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DISSIPATIVE DIRAC OPERATOR WITH GENERAL BOUNDARY CONDITIONS ON TIME SCALES

ДИСИПАТИВНИЙ ОПЕРАТОР ДІРАКА ІЗ ЗАГАЛЬНИМИ ГРАНИЧНИМИ УМОВАМИ НА ЧАСОВИХ ШКАЛАХ

In this paper, we consider the symmetric Dirac operator on bounded time scales. With general boundary conditions, we describe extensions (dissipative, accumulative, self-adjoint and the other) of such symmetric operators. We construct a self-adjoint dilation of dissipative operator. Hence, we determine the scattering matrix of dilation. Later, we construct a functional model of this operator and define its characteristic function. Finally, we prove that all root vectors of this operator are complete.

Розглядається симетричний оператор Дірака на обмежених часових шкалах. При загальних граничних умовах описано розширення (дисипативні, акумулятивні, самоспряжені та інші) таких симетричних операторів. Побудовано самоспряжене розширення дисипативного оператора та визначено матрицю розсіювання дилатації. Також побудовано функціональну модель цього оператора та визначено його характеристичну функцію. Насамкінець доведено, що всі кореневі вектори цього оператора є повними.

1. Introduction. Continuous systems and discrete systems are two important types of the dynamic systems. The time scale calculus is one of the main approaches in order to unify continuous and discrete analysis. It was founded by Stefan Hilger in his Ph. D. thesis [15]. Since then it has received much attention because it has important applications in some mathematical models of real processes and phenomena studied in physics, chemical technology, population dynamics, biotechnology and economics, neural networks and social sciences. For more information, we refer the reader to the references [16 – 21].

The spectral theory of Sturm – Liouville equations on time scales has been developed extensively during the past several years. We refer the reader to [26 – 37]. However, in the literature, there exists a few research about the Dirac system on time scales [25, 38]. In [25], Gulsen and Yılmaz studied an eigenvalue problem for the Dirac system with separated boundary conditions on an arbitrary time scale. They improved the results about the spectral theory of the classical Dirac system, such as the orthogonality of eigenfunctions and the simplicity of the eigenvalues. In [38], the author gave some sufficient conditions for the disconjugacy of Dirac systems and obtained a formula about the number of the eigenvalues of the problem. In this paper, we prove some theorems on the completeness of the system of root functions of the Dirac system on time scales. Hence, our study could fill an important gap in the spectral theory of the Dirac system on time scales.

The Dirac system is a cornerstone in the history of physics. It provides a natural description of the electron spin, predicts the existence of antimatter and is able to reproduce accurately the spectrum of the hydrogen atom. Dirac systems describes particles known as fermions, such as electrons. For more information, we refer the reader to the books [14, 22, 23].

In the operator theory, dissipative operators is one of the important class of operators. When we study the spectral analysis of dissipative operators, some of the basic methods are resolvent analysis, Riesz integrals and the theory of dilations with applications of functional models. Using a functional model of a dissipative operator, spectral properties of such operators were investigated in [3 – 5, 9 – 11]. In this article, we use this method for the one dimensional dissipative Dirac operator on time scales. Although the Dirac system on the time scale is not a particular example of the known general theory because Δ-differentiation is different from ordinary differentiation, the main results of this time scale version coincide with the general theory.

The paper is organized as follows. In Section 2, some preliminary concepts related to time scales are presented for the convenience of the reader. In Section 3, we describe all the maximal dissipative, maximal accumulative, self-adjoint and other extensions of minimal symmetric Dirac operator derived from the one dimensional Dirac system

$$-\Delta y_2^{\rho} + p(t)y_1 = \lambda y_1, \quad t \in \mathbb{T},$$

$$\Delta y_1 + r(t)y_2 = \lambda y_2,$$

where $p(.)$ and $r(.)$ are real-valued continuous functions defined on bounded time scale $\mathbb{T}$, $y_2^{\rho}(t) = y_2(\rho(t))$ and $p(.), r(.) \in L_{\Delta}^1(\mathbb{T})$. In Section 4, we construct a self-adjoint dilation of dissipative operator and its incoming and outgoing spectral representations of dilation. Hence, we determine the scattering matrix of the dilation according to the Lax and Phillips scheme [1, 2]. We construct a functional model of the maximal dissipative Dirac operator with general boundary conditions, using incoming spectral representations. Further, we determine characteristic function of this operator. Finally, in Section 5, we prove that all root vectors of the maximal dissipative Dirac operator are complete.

2. Preliminaries. Let $\mathbb{T}$ be a time scale, i.e., a nonempty closed subset of real numbers $\mathbb{R}$. The forward jump operator $\sigma\colon \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) = \inf\{s \in \mathbb{T}\colon s > t\}, \quad \text{where} \quad t \in \mathbb{T},$$

and the backward jump operator $\rho\colon \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) = \sup\{s \in \mathbb{T}\colon s < t\}, \quad \text{where} \quad t \in \mathbb{T}.$$

It is convenient to have graininess operators $\mu_\sigma\colon \mathbb{T} \to [0, \infty)$ and $\mu_\rho\colon \mathbb{T} \to (-\infty, 0]$ defined by

$$\mu_\sigma(t) = \sigma(t) - t$$

and

$$\mu_\rho(t) = \rho(t) - t,$$

respectively.

Definition 1. *A point $t \in \mathbb{T}$ is left scattered if $\mu_\rho(t) \neq 0$ and left dense if $\mu_\rho(t) = 0$. A point $t \in \mathbb{T}$ is right scattered if $\mu_\sigma(t) \neq 0$ and right dense if $\mu_\sigma(t) = 0$.*

Now, we introduce the sets $\mathbb{T}^k$, $\mathbb{T}_k$ which are derived form the time scale $\mathbb{T}$ as follows. If $\mathbb{T}$ has a left scattered maximum t_1, then $\mathbb{T}^k = \mathbb{T} - \{t_1\}$, otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right scattered minimum t_2, then $\mathbb{T}_k = \mathbb{T} - \{t_2\}$, otherwise $\mathbb{T}_k = \mathbb{T}$.

Definition 2. *A function f on $\mathbb{T}$ is said to be Δ-differentiable at some point $t \in \mathbb{T}^k$ if there is a number $f^{\Delta}(t)$ such that for every $\varepsilon > 0$ there is a neighborhood $U \subset \mathbb{T}$ of t such that*

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|, \quad \textit{where} \quad s \in U. \tag{1}$$

Analogously, one may define the notion of ∇-differentiability of some function using the backward jump ρ.

We note the following.

If $t \in \mathbb{T} \backslash \mathbb{T}^k$, then $f^{\Delta}(t)$ is not uniquely defined, since for such a point t, small neighborhoods U of t consist only of t, and besides, we have $\sigma(t) = t$. Therefore, (1) holds for an arbitrary number $f^{\Delta}(t)$(see [29]).

One can show (see [29])

$$f^{\Delta}(t) = f^{\nabla}(\sigma(t)), \quad f^{\nabla}(t) = f^{\Delta}(\rho(t))$$

for continuously differentiable functions.

