$, where

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^j \frac{\partial}{\partial y^j}$$

and $N_i^j(x,y)$ are the nonlinear differentiable functions on TM, called coefficients of the nonlinear connection.

Let $X = X^i \dfrac{\partial}{\partial x^i}$ be the local expression in $U \subset TM$ of a vector field X on M. Then the vertical lift X^v, the horizontal lift X^h and the complete lift X^C of X are given, with respect to the induced coordinates, by

$$X^v = X^i \frac{\partial}{\partial y^i},$$

$$X^h = X^i \frac{\partial}{\partial x^i} - y^s \Gamma^i_{sk} X^k \frac{\partial}{\partial y^i},$$

$$X^C = X^i \frac{\partial}{\partial x^i} + y^s \partial_s X^i \frac{\partial}{\partial y^i} = X^i \frac{\delta}{\delta x^i} + y^s \nabla_s X^i \frac{\partial}{\partial y^i},$$

where Γ^i_{jk} are the coefficients of the Levi–Civita connection ∇ of g.

Suppose that we are given a tensor field $S \in \Im^p_q(M)$, $q > 1$, where $\Im^p_q(M)$ is the set of all tensor fields of type (p,q) on M. We define a tensor field $\gamma S \in \Im^p_q(TM)$ on $\pi^{-1}(U)$ by

$$\gamma S = \left(y^e S^{j_1 \ldots j_p}_{e i_2 \ldots i_q}\right) \frac{\partial}{\partial y^{j_1}} \otimes \ldots \otimes \frac{\partial}{\partial y^{j_p}} \otimes dx^{i_2} \otimes \ldots \otimes dx^{i_q}$$

with respect to the induced coordinates (x^i, y^i). We easily see that γA has components, with respect to the induced coordinates (x^i, y^i),

$$(\gamma A) = \left(0, y^i A_i^j\right)$$

for any $A \in \Im^1_1(M)$ and $(\gamma A)(f^v) = 0$, $f \in \Im^0_0(M)$, i.e., γA is a vertical vector field on TM.

The bracket operation of vertical and horizontal vector fields is given by the following formulae:

$$\left[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right] = -y^s R^h_{sij} \frac{\partial}{\partial y^h},$$

$$\left[\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\right] = \Gamma^h_{ij} \frac{\partial}{\partial y^h},$$

$$\left[\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right] = 0,$$

where R^h_{sij} denotes the components of the Riemannian curvature tensor of g defined by

$$R(X,Y) = \left[\nabla_X, \nabla_Y\right] - \nabla_{[X,Y]}.$$

3. A lift metric on tangent bundle. Let (M, g) be a Riemannian manifold. In this section we introduce a new Riemannian or pseudo-Riemannian metric on TM derived from g. This metric is in some sense more general than the other lift metrics defined previously on TM. By mean of the dual basis $\{dx^i, \delta y^i\}$ analogously to the Riemannian geometry the tensors; $g_1 := g_{ij}dx^i dx^j$, $g_2 := 2g_{ij}dx^i\delta y^j$ and $g_3 := g_{ij}\delta y^i \delta y^j$ are all quadratic differential tensors defined globally on TM, see [3]. Now let's consider the Riemannian metric tensor g with the components $g_{ij}(x, y)$. The tensor field $\tilde{g} = \alpha g_1 + \beta g_2 + \mu g_3$ on TM, where the coefficient α, β and γ are real numbers, has the components

$$\begin{pmatrix} \alpha g & \beta g \\ \beta g & \mu g \end{pmatrix}$$

with respect to the dual basis of TM. From the linear algebra we have

$$\det \tilde{g} = (\alpha\mu - \beta^2)^n \det g^2.$$

Therefore, $\tilde{g}$ is nonsingular if $\alpha\mu - \beta^2 \neq 0$ and it is positive definite if α, μ are positive and $\alpha\mu - \beta^2 > 0$. Indeed $\tilde{g}$ defines respectively a pseudo-Riemannian or a Riemannian lift metric on TM.

