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GEODESIC COMPLETENESS OF THE LEFT-INVARIANT METRICS ON $\mathbb{R}H^n$*
ГЕОДЕЗИЧНА ПОВНОТА ЛІВОІНВАРІАНТНИХ МЕТРИК НА $\mathbb{R}H^n$

We give the full classification of left-invariant metrics of an arbitrary signature on the Lie group corresponding to the real hyperbolic space. We show that all metrics have constant sectional curvature and that they are geodesically complete only in the Riemannian case.

Наведено повну класифікацію лівоінваріантних метрик довільної сигнатури на групі Лі, що відповідає дійсному гіперболічному просторові. Показано, що всі такі метрики мають сталу кривизну перерізу і геодезично повні лише в рімановому випадку.

It is a well-known fact that the real hyperbolic space $\mathbb{R}H^n$ with the standard Riemannian metric of constant negative curvature has a structure of a Lie group such that the metric is left-invariant. Milnor [8] considered a special class of solvable Lie groups with the property that the commutator $[x, y]$ is a linear combination of x and y for any two elements from the corresponding Lie algebra. Moreover, he has shown that such an algebra is isomorphic to the Lie algebra of $\mathbb{R}H^n$ and that every left-invariant Riemannian metric on such group has a constant negative sectional curvature. In the Lorentz case, Wolf [13] showed that this group admits a flat metric, while Nomizu [9] proved that for an arbitrary $K \in \mathbb{R}$ there exists a left-invariant metric with sectional curvature K. In [10], Nomizu considered in detail these metrics that he called Lorentz – Poincaré metrics.

In this paper we classify all left-invariant metrics of arbitrary signature on the Lie group corresponding to the hyperbolic space. This problem has also been considered in [6]. According to Arnold [1], geodesic of an arbitrary left invariant metric on a Lie group G can be seen as a motion of a "generalized rigid body" with a configuration space G. In the Riemannian case all geodesic are complete, but Gudeiri [4] gave an example of the Lorentzian metric on four-dimensional Lie group with non-complete geodesics. Lauret [7] has classified all Riemannian left-invariant metrics on the four-dimensional nilpotent Lie groups, while the authors have generalized that result to an arbitrary signature [2]. Calvaruso [3] has classified Lorentzian left-invariant metrics on the four-dimensional Lie groups that are Einstein or Ricci-parallel. In [11] the Riemannian and Lorentzian left-invariant metrics on the Heisenberg – Lie group were classified.

In the preliminary section we introduce a basic notation and give an explanation what does it mean to classify the left-invariant metrics.

In the Theorem 2.2 we present classification of the arbitrary signature left-invariant metrics on the Lie group $\mathbb{R}H^n$. We show that in the Riemannian case the only metrics are the standard metrics of constant curvature $K < 0$ of hyperbolic space. In the Lorentz case, every metric from our classification is isometric either to the flat metric obtained by Wolf, or to the metric of constant curvature $K \neq 0$ obtained by Nomizu. Also, in the Theorem 2.3 we prove that every non-flat metric on $\mathbb{R}H^n$ is a metric of the constant sectional curvature.

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The geodesical completeness of metrics is considered in Section 3. We show that all geodesic curves are complete only in the Riemannian case (Theorems 3.1 and 3.2).

In Section 4, we exhibit isometric imbedding of $\mathbb{R}H^n$ into the space forms of the curvature K.

1. Preliminaries. The group structure on a half-space model of a real hyperbolic space $\mathbb{R}H^n = \{ (x_1, \ldots, x_n) \in \mathbb{R}^n | \, x_n > 0\}$ is given by

$$(x_1, \ldots, x_{n-1}, x_n)\,(y_1, \ldots, y_{n-1}, y_n) = \big(x_1 + x_n y_1, \ldots, x_{n-1} + x_n y_{n-1}, x_n y_n\big). \tag{1}$$