If $\mathbb{T} = \mathbb{R}$, then

$$f^{\Delta}(t) = f'(t).$$

If $\mathbb{T} = h\mathbb{Z}$, $h > 0$, then

$$f^{\Delta}(t) = \frac{f\,(t+h) - f(t)}{h}.$$

If $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$, then

$$f^{\Delta}(t) = \frac{f\,(qt) - f(t)}{(q-1)\,t}.$$

The product and quotient rules on time scales have the following form: If $f, g \colon \mathbb{T} \to \mathbb{R}$, then

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t),$$

$$(fg)^{\nabla}(t) = f^{\nabla}(t)g(t) + f(\rho(t))g^{\nabla}(t),$$

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g\,(\sigma(t))},$$

$$\left(\frac{f}{g}\right)^{\nabla}(t) = \frac{f^{\nabla}(t)g(t) - f(t))g^{\nabla}(t)}{g(t)g\,(\rho(t))}.$$

Let $f \colon \mathbb{T} \to \mathbb{R}$ be a function, and $a, b \in \mathbb{T}$. If there exists a function $F \colon \mathbb{T} \to \mathbb{R}$ such that $F^{\Delta}(t) = f(t)$ for all $t \in \mathbb{T}^k$, then F is a Δ-antiderivative of f. In this case the integral is given by the formula

$$\int_a^b f(t)\Delta t = F(b) - F(a) \quad \text{for} \quad a, b \in \mathbb{T}.$$

Analogously, one may define the notion of ∇-antiderivative of some function. If $\mathbb{T} = \mathbb{R}$ and f is continuous, then

$$\int_a^b f(t)\Delta t = \int_a^b f(t)dt.$$

If $\mathbb{T} = h\mathbb{Z},\ h > 0$, and $a = hx,\ b = hy,\ x < y$, then

$$\int_a^b f(t)\Delta t = h\sum_{k=x}^{y-1} f(hk).$$

If $\mathbb{T} = q^{\mathbb{N}_0},\ q > 1$, and $a = q^x,\ b = q^y,\ x < y$, then

$$\int_a^b f(t)\Delta t = (q-1)\sum_{k=x}^{y-1} q^k f\left(q^k\right).$$

Let $a < b$ be fixed points in $\mathbb{T}$ and $a \in \mathbb{T}^k,\ b \in \mathbb{T}_k$. Let $L^2_\Delta(\mathbb{T})$ be the space of all functions defined on $\mathbb{T}$ such that

$$\|f\| := \left(\int_a^b |f(t)|^2 \Delta t\right)^{1/2} < \infty.$$

The space $L^2_\Delta(\mathbb{T})$ is a Hilbert space with the inner product (see [24])

$$(f,g) := \int_a^b f(t)\overline{g(t)}\Delta t, \quad f,g \in L^2_\Delta(\mathbb{T}).$$

Now, we introduce convenient Hilbert space $H := L^2_\Delta\ (\mathbb{T}; E)\ (E := \mathbb{C}^2)$ of vector-valued functions using the inner product

$$(f,g)_H := \int_a^b (f(x), g(x))_E \Delta t.$$

3. All extensions of the symmetric Dirac operators on time scales. In this section, we construct a space of boundary value for minimal symmetric Dirac operator on time scales and describe all extensions (maximal dissipative, accumulative, self-adjoint and other) of such operators.

Let us consider the Dirac systems

$$My := \begin{cases} -\Delta y_2^\rho + p(t)y_1 \\ \Delta y_1 + r(t)y_2 \end{cases} = \lambda y = \begin{pmatrix} \lambda y_1 \\ \lambda y_2 \end{pmatrix}, \quad t \in \mathbb{T}, \tag{2}$$

where $\Delta f(t) = f^\Delta(t)$, λ is a complex eigenvalue parameter, $p(.)$ and $r(.)$ are real-valued continuous functions defined on bounded time scale $\mathbb{T}$ and $p(.), r(.) \in L^1_\Delta(\mathbb{T})$.

Let us consider the set $D_{\max}$ consisting of all vector-valued functions $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in H$ in which y_1 and y_2 are Δ-absolutely continuous functions on $\mathbb{T}$ and $My \in H$. We define the maximal operator $\Lambda_{\max}$ on the set $D_{\max}$ by the equality $\Lambda_{\max} y = My$.

Let $D_{\min}$ denote the linear set of all vectors $y \in D_{\max}$ satisfying the conditions

$$y_1(a) = y_2^\rho(a) = y_1(b) = y_2^\rho(b) = 0.$$

If we restrict the operator $\Lambda_{\max}$ to the set $D_{\min}$, then we obtain the minimal operator $\Lambda_{\min}$. It is clear that $\Lambda_{\min}^* = \Lambda_{\max}$, and $\Lambda_{\min}$ is a closed symmetric operator with deficiency indices $(2,2)$. In fact since $\mathbb{T}$ is a bounded time scale, the two linearly independent solutions of the Dirac systems defined by (2) both lie in $L_\Delta^2(\mathbb{T})$. Therefore, the deficiency indices of the symmetric operator $\Lambda_{\min}$ are $(2,2)$ (see [6]). Now we recall the following definitions.

Definition 3. *A linear operator S (with dense domain $D(S)$) acting on some Hilbert space H is called dissipative (accumulative) if* $\operatorname{Im}(Sf,f) \geq 0$ $(\operatorname{Im}(Sf,f) \leq 0)$ *for all $f \in D(S)$ and maximal dissipative (maximal accumulative) if it does not have a proper dissipative (accumulative) extension (see* [3–5]).

Definition 4 (space of boundary values). *A triplet $(\mathbb{H}, T_1, T_2)$ is called a space of boundary values of a closed symmetric operator S with equal deficiency numbers on a Hilbert space H if T_1 and T_2 are linear maps from $D(S^*)$ to $\mathbb{H}$, and such that:*

i) *for every $f, g \in D(S^*)$ we have*

$$(S^*f, g)_H - (f, S^*g)_H = (T_1 f, T_2 g)_{\mathbb{H}} - (T_2 f, T_1 g)_{\mathbb{H}};$$

ii) *for any $F_1, F_2 \in \mathbb{H}$ there is a vector $f \in D(A^*)$ such that $T_1 f = F_1$ and $T_2 f = F_2$ (see* [7]).

Now we have the following lemma.

Lemma 1 (Green's formula). *Let* $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in D_{\max}$. *Then we have*

$$(My, z) - (y, Mz) = [y,z]_b - [y,z]_a,$$

where $[y,z]_t := y_1(t)\overline{z_2^\rho(t)} - \overline{z_1(t)}y_2^\rho(t)$.

Proof. Let $y, z \in D_{\max}$. Then we obtain

$$(My,z)_H - (y,Mz)_H = \int_a^b \left(-\Delta y_2^\rho + p(t)y_1\right)\overline{z_1}\Delta t + \int_a^b \left(\Delta y_1 + r(t)y_2\right)\overline{z_2}\Delta t -$$

$$- \int_a^b y_1\overline{(-\Delta z_2^\rho + p(t)z_1)}\Delta t - \int_a^b y_2\overline{(\Delta z_1 + r(t)z_2)}\Delta t =$$

$$= -\int_a^b \left[(\Delta y_2^\rho)\overline{z_1} + y_2\overline{(\Delta z_1)}\right]\Delta t + \int_a^b \left[(\Delta y_1)\overline{z_2} + y_1\overline{(\Delta z_2^\rho)}\right]\Delta t.$$

Since

$$\Delta\left(\overline{z_1(t)}y_2^\rho(t)\right) = \overline{z_1(t)}\left(\Delta y_2^\rho(t)\right) + \left(y_2^\rho(t)\right)^\sigma\overline{\left(\Delta z_1(t)\right)} =$$

$$= \Delta y_2^\rho(t)\overline{z_1(t)} + y_2(t)\overline{(\Delta z_1(t))}$$

and

$$\Delta\left(\overline{z_2^\rho(t)}y_1(t)\right) = \overline{(\Delta z_2^\rho(t))}y_1(t) + \overline{\left(z_2^\rho(t)\right)^\sigma}\left(\Delta y_1(t)\right) =$$

$$= \overline{(\Delta z_2^\rho(t))} y_1(t) + \overline{z_2(t)} \Delta y_1(t).$$

Hence, we get

$$(My, z)_H - (y, Mz)_H = -\int_a^b \Delta\left(\overline{z_1(t)} y_2^\rho(t)\right) \Delta t + \int_a^b \Delta\left(y_1(t) \overline{z_2^\rho(t)}\right) \Delta t =$$

$$= \int_a^b \Delta\left[y_1(t)\overline{z_2^\rho(t)} - \overline{z_1(t)} y_2^\rho(t)\right] \Delta t = [y,z]_b - [y,z]_a .$$

Lemma 1 is proved.