Definition 3.1 [2]. *Let (M, g) be a Riemannian manifold. Consider tensor field $\tilde{g} = \alpha g_1 + \beta g_2 + \mu g_3$ on TM, where the coefficient α, μ and β are real numbers. If $\alpha\mu - \beta^2 \neq 0$, then $\tilde{g}$ is nonsingular and it can be regarded as a pseudo-Riemannian metric on TM. If α and μ are positive such that $\alpha\mu - \beta^2 > 0$, then $\tilde{g}$ is positive definite and it can consequently be regarded as a Riemannian metric on TM; $\tilde{g}$ is called the lift metric of g on TM.*

4. Main results. Let $\tilde{X}$ be a vector field on TM with components (v^h, w^h) with respect to the adapted frame $\left\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right\}$. Then $\tilde{X}$ is a fibre-preserving vector field on TM if and only if v^h depend only on the variables (x^h). Therefore, every fibre-preserving vector field $\tilde{X}$ on TM induces a vector field $X = v^h \frac{\partial}{\partial x^h}$ on M.

Let M be an n-dimensional manifold, X a vector field on M and $\{\phi_t\}$ a 1-parameter local group of local transformations of M generated by X. Take any tensor field S on M, and denote by $\phi_t^*(S)$ the pulled back of S by ϕ_t. Then the Lie derivation of S with respect to X is a tensor field $L_X S$ on M defined by

$$L_X S = \frac{\partial}{\partial t}\phi_t^*(S)\bigg|_{t=0} = \lim_{t\to 0}\frac{\phi_t^*(S) - (S)}{t},$$

on the domain of t. The mapping L_X which maps S to $L_X(S)$ is called the Lie derivation with respect to X. Then we obtain the following.

Lemma 4.1 [20, 21]. *The Lie derivations of the adapted frame and its dual basis with respect to $\tilde{X} = v^h \frac{\delta}{\delta x^h} + w^h \frac{\partial}{\partial y^h}$ are given as follows*:

1) $L_{\tilde{X}} \frac{\delta}{\delta x^h} = -\partial_h v^a \frac{\delta}{\delta x^a} - \left(v^b R^a_{bh} + w^b \Gamma^a_{bh} + \frac{\delta w^a}{\delta x^h}\right)\frac{\partial}{\partial y^a}$,

2) $L_{\tilde{X}} \frac{\partial}{\partial y^h} = \left(v^b \Gamma^a_{bh} - \frac{\partial w^a}{\partial y^h}\right)\frac{\partial}{\partial y^a}$,

3) $L_{\tilde{X}} dx^h = \partial_m v^h dx^m$,

4) $L_{\tilde{X}}\delta y^h = \left(v^b R^h_{bm} + w^b\Gamma^h_{bm} + \dfrac{\delta w^a}{\delta x^m}\right)dx^m - \left(v^b\Gamma^h_{bm} - \dfrac{\partial w^h}{\partial y^m}\right)\delta y^m.$

Lemma 4.2. *The Lie derivative $L_{\tilde{X}}\tilde{g}$ with respect to the fibre-preserving vector field $\tilde{X}$ are given as follows*:

$$L_{\tilde{X}}\tilde{g} = \left(\alpha L_X g_{ij} + 2\beta g_{ai}\left(v^p R^a_{pj} + w^p\Gamma^a_{pj} + \frac{\delta w^a}{\delta x^j}\right)\right)dx^i dx^j +$$

$$+2\left(\beta\left(L_X g_{ij} - g_{ia}\nabla_j v^a + g_{ai}\frac{\partial w^a}{\partial y^j}\right)+\right.$$

$$\left.+\mu g_{aj}\left(v^p R^a_{pi} + w^p\Gamma^a_{pi} + \frac{\delta w^a}{\delta x^i}\right)\right)dx^i\delta y^j +$$

$$+\mu\left(g_{aj}\frac{\partial w^a}{\partial y^i} + g_{ai}\frac{\partial w^a}{\partial y^j}\right)\delta y^i\delta y^j.$$

Proof. The statement is a direct consequence of Lemma 4.1.

Let TM be the tangent bundle over M with the lift metric $\tilde{g}$, and let $\tilde{X}$ be an infinitesimal fibre-preserving conformal transformation on $(TM, \tilde{g})$ such that

$$L_{\tilde{X}}\tilde{g} = 2\Omega\tilde{g}. \tag{4.1}$$

By means of Lemma 4.2, we have

$$2\alpha\Omega g_{ij}dx^i dx^j + 4\beta\Omega g_{ij}dx^i\delta y^j + 2\mu\Omega g_{ij}\delta y^i\delta y^j =$$

$$= \left(\alpha L_X g_{ij} + 2\beta g_{ai}\left(v^p R^a_{pj} + w^p\Gamma^a_{pj} + \frac{\delta w^a}{\delta x^j}\right)\right)dx^i dx^j +$$

$$+2\left(\beta\left(L_X g_{ij} - g_{ia}\nabla_j v^a + g_{ai}\frac{\partial}{\partial y^j}\right)+\right.$$

$$\left.+\mu g_{aj}\left(v^p R^a_{pi} + w^p\Gamma^a_{pi} + \frac{\delta w^a}{\delta x^i}\right)\right)dx^i\delta y^j +$$

$$+\mu\left(g_{aj}\frac{\partial w^a}{\partial y^i} + g_{ai}\frac{\partial w^a}{\partial y^j}\right)\delta y^i\delta y^j.$$