Denote by $\{e_1, \ldots, e_n\}$ a corresponding basis of the Lie algebra $\mathfrak{r}_n = \text{Lie}\,\mathbb{R}H^n$. It is a semidirect product $\mathbb{R}e_n \ltimes \mathfrak{n}$, where $\mathfrak{n} = \mathcal{L}(e_1, \ldots, e_{n-1})$ is an Abelian ideal and $\text{ad}(e_n)\,|_{\mathfrak{n}} = \text{id}$, i.e., the non-null commutators are

$$[e_n, e_k] = e_k, \quad k < n.$$

Denote by $\mathcal{S}(\mathfrak{r}_n)$ a set of non-equivalent inner products of an algebra $\mathfrak{r}_n$. With a basis of the Lie algebra $\mathfrak{r}_n$ fixed, the set $\mathcal{S}(\mathfrak{r}_n)$ is identified with symmetric matrices S of an arbitrary signature modulo the following action of the automorphism group:

$$S \mapsto F^T S F, \quad F \in \text{Aut}\,(\mathfrak{r}_n). \tag{2}$$

Here we denoted by $\text{Aut}\,(\mathfrak{r}_n)$ the group of automorphisms of the Lie algebra $\mathfrak{r}_n$ that is defined by

$$\text{Aut}\,(\mathfrak{r}_n) := \big\{F\colon \mathfrak{r}_n \to \mathfrak{r}_n \,|\, F \ \text{ linear, bijective,} \ \ [Fx, Fy] = F[x, y], \ \ x, y \in \mathfrak{r}_n\big\}.$$

It is easy to check the following lemma.

Lemma 1.1. *The group* $\text{Aut}\,(\mathfrak{r}_n)$ *of automorphisms of Lie algebra* $\mathfrak{r}_n$, *in basis* $\{e_1, e_2, \ldots, e_n\}$, *consists of real matrices of form*

$$\text{Aut}\,(\mathfrak{r}_n) = \left\{ \begin{pmatrix} A & a \\ 0 & 1 \end{pmatrix} \middle| \, A \in GL(n-1, \mathbb{R}), \ \ a \in \mathbb{R}^{n-1} \right\} \cong \text{Aff}_{n-1}(\mathbb{R}), \tag{3}$$

i.e., it is isomorphic to a group of affine transformations of $\mathbb{R}^{n-1}$.

Let $x \in \mathbb{R}H^n$ with $x = (x_1, \ldots, x_n)$. For the left translations L_x the differential dL_x in every point $y \in \mathbb{R}H^n$ is given by

$$(dL_x)\left(\frac{\partial}{\partial x_k}(y)\right) = x_n \frac{\partial}{\partial x_k} L_x(y).$$

Therefore, the left-invariant vector fields $X_1^L, \ldots, X_n^L$ are given by

$$X_k^L(x) = x_n \frac{\partial}{\partial x_k}, \quad k \leq n. \tag{4}$$

2. Left-invariant metrics. Let us denote by $I_{p,r}$ the diagonal matrix $\text{diag}(\epsilon_1, \epsilon_2, \ldots, \epsilon_{p+r})$, where $\epsilon_k = -1, \ 1 \leq k \leq p$, and $\epsilon_k = 1, \ p+1 \leq k \leq p+r$.

Theorem 2.1. *The set $\mathcal{S}(\mathfrak{r}_n)$ of non-equivalent inner products of an arbitrary signature on the algebra $\mathfrak{r}_n$ is represented by the following matrices*:

$$S_\lambda = \begin{pmatrix} I_{p,r} & 0 \\ 0 & \lambda \end{pmatrix}, \quad p+r=n-1, \quad \lambda \neq 0,$$

$$S_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & I_{p,r} & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad p+r=n-2.$$

Proof. Let $F \in \mathrm{Aut}\,(\mathfrak{r}_n)$ and denote by $\bar{S}$ the arbitrary symmetric matrix representing the inner product q in the basis $\{e_1, \ldots, e_n\}$. In the same basis F is represented by the matrix (3). We are looking for the new basis such that $F^T \bar{S} F$ has the simplest form.