Let us define by $T_1,\ T_2$ the linear maps from $D_{\max}$ to $\mathbb{C}^2$ by the formula

$$T_1 y = \begin{pmatrix} -y_1(a) \\ y_1(b) \end{pmatrix}, \quad T_2 y = \begin{pmatrix} y_2^\rho(a) \\ y_2^\rho(b) \end{pmatrix}. \tag{3}$$

Now we will prove the following theorem.

Theorem 1. *The triplet* $\left(\mathbb{C}^2, T_1, T_2\right)$ *defined by* (3) *is a boundary spaces of the operator* $\Lambda_{\min}$.

Proof. Let $y, z \in D_{\max}$. Then we obtain

$$(T_1 y, T_2 z)_{\mathbb{C}^2} - (T_2 y, T_1 z)_{\mathbb{C}^2} = -y_1(a)\overline{z}_2^\rho(a) + \overline{z_1}(a) y_2^\rho(a) +$$

$$+ y_1(b)\overline{z}_2^\rho(b) - \overline{z_1}(b) y_2^\rho(b).$$

By using Green's formula, we get

$$(T_1 y, T_2 z)_{\mathbb{C}^2} - (T_2 y, T_1 z)_{\mathbb{C}^2} = [y,z]_b - [y,z]_a .$$

Hence,

$$(\Lambda_{\max} y, z)_H - (y, \Lambda_{\max} z)_H = (T_1 y, T_2 z)_{\mathbb{C}^2} - (T_2 y, T_1 z)_{\mathbb{C}^2} .$$

Thus, we prove the first condition of the definition of a space of boundary value.

Now, we will prove the second condition. Let

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{C}^2.$$

Then the vector-valued function

$$y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \alpha_1(t) u_1(t) + \alpha_2(t) v_1(t) + \beta_1(t) u_2(t) + \beta_2(t) v_2(t),$$

where

$$\alpha_1(t) = \begin{pmatrix} \alpha_{11}(t) \\ \alpha_{12}(t) \end{pmatrix}, \quad \alpha_2(t) = \begin{pmatrix} \alpha_{21}(t) \\ \alpha_{22}(t) \end{pmatrix},$$

$$\beta_1(t) = \begin{pmatrix} \beta_{11}(t) \\ \beta_{12}(t) \end{pmatrix}, \quad \beta_2(t) = \begin{pmatrix} \beta_{21}(t) \\ \beta_{22}(t) \end{pmatrix} \in D_{\max}$$

satisfy the conditions

$$\alpha_{11}(a) = -1, \quad \alpha_{12}^{\rho}(a) = \alpha_{11}(b) = \alpha_{12}^{\rho}(b) = 0,$$

$$\alpha_{22}^{\rho}(a) = 1, \quad \alpha_{21}(a) = \alpha_{21}(b) = \alpha_{22}^{\rho}(b) = 0,$$

$$\beta_{11}(b) = 1, \quad \beta_{11}(a) = \beta_{12}^{\rho}(a) = \beta_{12}^{\rho}(b) = 0,$$

$$\beta_{22}^{\rho}(b) = 1, \quad \beta_{21}(a) = \beta_{21}(b) = \beta_{22}^{\rho}(a) = 0,$$

belong to the set $D_{\max}$ and $T_1 y = u$, $T_2 y = v$.

Theorem 1 is proved.

***Corollary* 1.** *For any contraction K in $\mathbb{C}^2$ the restriction of the operator $\Lambda_{\max}$ to the set of functions $y \in D_{\max}$ satisfying either*

$$(K - I)\, T_1 y + i\,(K + I)\, T_2 y = 0 \tag{4}$$

or

$$(K - I)\, T_1 y - i\,(K + I)\, T_2 y = 0 \tag{5}$$

is, respectively, the maximal dissipative and accumulative extension of the operator $\Lambda_{\min}$. Conversely, every maximal dissipative (accumulative) extension of the operator $\Lambda_{\min}$ is the restriction of $\Lambda_{\max}$ to the set of functions $y \in D_{\max}$ satisfying (4) ((5)), *and the extension uniquely determines the contraction K. Conditions* (4) ((5)), *in which K is an isometry describe the maximal symmetric extensions of $\Lambda_{\min}$ in H. If K is unitary, these conditions define self-adjoint extensions. In the latter case,* (4) *and* (5) *are equivalent to the condition*

$$(\cos S)\, T_1 y - (\sin S)\, T_2 y = 0,$$

where S is a self-adjoint operator in $\mathbb{C}^2$. The general form of dissipative and accumulative extension of the operator $\Lambda_{\min}$ is given by the conditions

$$K\,(T_1 y + i T_2 y) = T_1 y - i T_2 y, \quad T_1 y + i T_2 y \in D\,(K), \tag{6}$$

$$K\,(T_1 y - i T_2 y) = T_1 y + i T_2 y, \quad T_1 y - i T_2 y \in D\,(K), \tag{7}$$

where K is a linear operator with

$$\|Kf\| \le \|f\|\,, \quad f \in D\,(K).$$

The general form of symmetric extensions is given by formulae (6) *and* (7), *where K is an isometric operator. In particular, the boundary conditions*

$$y_2^{\rho}(a) - \sigma_1 y_1(a) = 0, \tag{8}$$

$$y_2^{\rho}(b) + \sigma_2 y_1(b) = 0 \tag{9}$$

with $\operatorname{Im}\sigma_1 \ge 0$ or $\sigma_1 = \infty$, $\operatorname{Im}\sigma_2 \ge 0$ or $\sigma_2 = \infty$ $(\operatorname{Im}\sigma_1 = 0$ or $\sigma_1 = \infty$, $\operatorname{Im}\sigma_2 = 0$ or $\sigma_2 = \infty)$ describe the maximal dissipative (*self-adjoint*) *extensions of $\Lambda_{\min}$ with separated boundary conditions. Note that if $\sigma_1 = \infty$ $(\sigma_2 = \infty)$, then the boundary condition* (8) ((9)) *should be replaced by $y_1(a) = 0$ $(y_1(b) = 0)$.*

Now, we study the maximal dissipative operator Λ_K, where K is the strict contraction in $\mathbb{C}^2$ generated by the expression M and boundary condition (4). Since K is a strict contraction, the operator $K+I$ must be invertible, and the boundary condition (4) is equivalent to the condition

$$T_2 y + G T_1 y = 0, \tag{10}$$

where $G = -i\,(K+I)^{-1}(K-I)$, $\operatorname{Im} G > 0$, and $-K$ is the Cayley transform of the dissipative operator G. We denote $\Lambda_G\,(=\Lambda_K)$ the dissipative operator generated by the expression M and boundary condition (10).

Let

$$G = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix},$$

where $\operatorname{Im}\sigma_1 > 0$, $\operatorname{Im}\sigma_2 > 0$ and $C^2 = 2\operatorname{Im} G$, $C > 0$. Then the boundary condition (10) coincides with the separated boundary conditions (8) and (9).

4. Self-adjoint dilation, incoming and outgoing spectral representations. In this section, we set up a self-adjoint dilation of the maximal dissipative Dirac operator on time scales and its incoming and outgoing spectral representations. Later, we determine the scattering matrix of the dilation according to the Lax and Phillips scheme [1, 2]. Using incoming spectral representations, we establish a functional model of this operator. Finally, we determine characteristic function of this operator.

Now, we consider the spaces $L^2((-\infty,0);E)$ and $L^2((0,\infty);E)$. The orthogonal sum $\mathcal{H} = L^2((-\infty,0);E) \oplus H \oplus L^2((0,\infty);E)$ is called *main Hilbert space of the dilation*.