Comparing both sides of the above equation, we obtain the following three relations:

$$\alpha L_X g_{ij} + 2\beta g_{ai}\left(v^p R^a_{pj} + w^p\Gamma^a_{pj} + \frac{\delta w^a}{\delta x^j}\right) = 2\alpha\Omega g_{ij}, \tag{4.2}$$

$$\beta\left(L_X g_{ij} - g_{ia}\nabla_j v^a + g_{ai}\frac{\partial}{\partial y^j}\right) + \mu g_{aj}\left(v^p R^a_{pi} + w^p\Gamma^a_{pi} + \frac{\delta w^a}{\delta x^i}\right) = 2\beta\Omega g_{ij}, \tag{4.3}$$

$$\mu\left(g_{aj}\frac{\partial w^a}{\partial y^i} + g_{ai}\frac{\partial w^a}{\partial y^j}\right) = 2\mu\Omega g_{ij}. \tag{4.4}$$

Theorem 4.1. *Let (M, g) be a Riemannian manifold and TM be its tangent bundle equipped with the lift metric $\tilde{g}$. A vector field $\tilde{X}$ on TM is a fiber-preserving conformal vector field with respect to $\tilde{g}$ if and only if*

$$\tilde{X} = X^C + \gamma A + B^v,$$

where $B, X \in \Im_0^1(M)$ and $A = (A_h^a) \in \Im_1^1(M)$ such that:

1) *if $\mu = 0$, then*

$$\beta L_B g_{ij} + \alpha L_X g_{ij} = 2\alpha\Omega g_{ij}, \tag{4.5}$$

$$v^p R_{ispj} + g_{ai}\nabla_j P_s^a = 0, \tag{4.6}$$

$$g_{aj}\nabla_i v^a + g_{ai} P_j^a = 2\Omega g_{ij}; \tag{4.7}$$

2) *if $\mu \neq 0$, then*

$$\beta L_B g_{ij} + \alpha L_X g_{ij} = 2\alpha\Omega g_{ij}, \tag{4.8}$$

$$v^p R_{ispj} + g_{ai}\nabla_j P_s^a = 0, \tag{4.9}$$

$$\beta\big(L_X g_{ij} - g_{ai}\nabla_j v^a + g_{ai} P_j^a\big) + \mu g_{aj}\nabla_i B^a = 2\beta\Omega g_{ij}, \tag{4.10}$$

$$g_{ai} P_j^a + g_{aj} P_i^a = 2\Omega g_{ij}, \tag{4.11}$$

where $A_h^a = P_h^a - \nabla_h X^a$.

Proof. We consider the 0-section $(y^i = 0)$ in the coordinate neighborhood $\pi^{-1}(U)$ in TM and its neighborhood W. For a vector field $\tilde{X} = v^h \dfrac{\delta}{\delta x^h} + w^h \dfrac{\partial}{\partial y^h}$ on TM, and $(x, y) = (x^i, y^i)$ in W, we can write, by Taylor's theorem,

$$v^h(x,y) = v^h(x,0) + (\partial_{\dot r} v^h)(x,0)y^r + \frac{1}{2}(\partial_{\dot r}\partial_{\dot s} v^i)(x,0)y^r y^s + \ldots + [*]_m^h,$$

$$w^h(x,y) = w^h(x,0) + (\partial_{\dot r} w^h)(x,0)y^r + \frac{1}{2}(\partial_{\dot r}\partial_{\dot s} w^i)(x,0)y^r y^s + \ldots + [*]_m^h,$$

where $[*]_m^h$, $h = 1, 2, \ldots, 2n$, is of the form

$$[*]_m^h = \frac{1}{m!}\left(\frac{\partial^m v^h}{\partial y^{i_1}\partial y^{i_2}\ldots\partial y^{i_m}}\right)(x^a, \theta(x,y)y^b)y^{i_1}y^{i_2}\ldots y^{i_m},$$