If we represent $\bar{S}$ in the following form:

$$\bar{S} = \begin{pmatrix} S & v \\ v^T & s \end{pmatrix},$$

where $S = S^T$ is $(n-1)\times(n-1)$ matrix, $v \in \mathbb{R}^{n-1}$ and $s \in \mathbb{R}$, then

$$F^T \bar{S} F = \begin{pmatrix} A^T S A & A^T(Sa+v) \\ (a^T S + v^T)A & a^T S a + v^T a + a^T v + s \end{pmatrix}.$$

Now, we distinguish between two cases.

Case 1. S is a regular matrix of signature (p,r). Since S is symmetric there exists $A \in GL(n-1, \mathbb{R})$ such that $A^T S A = I_{p,r}$, $p+r=n-1$. Setting $a = -S^{-1}v$, we get that the corresponding inner product is S_λ, with $\lambda = s - v^T S^{-1} v$. Since $\bar{S}$ is a non-singular matrix, $\lambda \neq 0$ must hold.

Case 2. If S is not regular, without loss of generality, we can assume that S has the form

$$\begin{pmatrix} 0 & 0 \\ 0 & \tilde{S} \end{pmatrix},$$

where $\tilde{S}$ is a regular matrix of signature (p,r), $p+r=n-2$. Then there exists a regular matrix $\widetilde{A}$ such that

$$A = \begin{pmatrix} \dfrac{1}{w} & 0 \\ 0 & \widetilde{A} \end{pmatrix} \qquad \text{and} \qquad A^T S A = \begin{pmatrix} 0 & 0 \\ 0 & I_{p,r} \end{pmatrix}.$$

For the vector $v = (w, \bar{v}^T)^T$, $w \neq 0$, $\bar{v} \in \mathbb{R}^{n-2}$, we set $a = (a_1, \bar{a}^T)^T$:

$$a_1 = \frac{\bar{v}^T \tilde{S}^{-1} \bar{v} - s}{2w} \in \mathbb{R}, \qquad \bar{a} = -\tilde{S}^{-1}\bar{v} \in \mathbb{R}^{n-2},$$

to obtain the inner product S_0.

Theorem 2.1 is proved.

The inner product q on $\mathfrak{r}_n$ gives rise to a left-invariant metric g on the corresponding Lie group. For global coordinates $(x_1, \ldots, x_n)$ on $\mathbb{R}H^n$, using the left-invariant vector fields (4), we can find a coordinate description of metrics defined in the previous theorem.

Theorem 2.2. *Each left-invariant metric on the group $\mathbb{R}H^n$, up to an automorphism of $\mathbb{R}H^n$, is isometric to one of the following*:

$$g_\lambda = \frac{1}{x_n^2}\left(-dx_1^2 - \ldots - dx_p^2 + dx_{p+1}^2 + \ldots + dx_{n-1}^2 + \lambda dx_n^2\right), \quad \lambda \neq 0,$$

$$g_0 = \frac{1}{x_n^2}\left(-dx_2^2 - \ldots - dx_{p+1}^2 + dx_{p+2}^2 + \ldots + dx_{n-1}^2 + 2dx_1 dx_n\right).$$

Wolf [13] showed that $\mathbb{R}H^n$ admits flat metrics and, in the four-dimensional case, according to classification of Jensen [5], we know that it also admits Einstein metrics. Later, Milnor [8] has shown that every left-invariant positive definite metric on $\mathbb{R}H^n$ has a negative sectional curvature, while Nomizu [9] proved that for every $K \in \mathbb{R}$ one can find the left-invariant Lorentz metric on $\mathbb{R}H^n$ with K as constant sectional curvature. Yet, we are able to prove more.

Theorem 2.3. *All left-invariant metrics of an arbitrary signature on $\mathbb{R}H^n$ have a constant sectional curvature.*

Proof. In order to prove the theorem we need to calculate the curvature tensor. We use the identification of the left-invariant vector fields X_k^L with their value in the unit element $X_k^L(e) = e_k$.