In the space $\mathcal{H}$, we define the operator Υ on the set $D(\Upsilon)$, its elements consisting of vectors $w = \langle \eta_-, y, \eta_+\rangle$, generated by the expression

$$\Upsilon\langle \eta_-, y, \eta_+\rangle = \left\langle i\frac{d\eta_-}{d\xi}, My, i\frac{d\eta_+}{d\varsigma}\right\rangle, \tag{11}$$

$$\eta_- \in W_2^1\left((-\infty,0);E\right), \quad \eta_+ \in W_2^1\left((0,\infty);E\right), \quad y \in H,$$

$$T_2 y + G T_1 y = C\eta_-(0), \quad T_2 y + G^* T_1 y = C\eta_+(0), \quad C^2 := 2\operatorname{Im} G, \quad C > 0, \tag{12}$$

where W_2^1 is the Sobolev space.

Theorem 2. *The operator Υ is self-adjoint in $\mathcal{H}$.*

Proof. We first prove that Υ is symmetric in $\mathcal{H}$. Let $f,g \in D(\Upsilon)$, $f = \langle \eta_-, y, \eta_+\rangle$ and $g = \langle \zeta_-, z, \zeta_+\rangle$. Then we have

$$(\Upsilon f, g)_{\mathcal{H}} - (f, \Upsilon g)_{\mathcal{H}} = i\left(\eta_-(0), \zeta_-(0)\right)_E - i\left(\eta_+(0), \zeta_+(0)\right)_E + [y,z]_b - [y,z]_a. \tag{13}$$

By direct computation, we get

$$i\left(\eta_-(0), \zeta_-(0)\right)_E - i\left(\eta_+(0), \zeta_+(0)\right)_E + [y,z]_b - [y,z]_a = 0.$$

Thus, $\Upsilon \subseteq \Upsilon^*$, i.e., Υ is a symmetric operator.

It is easy to check that Υ and Υ^* are generated by the same expression (11). Let us describe the domain of Υ^*. We shall compute the terms outside the integral sign, which are obtained by integration by parts in bilinear form $(\Upsilon f, g)_{\mathcal{H}}$, $f \in D(\Upsilon)$, $g \in D(\Upsilon^*)$. Their sum is equal to zero,

$$[y,z]_b - [y,z]_a + i\left(\eta_-(0),\zeta_-(0)\right)_E - i\left(\eta_+(0),\zeta_+(0)\right)_E = 0.$$

Further, solving the boundary conditions (12) for T_1y and T_2y, we find

$$T_1y = -iC^{-1}\left(\eta_-(0) - \eta_+(0)\right), \quad T_2y = C\eta_-(0) + iTC^{-1}\left(\eta_-(0) - \eta_+(0)\right).$$

Therefore, using (3), we find that (13) is equivalent to the equality

$$i\left(\eta_+(0),\zeta_+(0)\right)_E - i\left(\eta_-(0),\zeta_-(0)\right)_E =$$
$$= [y,z]_b - [y,z]_a = (T_1y, T_2z)_E - (T_2y, T_1z)_E =$$
$$= -i\left(C^{-1}\left(\eta_-(0) - \eta_+(0)\right), T_2z\right)_E - \left(C\eta_-(0), T_1z\right)_E -$$
$$-i\left(TC^{-1}\left(\eta_-(0) - \eta_+(0)\right), T_1z\right)_E.$$

Since the values $\eta_\mp(0)$ can be arbitrary vectors, a comparison of the coefficients of $\eta_{i\mp}(0)$, $i = 1,2$, on the left and right of the last equality proves that the vector $g = \langle \zeta_-, z, \zeta_+\rangle$ satisfies the boundary conditions (12), $T_2z + GT_1z = C\zeta_-(0)$, $T_2z + G^*T_1z = C\zeta_+(0)$. Therefore, $D(\Upsilon^*) \subseteq D(\Upsilon)$, and, hence, $\Upsilon = \Upsilon^*$.

Theorem 2 is proved.

Note that the self-adjoint operator Υ generates a unitary group $U_t = \exp(i\Upsilon t)$, $t \in \mathbb{R}$, on $\mathcal{H}$. Let denote by $\mathcal{P}\colon \mathcal{H} \to H$ and $\mathcal{P}_1\colon H \to \mathcal{H}$ the mapping acting according to the formulae $\mathcal{P}\colon \langle \eta_-, y, \eta_+\rangle \to y$ and $\mathcal{P}_1\colon y \to \langle 0, y, 0\rangle$. Let $Z_t := \mathcal{P}U_t\mathcal{P}_1$, $t \geq 0$, by using U_t. The family $\{Z_t\}$, $t \geq 0$ of operators is a strongly continuous semigroup of completely nonunitary contraction on H. Let us denote by$B$ the generator of this semigroup: $By = \lim_{t\to+0}\left(\dfrac{Z_ty - y}{it}\right)$. The domain of B consists of all the vectors for which the limit exists. The operator B is dissipative. The operator Υ is called the *self-adjoint dilation* of B. Then we have the following theorem.

Theorem 3. *The operator Υ is a self-adjoint dilation of the operator Λ_G $(= \Lambda_K)$.*

Proof. We will show that $B = \Lambda_G$, hence Υ is self-adjoint dilation of B. If we prove that the equality

$$\mathcal{P}(\Upsilon - \lambda I)^{-1}\mathcal{P}_1y = (\Lambda_G - \lambda I)^{-1}y, \quad y \in H, \quad \operatorname{Im}\lambda < 0,$$

the assertion follows (see [8]). For this purpose, we set $(\Upsilon - \lambda I)^{-1}\mathcal{P}_1y = g = \langle \zeta_-, z, \zeta_+\rangle$ implies that $(\Upsilon - \lambda I)g = \mathcal{P}_1y$, and, hence, $Mz - \lambda z = y$, $\zeta_-(\xi) = \zeta_-(0)e^{-i\lambda\xi}$ and $\zeta_+(\xi) = \zeta_+(0)e^{-i\lambda\xi}$. Since $g \in D(\Upsilon)$, then $\zeta_- \in W_2^1((-\infty,0);E)$, it follows that $\zeta_-(0) = 0$, and consequently z satisfies the boundary condition $T_2z + GT_1z = 0$. Therefore, $z \in D(\Lambda_G)$, and since point λ with $\operatorname{Im}\lambda < 0$ cannot be an eigenvalue of dissipative operator, then $z = (\Lambda_G - \lambda I)^{-1}y$. Thus,

$$(\Upsilon - \lambda I)^{-1}\mathcal{P}_1y = \langle 0, (\Lambda_G - \lambda I)^{-1}y, C^{-1}(T_2y + G^*T_1y)e^{-i\lambda\xi}\rangle$$

for $y \in H$ and $\operatorname{Im}\lambda < 0$. On applying the mapping $\mathcal{P}$, we get

$$(\Lambda_G - \lambda I)^{-1} = \mathcal{P}(\Upsilon - \lambda I)^{-1}\mathcal{P}_k = -i\mathcal{P}\int\limits_0^\infty U_te^{-i\lambda t}dt\mathcal{P}_k =$$

$$= -i\int_0^\infty Z_t e^{-i\lambda t}dt = (B-\lambda I)^{-1}, \quad \operatorname{Im}\lambda < 0,$$

i.e., $\Lambda_G = B$.

Theorem 3 is proved.

On the other hand, the unitary group $\{U_t\}$ has an important property which makes it possible to apply it to the Lax–Phillips theory (see [1]). It has orthogonal incoming and outgoing subspaces $D_- = \langle L^2((-\infty,0);E),0,0\rangle$ and $D_+ = \langle 0,0,L^2((0,\infty);E)\rangle$ and they have following properties.