where $1 \leq i_1, \ldots, i_m \leq n$. Tanno [19] Proved that in this situation the following:

$$X = \big(X^i(x)\big) = \big(v^i(x,0)\big), \qquad Y = \big(Y^i(x)\big) = \big(w^i(x,0)\big),$$

$$K = \big(K_r^i(x)\big) = \big((\partial_{\dot r}\partial_{\dot s} v^i)(x,0)\big), \qquad E = \big(E_{rs}^i(x)\big) = \big((\partial_{\dot r}\partial_{\dot s} v^i)(x,0)\big),$$

$$P = \big(P_r^i(x)\big) = \big((\partial_{\dot r} w^i)(x,0) - (\partial_{\dot r} v^i)(x,0)\big)$$

are tensor fields on M. So for a fibre-preserving vector field $\tilde{X} = v^h \dfrac{\delta}{\delta x^h} + w^h \dfrac{\partial}{\partial y^h}$ on TM, we can write

$$v^h(x,y) = X^h, \tag{4.12}$$

$$w^h(x,y) = B^h + P^h_r y^r + \frac{1}{2} Q^h_{rs} y^r y^s + \ldots + [*]^h_m, \tag{4.13}$$

where P^h_r and Q^h_{rs} are given by $P^h_r = (\partial_r w^h)(x,0)$ and $Q^h_{rs} = \partial_r \partial_s w^h)(x,0)$. By putting (4.12) and (4.13) into (4.2)–(4.4), we have

$$\alpha L_X g_{ij} + \beta\big(g_{ai}\nabla_j B^a + g_{aj}\nabla_i B^a\big) + 2\beta g_{ai} y^s \big(v^p R^a_{spj} + \nabla_j P^a_s - N^t_j \partial_t P^a_s\big) +$$
$$+ \beta g_{ai} y^s y^r \big(\nabla_j Q^a_{rs} + N^t_j \partial_t Q^a_{rs}\big) = 2\alpha\Omega g_{ij}, \tag{4.14}$$

$$\beta\big(L_X g_{ij} - g_{ai}\nabla_j v^a + g_{ai} P^a_j\big) + \beta y^s g_{ai}\big(\partial_j P^a_s + Q^a_{js}\big) + \mu y^s g_{aj}\big(v^p R^a_{spi} + \nabla_i P^a_s - N^t_i \partial_t P^a_r\big) +$$
$$+ \mu g_{ai}\nabla_i B^a + \frac{\beta}{2} g_{ai} y^r y^s \partial_j Q^a_{rs} + \frac{\mu}{2} g_{aj} y^r y^s \big(\nabla_i Q^a_{rs} - N^t_i \partial_t Q^a_{rs}\big) = 2\beta\Omega g_{ij}, \tag{4.15}$$

$$\mu y^r \big(g_{aj}(Q^a_{ir} + \partial_i P^a_r) + g_{ai}(Q^a_{jr} + \partial_j P^a_r)\big) +$$
$$+ \mu y^r y^s (g_{aj}\partial_i Q^a_{rs} + g_{ai}\partial_j Q^a_{rs}) + \mu\big(P^a_i g_{aj} + P^a_j g_{ai}\big) = 2\mu\Omega g_{ij}. \tag{4.16}$$

Case 1. If $\mu = 0$, since $\alpha\mu - \beta^2 \neq 0$, we have $\beta \neq 0$, so from (4.15) we get

$$\big(L_X g_{ij} - g_{ai}\nabla_j v^a + g_{ai} P^a_j\big) + y^s g_{ai}\big(\partial_j P^a_s + Q^a_{js}\big) + \frac{1}{2} g_{ai} y^r y^s \partial_j Q^a_{rs} = 2\Omega g_{ij}. \tag{4.17}$$

Putting (4.17) into (4.14), and taking into account the part which does not contain y^r, we obtain

$$\beta L_B g_{ij} + \alpha g_{ai}\nabla_j v^a = \alpha g_{ai} P^a_j. \tag{4.18}$$

Therefore, P^a_j depends only on the variables (x^h).