Recall that the curvature operators $R(e_i, e_j)$ belong to the algebra $\mathrm{so}(q)$ preserving the inner product q, i.e.,

$$\mathrm{so}(q) := \left\{A \in gl(\mathfrak{r}_n) \mid AS + SA^T = 0\right\},$$

where S denotes the matrix of q. This algebra can be identified with the space $\Lambda^2\mathfrak{r}_n$ of bivectors, whose action on $\mathfrak{r}_n$ is given by

$$(x \wedge y)z := q(y,z)x - q(x,z)y, \quad x, y, z \in \mathfrak{r}_n.$$

By using standard calculations, for the metric g_λ, we get that the connection is given by the non-zero expressions

$$\nabla_{e_i} e_i = \frac{\epsilon_i}{\lambda} e_n, \quad \epsilon_i \in \{-1, 1\}, \qquad \nabla_{e_i} e_n = -e_i, \quad i < n, \tag{5}$$

and the curvature operators are given by

$$R(e_i, e_j) = -\frac{1}{\lambda} e_i \wedge e_j.$$

From the previous is apparent that the sectional curvature is constant $K = -\dfrac{1}{\lambda}$.

For metric g_0 all components of curvature tensor R vanish, thus the metric is flat and $K = 0$.

Theorem 2.3 is proved.

3. Geodesics. Every C^1 curve $c(t)$ on the Lie group G, up to the left translations, gives rise to the curve

$$\gamma(t) = L_{c(t)*}^{-1} \dot{c}(t) \tag{6}$$

on the corresponding Lie algebra $\mathfrak{g}$. The curves of $\mathfrak{g}$ associated to geodesics are solutions of the equations

$$\dot{x} = \operatorname{ad}^*_x x, \tag{7}$$

where ad^*_x stands for the adjoint of ad_x relative to the inner product on $\mathfrak{g}$.

First, let us consider the Einstein metric g_λ on $\mathbb{R}H^n$. Fixing the basis $\{e_1, \ldots, e_n\} \in \mathfrak{r}_n$, an easy computation gives us

$$\operatorname{ad}^*_{e_k} e_k = -\frac{\epsilon_k}{\lambda} e_n, \qquad \operatorname{ad}^*_{e_n} e_k = e_k, \quad k < n.$$

In local coordinates $(x_1, \ldots, x_n)$, for $\gamma(t) = \sum_{k=1}^{n} x_k(t) e_k$, from the equation (7) we obtain the system

$$\dot{x}_k = x_k x_n, \quad k < n, \qquad \dot{x}_n = -\frac{1}{\lambda} \sum_{j=1}^{n-1} \epsilon_j x_j^2. \tag{8}$$

Let $C_1, \ldots, C_n \in \mathbb{C}$ and let us denote by $C_{n+1}^2 = -\frac{1}{\lambda} \sum_{k=1}^{n-1} \epsilon_k C_k^2$. Then the solutions $\gamma(t) = (x_1(t), \ldots, x_n(t))$ of the system (8) are given by

$$\begin{gathered} x_k(t) = \frac{C_k}{\cos(C_{n+1} t + C_n)}, \quad k < n, \\ x_n(t) = C_{n+1} \tan(C_{n+1} t + C_n), \quad \text{for} \quad C_{n+1}^2 \neq 0, 1, \end{gathered} \tag{9}$$

$$x_k(t) = \frac{C_k}{t + C_n}, \quad k < n, \qquad x_n(t) = -\frac{1}{t + C_n}, \quad \text{for} \quad C_{n+1}^2 = 1, \tag{10}$$

$$x_k(t) = C_k e^{C_n t}, \quad k < n, \qquad x_n(t) = C_n, \quad \text{for} \quad C_{n+1} = 0. \tag{11}$$

Note that constants $C_1, \ldots, C_n$ must be real in case of the solutions (10) and (11). For the solution (9) they can be either real or complex, but they need to satisfy the additional constraints which will be explained in detail in the proof of the following theorem.