Lemma 2. *$U_tD_- \subset D_-$, $t \leq 0$, and $U_tD_+ \subset D_+$, $t \geq 0$.*

Proof. We will just prove for D_+ since the proof for D_- is similar. Set $\mathcal{R}_\lambda = (\Upsilon - \lambda I)^{-1}$. Then, for all λ, with $\operatorname{Im}\lambda < 0$, we have

$$\mathcal{R}_\lambda f = \left\langle 0,0,-ie^{-i\lambda\xi}\int_0^\xi e^{i\lambda s}\eta_+(s)\,ds\right\rangle, \quad f = \langle 0,0,\eta_+\rangle \in D_+.$$

Hence, we have $\mathcal{R}_\lambda f \in D_+$. If $g \perp D_+$, then we get

$$0 = (\mathcal{R}_\lambda f, g)_{\mathcal{H}} = -i\int_0^\infty e^{-i\lambda t}(U_tf,g)_{\mathcal{H}}\,dt, \quad \operatorname{Im}\lambda < 0.$$

Thus, we obtain $(U_tf,g)_{\mathcal{H}} = 0$ for all $t \geq 0$, i.e., $U_tD_+ \subset D_+$ for $t \geq 0$.

Lemma 2 is proved.

Lemma 3. $\bigcap_{t\leq 0} U_tD_- = \bigcap_{t\geq 0} U_tD_+ = \{0\}$.

Proof. Let us define the mappings $\mathcal{P}^+ : \mathcal{H} \to L^2((0,\infty);E)$ and $\mathcal{P}_1^+ : L^2((0,\infty);E) \to D_+$ as follows $\mathcal{P}^+ : \langle \eta_-, \widehat{y}, \eta_+\rangle \to \eta_+$ and $\mathcal{P}_1^+ : \eta \to \langle 0,0,\eta\rangle$, respectively. We take into consider that the semigroup of isometries $U_t^+ := \mathcal{P}^+U_t\mathcal{P}_1^+$, $t \geq 0$, is a one-sided shift in $L^2((0,\infty);E)$. Indeed, the generator of the semigroup of the one-sided shift V_t in $L^2((0,\infty);E)$ is the differential operator $i\dfrac{d}{d\xi}$ with the boundary condition $\eta(0) = 0$. On the other hand, the generator S of the semigroup of isometries U_t^+, $t \geq 0$, is the operator

$$S\eta = \mathcal{P}^+\Upsilon\mathcal{P}_1^+\eta = \mathcal{P}^+\Upsilon\langle 0,0,\eta\rangle = \mathcal{P}^+\left\langle 0,0,i\frac{d\eta}{d\xi}\right\rangle = i\frac{d\eta}{d\xi},$$

where $\eta \in W_2^1((0,\infty);E)$ and $\eta(0) = 0$. Since a semigroup is uniquely determined by its generator, it follows that $U_t^+ = V_t$, and, therefore, we obtain

$$\bigcap_{t\geq 0} U_tD_+ = \langle 0,0,\bigcap_{t\leq 0} V_tL^2((0,\infty);E)\rangle = \{0\}.$$

Lemma 3 is proved.

Definition 5. *The linear operator T acting in the Hilbert space H is called simple if there is no subspace $N \neq \{0\}$ invariant for T such that $\overline{N \cap D(T)} = N$ and the restriction of T to $N \cap D(T)$ is self-adjoint on N.*

Lemma 4. *The operator Λ_G is simple.*

Proof. Let $H' \subset H$ be a nontrivial subspace in which Λ_G induces a self-adjoint operator Λ'_G with domain $D(\Lambda'_G) = H' \cap D(\Lambda_G)$. If $f \in D(\Lambda'_G)$, then $f \in D(\Lambda^*_G)$ and

$$0 = \frac{d}{dt}\|e^{i\Lambda_G t} f\|^2_H = \frac{d}{dt}\left(e^{i\Lambda_G t} f, e^{i\Lambda_G t} f\right)_H =$$
$$= -2\left(\operatorname{Im} G T_1 e^{i\Lambda_G t} f, T_1 e^{i\Lambda_G t} f\right)_E.$$

Consequently, we obtain $T_1 e^{i\Lambda_G t} f = 0$. For eigenvectors $y \in H'$ of the operator Λ_G we get $T_1 y = 0$. By using this result with boundary condition $T_2 y + G T_1 y = 0$, we have $T_2 y = 0$, i.e., $y = 0$. Since all solutions of $My = \lambda y$ belong to $L^2_q((0,a);E)$, from this it can be concluded that the resolvent $R_\lambda(\Lambda_G)$ is a compact operator, and the spectrum of Λ_G is purely discrete. Consequently, by the theorem on expansion in the eigenvectors of the self-adjoint operator Λ'_G, we obtain $H' = \{0\}$. Hence, the operator Λ_G is simple.

Lemma 4 is proved.

Now, we set $H_- = \overline{\bigcup_{t \geq 0} U_t D_-}$, $H_+ = \overline{\bigcup_{t \leq 0} U_t D_+}$. Then we have the following lemma.

Lemma 5. *The equality* $\overline{H_- + H_+} = \mathcal{H}$ *holds.*

Proof. From Lemma 2, it is easy to show that the subspace $\mathcal{H}' = \mathcal{H} \ominus (H_- + H_+)$ is invariant relative to the group $\{U_t\}$ and has the form $\mathcal{H}' = \langle 0, H', 0\rangle$, where H' is a subspace in H. Therefore, if the subspace $\mathcal{H}'$ (and also H') were nontrivial, then the unitary group $\{U'_t\}$ restricted to this subspace would be a unitary part of the group $\{U_t\}$, and, hence, the restriction Λ'_G of Λ_G to H' would be a self-adjoint operator in H'. Since the operator Λ_G is simple, it follows that $H' = \{0\}$.

Lemma 5 is proved.

Assume that $\chi(\lambda)$ and $\omega(\lambda)$ are solutions of $My = \lambda y$ satisfying the conditions

$$\chi_1(a,\lambda) = 0, \quad \chi^\rho_2(a,\lambda) = -1, \quad \omega_1(a,\lambda) = 1, \quad \omega^\rho_2(a,\lambda) = 0.$$

We denote by $m(\lambda)$ the matrix-valued function satisfying the conditions

$$m(\lambda) T_1 \chi = T_2 \chi, \quad m(\lambda) T_1 \omega = T_2 \omega;$$

$m(\lambda)$ is a meromorphic function on the complex plane $\mathbb{C}$ with a countable number of poles on the real axis. Further, it is possible to show that the function $m(\lambda)$ possesses the following properties: $\operatorname{Im} m(\lambda) \leq 0$ for all $\operatorname{Im}\lambda \neq 0$, and $m^*(\lambda) = m\left(\overline{\lambda}\right)$ for all $\lambda \in \mathbb{C}$, except the real poles $m(\lambda)$.

We denote by $\varsigma_j(x,\lambda)$ and $\tau_j(x,\lambda)$, $j = 1,2$, the solutions of system $My = \lambda y$ which satisfy the conditions

$$T_1 \varsigma_j = (m(\lambda) + G)^{-1} C e_j, \quad T_1 \tau_j = (m(\lambda) + G^*)^{-1} C e_j, \quad j = 1,2,$$

where e_1 and e_2 are an orthonormal basis for E.

We set

$$U^-_{\lambda j}(x,\xi,\rho) = \langle e^{-i\lambda\xi} e_j, \varsigma_j(x,\lambda), C^{-1}(m + G^*)(m + G)^{-1} C e^{-i\lambda\rho} e_j\rangle, \quad j = 1,2.$$

We note that the vectors $U^-_{\lambda j}(x,\xi,\rho)$, $j = 1,2$, for real λ do not belong to the space $\mathcal{H}$. However, $U^-_{\lambda j}(x,\xi,\rho)$, $j = 1,2$, satisfies the equation $\Upsilon U = \lambda U$ and the corresponding boundary conditions for the operator Υ.

By means of vector $U^{-}_{\lambda j}(x,\xi,\rho),\ j=1,2,$ we define the transformation $\mathcal{F}_{-}: f \to \widetilde{f}_{-}(\lambda)$ by

$$(\mathcal{F}_{-}f)(\lambda) := \widetilde{f}_{-}(\lambda) := \frac{1}{\sqrt{2\pi}}\sum_{j=1}^{2}(f, U_{\overline{\lambda j}})_{\mathcal{H}}\, e_j$$

on the vectors $f=\langle \eta_{-}, y, \eta_{+}\rangle$ in which $\eta_{-},\ \eta_{+},\ y$ are smooth, compactly supported functions.