Taking into account the parts which contain y^r and $y^r y^s$, we get

$$v^p R^a_{spj} + \nabla_j P^a_s = 0,$$
$$Q^a_{js} = 0.$$

Since $\partial_j P^a_s = Q^a_{js} = 0$. So, (4.17) turns to

$$g_{aj}\nabla_i v^a + g_{ai} P^a_j = 2\Omega g_{ij}. \tag{4.19}$$

From (4.18) and (4.19), we have

$$\beta L_B g_{ij} + \alpha L_X g_{ij} = 2\alpha\Omega g_{ij}.$$

Case 2. If $\mu \neq 0$, then from (4.16) we have

$$(P^a_i g_{aj} + P^a_j g_{ai}) + 2y^r (g_{aj} Q^a_{ir} + g_{ai} Q^a_{jr}) +$$
$$+ y^r y^s (g_{aj}\partial_i Q^a_{rs} + g_{ai}\partial_j Q^a_{rs}) = 2\Omega g_{ij}. \tag{4.20}$$

Putting (4.20) into (4.14) and taking into account the part which does not contain y^r, we obtain

$$\alpha L_X g_{ij} + \beta L_B g_{ij} = \alpha\big(g_{ai}P^a_j + g_{aj}P^a_i\big),$$

therefore, P^a_j depends only on the variables (x^h). We get

$$\partial_{\dot{j}} P^a_s = Q^a_{js} = 0. \tag{4.21}$$

Thus (4.20) turns to

$$g_{ai}P^a_j + g_{aj}P^a_i = 2\Omega g_{ij}. \tag{4.22}$$

Putting (4.20) into (4.15), and taking into account the part which does not contain y^r and (4.22), we obtain

$$\beta\big(L_X g_{ij} g_{ai}\nabla_j v^a + g_{ai}P^a_j\big) + \mu g_{aj}\nabla_i B^a = \beta\big(g_{ai}P^a_j + g_{aj}P^a_i\big) = 2\beta\Omega g_{ij}.$$

Taking into account the part which contains y^r and (4.21), we have

$$v^p R^a_{spj} + \nabla_j P^a_s = 0.$$

In both situation, we set $A^a_j := P^a_j - \nabla_j v^a$ and obtain

$$\tilde{X} = v^h \frac{\delta}{\delta x^h} + \left(B^h + P^h_s y^s + \frac{1}{2}Q^a_{rs}y^r y^s\right)\frac{\partial}{\partial y^h} =$$

$$= v^h \frac{\delta}{\delta x^h} + y^s\nabla_s v^h \frac{\partial}{\partial y^h} + y^s A^h_s \frac{\partial}{\partial y^h} + B^h \frac{\partial}{\partial y^h} =$$

$$= X^C + \gamma A + B^v.$$

Conversely, if $\tilde{X} = X^C + \gamma A + B^v$ is given such that X, B and A satisfy in (4.5) – (4.7) or (4.8) – (4.11), with a simple calculation we see that

$$L_{\tilde{X}}\tilde{g} = 2\Omega\tilde{g},$$

thus $\tilde{X}$ is a fiber-preserving conformal vector field with respect to $\tilde{g}$.

Theorem 4.1 is proved.

***Corollary* 4.1.** *Let (M, g) be a C^∞ Riemannian manifold, TM its tangent bundle and $\tilde{g} = \alpha g + 2\beta g + \mu g$ the Riemannian (or pseudo-Riemannian) metric on TM derived from g. Every infinitesimal fibre-preserving conformal transformation on $(TM, \tilde{g})$ is inessential.*

Proof. By taking into account proof of Theorem 4.1, we deduce that Ω depends only on the variables (x^h). Thus $\tilde{X}$ is inessential with respect to the induced coordinates (x^i, y^i) on TM.

***Corollary* 4.2.** *Let (M, g) be a C^∞ connected Riemannian manifold, TM its tangent bundle and $\tilde{g} = \alpha g + 2\beta g + \mu g$ the Riemannian (or pseudo-Riemannian) metric on TM derived from g. Every infinitesimal fibre-preserving conformal transformation on $(TM, \tilde{g})$ is homothetic if $\mu \neq 0$.*

Proof. By using Corollary 4.1, we deduce that Ω depends on the variables (x^h). Applying the covariant derivative ∇_s to the both sides of part (4.11) from Theorem 4.1, we obtain

$$\nabla_s P_i^a g_{aj} + \nabla_s P_j^a g_{ai} = 2 g_{ij} \nabla_s \Omega.$$

By using (4.9), we get

$$g_{ai} \nabla_s P_j^a = -v^p R_{ijps}.$$

Therefore,

$$2 g_{ij} \nabla_s \Omega = -v^p R_{ijps} - v^p R_{jips} = 0.$$

We get $\nabla_s \Omega = \dfrac{\partial \Omega}{\partial x^s} = 0$. Since M is connected, the scalar function Ω is constant.