Theorem 3.1. *The left-invariant metric g_λ on $\mathbb{R}H^n$ is geodesically complete if and only if it is positive definite.*

Proof. First, note that because of the left-invariance we may consider only the curves $\gamma(t)$ in the Lie algebra $\mathfrak{r}_n$ defined by (6).

A geodesic curve whose tangent vector is e_n corresponds to the solution (11) with $C_k = 0$, $k < n$, and $C_n = 1$. Those are the vertical lines ending on the hyperplane $x_n = 0$ and they are complete in every signature.

Let $v = \gamma(0) \neq e_n$ be a tangent vector of a geodesic curve. Notice that from (5) follows that the two-dimensional plane $\alpha = \mathcal{L}(v, e_n)$ is totally geodesic. Therefore, it is enough to discuss the induced signature in that plane.

It is not difficult to calculate that for the solution (9) $|v|^2 = -\lambda C_{n+1}^2$, for (10) $|v|^2 = 0$ and in the last case (11) $|v|^2 = \lambda C_n^2$. If the plane α is non-degenerate then we consider solutions (9) and (10), while the solution (11) occurs only if α is degenerate.

Case 1. Suppose that the plane α is Riemannian. Then $C_{n+1}^2 < 0$, i.e., $C_{n+1} = iD,\ D \in \mathbb{R}$. In order to determine the constant C_n we must consider the Gram determinant associated to the plane α. One can calculate that

$$G = -\lambda^2 C_{n+1}^2 (1 + \tan^2 C_n). \tag{12}$$

In Riemannian case G must be positive, which yields $C_n = iC,\ C \in \mathbb{R}$. Note that in order to obtain the real solutions, all the other constants $C_k, k < n$, must be real. The trigonometric functions in (9) become hyperbolic functions $\cosh$ and $-\tanh$, so the curves are complete. It is easy to check that the corresponding geodesic curves on the Lie group $\mathbb{R}H^n$ are half-ellipses with centers on the hyperplane $x_n = 0$. The plane α is isometric to the standard hyperbolic plane.

Case 2. Suppose that the plane α is Lorentzian.

If $|v|^2 \neq 0$, then the solution is given by (9) and we distinguish between two cases.

When the vectors v and e_n are of the same character, i.e., of the same signature, then $C_{n+1}^2 < 0$. Therefore, we must take $C_{n+1} = iD,\ D \in \mathbb{R}$. Here, the Gram determinant (12) must be negative, thus $C_n = iC + \dfrac{\pi}{2},\ C \in \mathbb{R}$. Also, all the constants $C_k,\ k < n$, must be purely imaginary. The trigonometric functions in (9) become hyperbolic functions $\sinh$ and $-\coth$.

When the vectors v and e_n are of the different character, then $C_{n+1}^2 > 0$ and we have the solution (9) where all the constants are real. In both cases the geodesics are incomplete. The corresponding geodesic curves on $\mathbb{R}H^n$ are branches of hyperbolas satisfying $x_n > 0$.

If $|v|^2 = 0$, then the corresponding curves are given by (10). In $\mathbb{R}H^n$ those are the straight lines ending on the hyperplane $x_n = 0$ and they are geodesically incomplete.

Case 3. Suppose that the plane α is degenerate. The vector $u = v - C_n e_n$ is a null vector orthogonal to all vectors from α. The corresponding solutions are complete geodesics given by (11). These are parabolas on $\mathbb{R}H^n$.

We can conclude that the metric is complete if and only if for every tangent vector v the corresponding plane α is Riemannian.

Theorem 3.1 is proved.

Similarly, for the flat metric g_0, in local coordinates, we have the system

$$\dot{x}_1 = -x_1 x_n - \sum_{j=2}^{n-1} \epsilon_j x_j^2, \qquad \dot{x}_k = x_k x_n, \quad 1 < k \leq n.$$

The solutions to the system above are given by

$$x_1(t) = C_1(t + C_n) + \frac{C_0}{2(t + C_n)},$$

$$x_k(t) = \frac{C_k}{t + C_n}, \quad 2 \leq k < n, \qquad x_n(t) = -\frac{1}{t + C_n},$$

$$x_1(t) = C_1 - tC_0, \qquad x_k(t) = C_k, \quad 2 \leq k < n, \quad x_n(t) = 0,$$

with $C_0 = \sum_{k=2}^{n-1} \epsilon_k C_k^2$ and $C_k \in \mathbb{R},\ k \leq n$.