Lemma 6. *The transformation $\mathcal{F}_{-}$ isometrically maps H_{-} onto $L^2(\mathbb{R};E)$. For all vectors $f,g\in H_{-}$ the Parseval equality and the inversion formulae hold*:

$$(f,g)_{\mathcal{H}} = (\widetilde{f}_{-},\widetilde{g}_{-})_{L^2} = \int\limits_{-\infty}^{\infty}\sum_{j=1}^{2}\widetilde{f}_{j-}(\lambda)\overline{\widetilde{g}_{j-}(\lambda)}d\lambda,$$

$$f = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}\sum_{j=1}^{2}\widetilde{f}_{j-}(\lambda)U_{\overline{\lambda j}}d\lambda,$$

where $\widetilde{f}_{-}(\lambda)=(\mathcal{F}_{-}f)(\lambda)$ *and* $\widetilde{g}_{-}(\lambda)=(\mathcal{F}_{-}g)(\lambda)$.

Proof. By Paley – Wiener theorem, we get

$$\widetilde{f}_{j-}(\lambda) = \frac{1}{\sqrt{2\pi}}(f, U_{\overline{\lambda j}})_{\mathcal{H}} =$$

$$= \frac{1}{2\pi}\int\limits_{-\infty}^{0}\left(\eta_{-}(\xi), e^{-i\lambda\xi}e_j\right)_E d\xi \in H^2_{-}(E), \quad j=1,2,$$

where $f=\langle \eta_{-},0,0\rangle,\ g=\langle \zeta_{+},0,0\rangle\in D_{-}$. If we will use the Parseval equality for Fourier integrals, then we obtain

$$(f,g)_{\mathcal{H}} = \int\limits_{-\infty}^{\infty}(\eta_{-}(\xi),\zeta_{-}(\xi))_E\, d\xi = \int\limits_{-\infty}^{\infty}(\widetilde{f}_{-}(\lambda),\widetilde{g}_{-}(\lambda))_E d\lambda = (\mathcal{F}_{-}f,\mathcal{F}_{-}g)_{L^2},$$

where $H^2_{\pm}$ denote the Hardy classes in $L^2(\mathbb{R};E)$ consisting of the functions analytically extendible to the upper and lower half-planes, respectively. We now extend to the Parseval equality to the whole of H_{-}. We consider in H_{-} the dense set of H'_{-} of the vectors obtained as follows from the smooth, compactly supported functions in $D_{-}: f\in H'_{-}$ if $f=U_T\ f_0,\ f_0=\langle\eta_{-},0,0\rangle,$ $\eta_{-}\in C_0^{\infty}((-\infty,0);E)$, where $T=T_f$ is a nonnegative number depending on f. If $f,g\in H'_{-}$, then, for $T>T_f$ and $T>T_g$, we have $U_{-T}f,\ U_{-T}g\in D_{-}$, moreover, the first components of these vectors belong to $C_0^{\infty}((-\infty,0);E)$. Therefore, since the operators $U_t,\ t\in\mathbb{R}$, are unitary, by the equality

$$\mathcal{F}_{-}U_t f = \frac{1}{\sqrt{2\pi}}\sum_{j=1}^{2}(U_t f, U_{\overline{\lambda j}})_{\mathcal{H}}\, e_j = e^{i\lambda t}\mathcal{F}_{-}f,$$

we have

$$(f,g)_{\mathcal{H}} = (U_{-T}f, U_{-T}g)_{\mathcal{H}} = (\mathcal{F}_{-}U_{-T}f, \mathcal{F}_{-}U_{-T}\ g)_{L^2} =$$

$$= (e^{-i\lambda T}\mathcal{F}_{-}f, e^{-i\lambda T}\mathcal{F}_{-}g)_{L^2} = (\widetilde{f}, \widetilde{g})_{L^2}.$$

By taking the closure, we obtain the Parseval equality for the space H_{-}. The inversion formula is obtained from the Parseval equality if all integrals in it are considered as limits in the of integrals over finite intervals. Finally, we obtain desired result

$$\mathcal{F}_{-}H_{-} = \overline{\bigcup_{t\geq 0}\mathcal{F}_{-}U_t D_{-}} = \overline{\bigcup_{t\geq 0} e^{i\lambda t}H_{-}^2} = L^2(\mathbb{R}; E).$$

Lemma 6 is proved.

Now, we set

$$U_{\lambda j}^{+}(x,\xi,\rho) = \langle S_G(\lambda)\, e^{-i\lambda\xi}e_j, \tau_j(x,\lambda), e^{-i\lambda\rho}e_j\rangle, \quad j = 1,2,$$

where

$$S_G(\lambda) = C^{-1}(m(\lambda) + G)(m(\lambda) + G^*)^{-1} C. \tag{14}$$

We note that the vectors $U_{\lambda j}^{+}(x,\xi,\rho)$ for real λ do not belong to the space $\mathcal{H}$. However, $U_{\lambda j}^{+}(x,\xi,\rho)$ satisfies the equation $\Upsilon U = \lambda U$ and the corresponding boundary conditions for the operator Υ. With the help of vector $U_{\lambda j}^{+}(x,\xi,\rho)$, we define the transformation $\mathcal{F}_{+} : f \to \widetilde{f}_{+}(\lambda)$ by

$$(\mathcal{F}_{+}f)(\lambda) := \widetilde{f}_{+}(\lambda) := \sum_{j=1}^{2}\widetilde{f}_{j+}(\lambda)e_j := \frac{1}{\sqrt{2\pi}}\sum_{j=1}^{2}\left(f, U_{\lambda j}^{+}\right)_{\mathcal{H}} e_j$$

on the vectors $f = \langle \eta_{-}, y, \eta_{+}\rangle$ in which η_{-}, η_{+} and y are smooth, compactly supported functions.

Lemma 7. *The transformation $\mathcal{F}_{+}$ isometrically maps H_{+} onto $L^2(\mathbb{R}; E)$. For all vectors $f, g \in H_{+}$ the Parseval equality and the inversion formula hold*:

$$(f,g)_{\mathcal{H}} = (\widetilde{f}_{+}, \widetilde{g}_{+})_{L^2} = \int_{-\infty}^{\infty}\sum_{j=1}^{2}\widetilde{f}_{j+}(\lambda)\overline{\widetilde{g}_{j+}(\lambda)}d\lambda,$$

$$f = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sum_{j=1}^{2}\widetilde{f}_{j+}(\lambda)U_{\lambda j}^{+}d\lambda,$$

where $\widetilde{f}_{+}(\lambda) = (\mathcal{F}_{+}f)(\lambda)$ and $\widetilde{g}_{+}(\lambda) = (\mathcal{F}_{+}g)(\lambda)$.

Proof. The proof is similar to Lemma 4.

It is clear that the matrix-valued function $S_G(\lambda)$ is meromorphic in $\mathbb{C}$ and all poles are in the lower half-plane. From (14), we obtain $\|S_G(\lambda)\| \leq 1$ for $\operatorname{Im}\lambda > 0$; and $S_G(\lambda)$ is the unitary matrix for all $\lambda \in \mathbb{R}$. Therefore, we have

$$U_{\lambda j}^{+} = \sum_{k=1}^{2} S_{jk}(\lambda)U_{\lambda k}^{-}, \quad j = 1,2, \tag{15}$$

where S_{jk}, $j,k = 1,2$, are elements of the matrix $S_G(\lambda)$. From Lemmas 4 and 5, we get $H_{-} = H_{+}$. With Lemma 5 this shows that $H_{-} = H_{+} = \mathcal{H}$, therefore we obtain been proved the following lemma for the incoming and outgoing subspaces (for D_{-} and D_{+}).