Corollary 4.2 is proved.

***Corollary* 4.3.** *Let (M, g) be a Riemannian manifold and TM be its tangent bundle equipped with the Riemannian (or pseudo-Riemannian) lift metric $\tilde{g}$. A vector field $\tilde{X}$ on TM is a fiber-preserving Killing vector field with respect to $\tilde{g}$ if and only if*

$$\tilde{X} = X^C + \gamma A + B^v,$$

where $B, X \in \Im_0^1(M)$ and $A = (A_h^a) \in \Im_1^1(M)$ such that:

1) *if $\mu = 0$, then*

$$A = (A_h^a) = -g^{ia} L_X g_{ih},$$

$$\alpha L_X g_{ij} + \beta L_B g_{ij} = 0,$$

$$L_X \Gamma_{ij}^a = 0;$$

2) *if $\mu \neq 0$, then*

$$\alpha L_X g_{ij} + \beta L_B g_{ij} = 0,$$

$$L_X \Gamma_{ij}^a = 0,$$

$$\beta \big(g_{aj} \nabla_i v^a + g_{ai} P_j^a \big) + \mu g_{aj} \nabla_i B^a = 0,$$

$$g_{ai} P_j^a + g_{aj} P_i^a = 0,$$

where $A_j^a := P_j^a - \nabla_j v^a$.

Proof. A vector field $\tilde{X}$ is a Killing vector field on TM with respect to $\tilde{g}$ if and only if $L_{\tilde{X}} \tilde{g} = 0$. By Theorem 4.1, we say that $\tilde{X} = v^h \dfrac{\delta}{\delta x^h} + w^h \dfrac{\partial}{\partial y^h}$ is a fiber-preserving Killing vector field on TM with respect to $\tilde{g}$ if and only if the following relations hold:

$$\tilde{X} = X^C + \gamma A + B^v,$$

where $B, X \in \Im_0^1(M)$ and $A = (A_h^a) \in \Im_1^1(M)$.

Case 1. If $\mu = 0$, then

$$\beta L_B g_{ij} + \alpha L_X g_{ij} = 0, \tag{4.23}$$

$$v^p R_{ispj} + g_{ai} \nabla_j P^a_s = 0, \tag{4.24}$$

$$g_{aj} \nabla_i v^a + g_{ai} P^a_j = 0. \tag{4.25}$$

Since $\alpha\mu - \beta^2 \neq 0$, then we get $\beta \neq 0$. Thus, by (4.25) we have

$$P^a_j = \nabla_j v^a - g^{ai} L_X g_{ij}, \tag{4.26}$$

Thus we set $A^a_j := -g^{ai} L_X g_{ij}$.

Now, by putting (4.26) into (4.24), we get

$$0 = v^p R_{ispj} + g_{ia} \nabla_j \nabla_s v^a - \nabla_j L_X g_{is} =$$

$$= v^p R_{ispj} + g_{ia}(L_X \Gamma^a_{js} - v^p R_{ispj}) + g_{ja}(L_X \Gamma^a_{is} - v^p R_{jspi}) =$$

$$= -g_{as} L_X \Gamma^a_{ij}$$

from which it follows that $L_X \Gamma^a_{ij} = 0$, i.e., X is an infinitesimal affine transformation on M.

Case 2. If $\mu \neq 0$, then

$$\beta L_B g_{ij} + \alpha L_X g_{ij} = 0,$$

$$v^p R_{ispj} + g_{ai} \nabla_j P^a_s = 0,$$

$$\beta\big(L_X g_{ij} - g_{ai} \nabla_j v^a + g_{ai} P^a_j\big) + \mu g_{aj} \nabla_i B^a = 0,$$

$$g_{ai} P^a_j + g_{aj} P^a_i = 0,$$

where $A^a_h = P^a_h - \nabla_h X^a$. By the same method used in case 1, we have

$$\alpha L_X g_{ij} + \beta L_B g_{ij} = 0,$$

$$L_X \Gamma^a_{ij} = 0,$$

$$\beta(g_{aj} \nabla_i v^a + g_{ai} P^a_j) + \mu g_{aj} \nabla_i B^a = 0,$$

$$g_{ai} P^a_j + g_{aj} P^a_i = 0,$$

where $A^a_j := P^a_j - \nabla_j v^a$.

Corollary 4.3 is proved.

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