Consequently, the following theorem holds.

Theorem 3.2. *The pseudo-Riemannian metric g_0 on the Lie group $\mathbb{R}H^n$ is geodesically incomplete.*

4. Isometric imbedding into the space forms. Denote by $\mathbb{R}^n_p$ the space $\mathbb{R}^n$ with the pseudo-Riemannian metric $g(X,Y) = -\sum_{k=1}^{p} x_k y_k + \sum_{k=p+1}^{n} x_k y_k$ for every $X, Y \in \mathbb{R}^n$.

Let $S^n_p \subseteq \mathbb{R}^{n+1}_p$ be the de Sitter space

$$S^n_p = \big\{u = (u_0, \ldots, u_n) \;\big|\; -u_0^2 - \ldots - u_{p-1}^2 + u_p^2 + \ldots + u_n^2 = -\lambda, \;\; \lambda < 0\big\}.$$

This is the hypersurface in $\mathbb{R}^{n+1}_p$ with its induced metric of signature $(p, n-p)$ of constant sectional curvature $K = -\lambda^{-1} > 0$.

Similarly, denote by $H^n_p \subseteq \mathbb{R}^{n+1}_{p+1}$ the anti-de Sitter space

$$H^n_p = \big\{u = (u_0, \ldots, u_n) \;\big|\; -u_0^2 - \ldots - u_p^2 + u_{p+1}^2 + \ldots + u_n^2 = -\lambda, \;\; \lambda > 0\big\}$$

with its induced metric of signature $(p, n-p)$ and constant sectional curvature $K = -\lambda^{-1} < 0$.

Define $\tilde{S}^n_p$ and $\tilde{H}^n_p$ to be the respective connected, simply connected manifolds corresponding to S^n_p and H^n_p.

According to Wolf [12] every complete connected pseudo-Riemannian manifold of signature $(p, n-p)$ and constant sectional curvature K has an universal pseudo-Riemannian covering $\tilde{S}^n_p$ if $K > 0$, $\tilde{H}^n_p$ if $K < 0$, and $\mathbb{R}^n_p$ if $K = 0$. Our metrics g_λ and g_0 have constant sectional curvature and, although they are not always complete, we are interested in finding a local isometry into the space forms.

Theorem 4.1. *$(\mathbb{R}H^n, g_\lambda)$ of signature $(p, n-p)$ is isometric to the part of S^n_p (if $\lambda < 0$) and H^n_p (if $\lambda > 0$) determined by the condition $u_0 + u_n > 0$.*

Proof. Suppose that $\lambda < 0$. Then the metric g_λ has the form

$$g_\lambda = \frac{1}{x_n^2}\left(-dx_1^2 - \ldots - dx_{p-1}^2 + dx_p^2 + \ldots + dx_{n-1}^2 + \lambda dx_n^2\right) =$$

$$= \frac{1}{x_n^2}\left(\sum_{k=1}^{n-1} \epsilon_k x_k^2 + \lambda x_n^2\right), \quad \lambda < 0.$$

We define an isometric imbedding $f : \mathbb{R}H^n \to S^n_p$ by

$$f(x) = f(x_1, \ldots, x_n) = (u_0, u_1, \ldots, u_n) = u,$$

where

$$u_0 = \frac{1 + \left(\sum_{k=1}^{n-1} \epsilon_k x_k^2 + \lambda x_n^2\right)}{2x_n}, \qquad u_k = \frac{x_k}{x_n}, \quad 1 \le k < n,$$

$$u_n = \frac{1 - \left(\sum_{k=1}^{n-1} \epsilon_k x_k^2 + \lambda x_n^2\right)}{2x_n}.$$

The image $f(\mathbb{R}H^n)$ is an open submanifold

$$\big\{u = (u_0, \ldots, u_n) \in S^n_p \;\big|\; u_0 + u_n > 0\big\}$$

The proof of the case when $\lambda > 0$ is similar, only replacing S^n_p with H^n_p.