Lemma 8. $\overline{\bigcup_{t\geq 0} U_t D_-} = \overline{\bigcup_{t\leq 0} U_t D_+} = \mathcal{H}.$

Lemma 9. $D_- \perp D_+$.

Proof. It is clear.

Thus, the transformation $\mathcal{F}_-$ isometrically maps H_- onto $L^2(\mathbb{R})$ with the subspace D_- mapped onto H_-^2 and the operators U_t are transformed into the operators of multiplication by $e^{i\lambda t}$. This means that $\mathcal{F}_-$ is the incoming spectral representation for the group $\{U_t\}$. Similarly, $\mathcal{F}_+$ is the outgoing spectral representation for the group $\{U_t\}$. It follows from (15) that the passage from the $\mathcal{F}_-$ representation of an element $f \in \mathcal{H}$ to its $\mathcal{F}_+$ representation is accomplished as $\widetilde{f}_+(\lambda) = S_G^{-1}(\lambda)\widetilde{f}_-(\lambda)$. Consequently, according to [1], we have proved the following theorem.

Theorem 4. *The function $S_G^{-1}(\lambda)$ is the scattering matrix of the group $\{U_t\}$ (of the self-adjoint operator Υ).*

Now, we recall the following definition.

Definition 6 [2]**.** *The analytic matrix-valued function $S(\lambda)$ on the upper half-plane $\mathbb{C}_+$ is called inner function on $\mathbb{C}_+$ if $\|S(\lambda)\| \leq 1$ for all $\lambda \in \mathbb{C}_+$ and $S(\lambda)$ is a unitary matrix for almost all $\lambda \in \mathbb{R}$.*

Let $S(\lambda)$ be an arbitrary nonconstant inner function on the upper half-plane. Let us define K by the formula $K = H_+^2 \ominus SH_+^2$. It is clear that $K \neq \{0\}$ is a subspace of the Hilbert space H_+^2. We consider the semigroup of operators Z_t $(t \geq 0)$ acting in K according to the formula

$$Z_t\varphi = \mathcal{P}[e^{i\lambda t}\varphi], \quad \varphi = \varphi(\lambda) \in K,$$

where $\mathcal{P}$ is the orthogonal projection from H_+^2 onto K. The generator of the semigroup $\{Z_t\}$ is denoted by

$$T\varphi = \lim_{t\to+0}\left(\frac{Z_t\varphi - \varphi}{it}\right)$$

which T is a maximal dissipative operator acting in K and with the domain $D(T)$ consisting of all functions $\varphi \in K$, such that the limit exists. The operator T is called *a model dissipative operator*. This model dissipative operator is a special case of a more general model dissipative operator constructed by Nagy and Foiaş [2], which is associated with the names of Lax – Phillips [1]. Here, the basic assertion is that $S(\lambda)$ is the *characteristic function* of the operator T.

Let $K = \langle 0, H, 0\rangle$, so that $\mathcal{H} = D_- \oplus K \oplus D_+$. From the explicit form of the unitary transformation $\mathcal{F}_-$ under the mapping $\mathcal{F}_-$, we obtain

$$\begin{gathered}
\mathcal{H} \to L^2(\mathbb{R}; E), \quad f \to \widetilde{f}_-(\lambda) = (\mathcal{F}_- f)(\lambda), \\
D_- \to H_-^2(E), \quad D_+ \to S_G H_+^2(E), \\
K \to H_+^2(E) \ominus S_G H_+^2(E), \\
U_t \to (\mathcal{F}_- U_t \mathcal{F}_-^{-1} \widetilde{f}_-)(\lambda) = e^{i\lambda t}\widetilde{f}_-(\lambda).
\end{gathered} \tag{16}$$

Formulae (16) show that operator Λ_G (Λ_K) is a unitarily equivalent to the model dissipative operator with the characteristic function $S_G(\lambda)$. Since the characteristic functions of unitary equivalent dissipative operator coincide (see [2]), we have thus proved the following theorem.

Theorem 5. *The function $S_G(\lambda)$ defined* (14) *coincides with the characteristic function of the maximal dissipative operator Λ_G (Λ_K).*

5. Completeness of root vectors of the maximal dissipative Dirac operator on time scales. In this section, we prove that all root vectors of the maximal dissipative Dirac operator on time scales are complete. we will prove the characteristic function $S_G(\lambda)$ is a Blaschke – Potapov product. Because we know that the absence of the singular factor in the factorization of the characteristic function is guarantee the completeness of the system of root vectors of maximal dissipative operators [2, 12].

Lemma 10 [4]. *The characteristic function* $\widetilde{S}_K(\lambda)$*of the operator* Λ_K *has the form*

$$\widetilde{S}_K(\lambda) := S_G(\lambda) = X_1 \left(I - K_1 K_1^*\right)^{1/2} \left(\Theta(\xi) - K_1\right) \left(I - K_1^* \Theta(\xi)\right)^{-1} \left(I - K_1 K_1^*\right)^{1/2} X_2,$$

where $K_1 = -K$ *is the Cayley transformation of the dissipative operator* G, *and* $\Theta(\xi)$ *is the Cayley transformation of the matrix-valued function* $m(\lambda)$,

$$\xi = (\lambda - i)(\lambda + i)^{-1}$$

and

$$X_1 := (\operatorname{Im} G)^{-1/2} (I - K_1)^{-1} \left(I - K_1 K_1^*\right)^{1/2},$$

$$X_2 := \left(I - K_1^* K_1\right)^{-1/2} \left(I - K_1^*\right)^{-1} (\operatorname{Im} G)^{1/2},$$

$$|\det X_1||\det X_2| = 1.$$

Recall that the inner matrix-valued function $\widetilde{S}_G(\lambda)$ is a Blaschke – Potapov product if and only if $\det \widetilde{S}_G(\lambda)$ is a Blaschke product (see [2, 12]). By Lemma 10, the characteristic function $\widetilde{S}_G(\lambda)$ is a Blaschke – Potapov product if and only if the matrix-valued function

$$X_K(\xi) = \left(I - K_1 K_1^*\right)^{1/2} \left(\Theta(\xi) - K_1\right) \left(I - K_1^* \Theta(\xi)\right)^{-1} \left(I - K_1 K_1^*\right)^{1/2}$$

is a Blaschke – Potapov product in the unit disk.

Definition 7 [13]. *Let* $\widetilde{E}$ *be an* n*-dimensional* $(n < \infty)$. *In* $\widetilde{E}$ *we fix an orthonormal basis* $e_1, e_2, \ldots, e_n$ *and denote by* E_k, $k = 1, 2, \ldots, n$, *the linear span of vectors* $e_1, e_2, \ldots, e_k$. *If* $L \subset E_k$, *then the population of* $x \in E_{k-1}$ *with the property*

$$\operatorname{Cap}\{\lambda : \lambda \in \mathbb{C}, (x + \lambda e_k) \subset L\} > 0$$

will be shown by $\Gamma_{k-1} L$ ($\operatorname{Cap} G$ *is the inner logarithmic capacity of a set* $G \subset \mathbb{C}$). *The* Γ*-capacity of a set* $L \subset \widetilde{E}$ *is a number*

$$\Gamma - \operatorname{Cap} L := \sup \operatorname{Cap} \{\lambda : \lambda e_1 \subset \Gamma_1 \Gamma_2 \ldots \Gamma_{n-1} L\},$$

where supremum is taken with respect to all orthonormal basis in $\widetilde{E}$.

It is known that every set $L \subset \widetilde{E}$ of zero Γ-capacity has zero $2n^2$-dimensional Lebesque measure, however the converse is not true.

Denote by $[E]$ the set of all linear operators in E. To convert $[E]$ into an 4-dimensional Hilbert space, we give the inner product $\langle T, S\rangle = \operatorname{tr} S^* T$ for $T, S \in [E]$ ($\operatorname{tr} S^* T$ is the trace of the operators $S^* T$). Hence we may give the Γ-capacity of a set in E.