Theorem 4.1 is proved.

***Remark* 4.1.** The previous theorem has been proven by Nomizu [9] in the Lorentz case. Following the reasoning from the same paper, one can show that there exists an isomorphism $h\colon \mathbb{R}H^n \to SO^+(p, n-p)$ such that the mapping f is equivariant, meaning that the following diagram commutes for every $g \in \mathbb{R}H^n$:

$$\begin{array}{ccc} \mathbb{R}H^n & \xrightarrow{L_g} & \mathbb{R}H^n \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} \\ S^n_p & \xrightarrow{h(g)} & S^n_p \end{array}$$

$\big($the same holds if we replace S^n_p with $H^n_p\big)$.

***Remark* 4.2.** Note that a geodesic curve $c(t)$ in $\mathbb{R}H^n$ is incomplete if and only if $f(c(t))$ reaches the boundary $u_0 + u_n = 0$ for a finite value of the affine parameter t. In the Figure 1 we illustrate this with the example of geodesics on $\mathbb{R}H^2$.

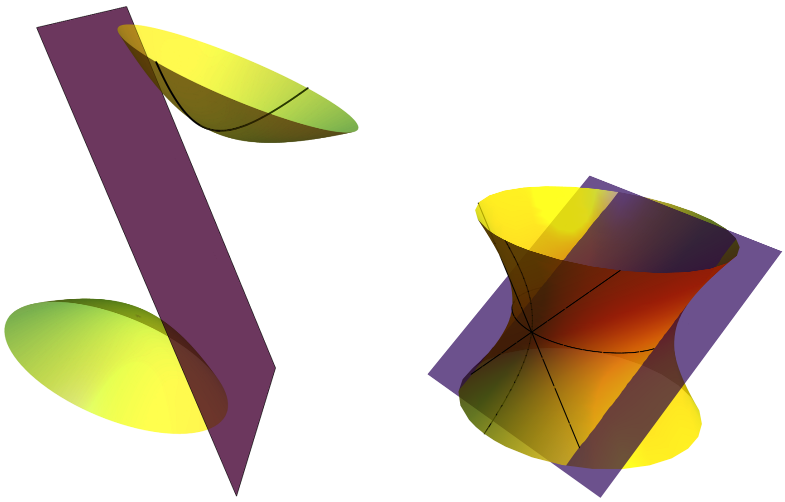

Fig. 1. Geodesics on $\mathbb{R}H^2$: Riemannian case (left), Lorentz case (right).

Theorem 4.2. *$\big(\mathbb{R}H^n, g_0\big)$ of signature $(p, n-p)$ is isometric to the part of $\mathbb{R}^n_p$ determined by the condition $y_1 + y_n > 0$.*

Proof. We can define the following change of coordinates, i.e., the map from $\mathbb{R}H^n \subset \mathbb{R}^n_p$ to $\mathbb{R}^n_p$:

$$y_1 = \frac{1 + \left(2x_1x_n + \sum_{k=2}^{n-1} \epsilon_k x_k^2\right)}{2x_n}, \qquad y_k = \frac{x_k}{x_n}, \quad 1 < k < n,$$

$$y_n = \frac{1 - \left(2x_1x_n + \sum_{k=2}^{n-1} \epsilon_k x_k^2\right)}{2x_n}.$$

In this new coordinates, the metric g_0 has the form

$$g_0' = -dy_1^2 + \sum_{k=2}^{n-1} \epsilon_k dy_k^2 + dy_n^2.$$

This is a part of an open half-space of the flat space form satisfying the relation $y_1 + y_n > 0$.

Theorem 4.2 is proved.

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