We use the following result of [12].

Lemma 11. *Let $X(\xi)$, $|\xi| < 1$, be a analytic function with the values to be contractive operators in $[E]$ $(\|X(\xi)\| \leq 1)$. Then for Γ-quasievery strictly contractive operators (i.e., for all strictly contractive $K \in [E]$ possible with the exception of a set of Γ of zero capacity) the inner part of the contractive function*

$$X_K(\xi) = (I - K_1 K_1^*)^{1/2} (X(\xi) - K_1) (I - K_1^* X(\xi))^{-1} (I - K_1 K_1^*)^{1/2}$$

is a Blaschke – Potapov product.

By summing all obtained result for the dissipative operator $\Lambda_K(\Lambda_G)$, we have proved the following theorem.

Theorem 6. *For Γ-quasievery strictly contractive $K \in [E]$ the characteristic function $\widetilde{S}_K(\lambda)$ of the dissipative operator Λ_K is a Blaschke – Potapov product and spectrum of Λ_K is purely discrete and belongs to the open upper half-plane. For Γ-quasievery strictly contractive $K \in [E]$ the operator Λ_K has an countable number of isolated eigenvalues with finite multiplicity and limit points at infinity, and the system of eigenvectors and associated vectors (or root vectors) of this operators is complete in H.*

References

1. P. D. Lax, R. S. Phillips, *Scattering theory*, Acad. Press, New York (1967).
2. B. Sz. Nagy, C. Foiaş, *Analyse Harmonique des Operateurs de L'espace de Hilber*t, Masson, Akad. Kiado, Paris, Budapest (1967).
3. B. P. Allahverdiev, *Spectral problems of nonself-adjoint 1D singular Hamiltonian systems*, Taiwanese J. Math., **17**, № 5, 1487 – 1502 (2013).
4. B. P. Allahverdiev, *Extensions, dilations and functional models of Dirac operators*, Integral Equat. and Oper. Theory, **51**, 459 – 475 (2005).
5. B. P. Allahverdiev, *Spectral analysis of dissipative Dirac operators with general boundary conditions*, J. Math. Anal. and Appl., **283**, 287 – 303 (2003).
6. M. A. Naimark, *Linear differential operators*, 2nd ed., Nauka, Moscow (1969).
7. M. L. Gorbachuk, V. I. Gorbachuk, *Boundary value problems for operator differential equations*, Naukova Dumka, Kiev (1984).
8. A. Kuzhel, *Characteristic functions and models of nonself-adjoint operators*, Kluwer Acad., Dordrecht (1996).
9. B. S. Pavlov, *Self-adjoint dilation of a dissipative Schrödinger operator and eigenfunction expansion*, Funct. Anal. and Appl., **98**, 172 – 173 (1975).
10. B. S. Pavlov, *Self-adjoint dilation of a dissipative Schrödinger operator and its resolution in terms of eigenfunctions*, Math. USSR Sbornik, **31**, № 4, 457 – 478 (1977).
11. B. S. Pavlov, *Dilation theory and spectral analysis of nonself-adjoint differential operators*, Proc. 7th Winter School, Drogobych (1974), 3-69 (1976) (in Russian); English transl: Transl. II. Ser., Amer. Math. Soc., **115**, 103 – 142 (1981).
12. Yu. P. Ginzburg, N. A. Talyush, *Exceptional sets of analytical matrix-functions, contracting and dissipative operators*, Izv. Vyssh. Uchebn. Zaved. Math., **267**, 9 – 14 (1984).
13. L. I. Ronkin, *Introduction to the theory of entire functions of several variables*, Nauka, Moscow (1971).
14. J. Weidmann, *Spectral theory of ordinary differential operators*, Lect. Notes Math., **1258** (1987).
15. S. Hilger, Ein Maßkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten (Ph. D. Thesis), Univ. Würzburg (1988).
16. D. R. Anderson, G. Sh. Guseinov, J. Hoffacker, *Higher-order self-adjoint boundary-value problems on time scales*, J. Comput. and Appl. Math., **194**, № 2, 309 – 342 (2006).
17. F. Atici Merdivenci, G. Sh. Guseinov, *On Green's functions and positive solutions for boundary value problems on time scales*, J. Comput. and Appl. Math., **141**, № 1-2, 75 – 99 (2002).
18. M. Bohner, A. Peterson, *Dynamic equations on time scales,* Birkhäuser, Boston (2001).
19. M. Bohner, A. Peterson (Eds.), *Advances in dynamic equations on time scales,* Birkhäuser, Boston (2003).

20. G. Sh. Guseinov, *Self-adjoint boundary value problems on time scales and symmetric Green's functions*, Turkish J. Math., **29**, № 4, 365 – 380 (2005).
21. V. Lakshmikantham, S. Sivasundaram, B. Kaymakcalan, *Dynamic systems on measure chains*, Kluwer Acad. Publ., Dordrecht (1996).
22. B. M. Levitan, I. S. Sargsjan, *Sturm – Liouville and Dirac operators*, Math. and Appl. (Soviet Series), Kluwer Acad. Publ. Group, Dordrecht (1991).
23. B. Thaller, *The Dirac equation,* Springer (1992).
24. B. P. Rynne, *L^2 spaces and boundary value problems on time scales*, J. Math. Anal. and Appl., **328**, 1217 – 1236 (2007).
25. T. Gulsen, E. Yilmaz, *Spectral theory of Dirac system on time scales*, Appl. Anal., **96**, № 16, 2684 – 2694 (2017).
26. G. Sh. Guseinov, *An expansion theorem for a Sturm – Liouville operator on semi-unbounded time scales*, Adv. Dyn. Syst. and Appl., **3**, 147 – 160 (2008).
27. G. Sh. Guseinov, *Eigenfunction expansions for a Sturm – Liouville problem on time scales*, Int. J. Different. Equat., **2**, 93 – 104 (2007).
28. A. Huseynov, E. Bairamov, *On expansions in eigenfunctions for second order dynamic equations on time scales*, Nonlinear Dyn. Syst. Theory, **9**, 77 – 88 (2009).
29. B. P. Allahverdiev, A. Eryilmaz, H. Tuna, *Dissipative Sturm – Liouville operators with a spectral parameter in the boundary condition on bounded time scales*, Electron. J. Different. Equat., **95**, 1 – 13 (2017).
30. B. P. Allahverdiev, *Extensions of symmetric singular second-order dynamic operators on time scales*, Filomat, **30**, № 6, 1475 – 1484 (2016).
31. B. P. Allahverdiev, *Non-self-adjoint singular second-order dynamic operators on time scale*, Math. Meth. Appl. Sci., **42**, 229 – 236 (2019).
32. B. P. Allakhverdiev, H. Tuna, *Spectral analysis of singular Sturm – Liouville operators on time scales*, Ann. Univ. Mariae Curie-Sklodowska, Sect. A, **72**, № 1, 1 – 11 (2018).
33. H. Tuna, *Dissipative Sturm – Liouville operators on bounded time scales*, Mathematica, **56**, 80 – 92 (2014).
34. H. Tuna, *Completeness of the root vectors of a dissipative Sturm – Liouville operators in time scales*, Appl. Math. and Comput., **228**, 108 – 115 (2014).
35. H. Tuna, *Completeness theorem for the dissipative Sturm – Liouville operators on bounded time scales*, Indian J. Pure and Appl. Math., **47**, № 3, 535 – 544 (2016).
36. H. Tuna, M. A. Özek, *The one-dimensional Schrödinger operator on bounded time scales*, Math. Meth. Appl. Sci., **40**, № 1, 78 – 83 (2017).
37. A. Huseynov, *Limit point and limit circle cases for dynamic equations on time scales*, Hacet. J. Math. Stat., **39**, 379 – 392 (2010).
38. A. S. Özkan, *Parameter-dependent Dirac systems on time scales*, Cumhuriyet Sci. J., **39**, № 4, 864 – 870 (2018).

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