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A MATRIX APPLICATION OF POWER INCREASING SEQUENCES TO INFINITE SERIES AND FOURIER SERIES

МАТРИЧНЕ ЗАСТОСУВАННЯ ЗРОСТАЮЧИХ СТЕПЕНЕВИХ ПОСЛІДОВНОСТЕЙ ДО НЕСКІНЧЕННИХ РЯДІВ І РЯДІВ ФУР'Є

The aim of the paper is a generalization, under weaker conditions, of the main theorem on quasi-σ-power increasing sequences applied to $|A,\theta_n|_k$ summability factors of infinite series and Fourier series. We obtain some new and known results related to basic summability methods.

Метою даної роботи є узагальнення основної теореми про застосування зростаючих квазі-σ-степеневих послідовностей до коефіцієнтів підсумовування $|A,\theta_n|_k$ нескінченних рядів і рядів Фур'є при слабших умовах. Отримано деякі нові та відомі результати, що відносяться до базових методів підсумовування.

1. Introduction.

Definition 1.1. *A positive sequence* (b_n) *is said to be an almost increasing sequence if there exists a positive increasing sequence* (c_n) *and two positive constants* M *and* N *such that* $Mc_n \leq b_n \leq Nc_n$ (*see* [1]).

Definition 1.2. *A positive sequence* (X_n) *is said to be quasi-*σ*-power increasing sequence if there exists a constant* $K = K(\sigma, X) \geq 1$ *such that* $Kn^\sigma X_n \geq m^\sigma X_m$ *for all* $n \geq m \geq 1$.

Every almost increasing sequence is a quasi-σ-power increasing sequence for any nonnegative σ, but the converse is not true for $\sigma > 0$ (see [13]). For any sequence (λ_n) we write that $\Delta^2\lambda_n = \Delta\lambda_n - \Delta\lambda_{n+1}$ and $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$.

Definition 1.3. *The sequence* (λ_n) *is said to be of bounded variation, denoted by* $(\lambda_n) \in \mathcal{BV}$, *if* $\sum_{n=1}^{\infty} |\Delta\lambda_n| < \infty$.

Let $\sum a_n$ be a given infinite series with the partial sums (s_n). By u_n^α and t_n^α we denote the nth Cesàro means of order α, with $\alpha > -1$, of the sequence (s_n) and (na_n), respectively, that is (see [8])

$$u_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v \quad \text{and} \quad t_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} v a_v,$$

where

$$A_n^\alpha = \frac{(\alpha+1)(\alpha+2)\ldots(\alpha+n)}{n!} = O(n^\alpha), \qquad A_{-n}^\alpha = 0 \quad \text{for} \quad n > 0.$$

Definition 1.4. *The series* $\sum a_n$ *is said to be summable* $|C,\alpha|_k$, $k \geq 1$, *if* (*see* [10, 12])

$$\sum_{n=1}^{\infty} n^{k-1} |u_n^\alpha - u_{n-1}^\alpha|^k = \sum_{n=1}^{\infty} \frac{1}{n} |t_n^\alpha|^k < \infty.$$

If we take $\alpha = 1$, then $|C,\alpha|_k$ summability reduces to $|C,1|_k$ summability. Let (p_n) be a sequence of positive real numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad \text{as} \quad n \to \infty \quad (P_{-i} = p_{-i} = 0, \quad i \geq 1).$$

The sequence-to-sequence transformation

$$w_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v$$

defines the sequence (w_n) of the Riesz mean or simply the $(\bar{N}, p_n)$ mean of the sequence (s_n) generated by the sequence of coefficients (p_n) (see [11]).

Definition 1.5. *The series* $\sum a_n$ *is said to be summable* $|\bar{N}, p_n|_k$, $k \geq 1$, *if* (*see* [2])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{k-1} |w_n - w_{n-1}|^k < \infty.$$

In the special case when $p_n = 1$ for all values of n (resp., $k = 1$), $|\bar{N}, p_n|_k$ summability is the same as $|C,1|_k$ (resp., $|\bar{N}, p_n|$) summability.

2. Known results. The following theorem is dealing with $|\bar{N}, p_n|_k$ summability factors of infinite series under weaker conditions.

Theorem 2.1 [7]. *Let* (X_n) *be a quasi-*σ*-power increasing sequence. If the sequences* (X_n), (λ_n) *and* (p_n) *satisfy the conditions*

$$\lambda_m X_m = O(1) \quad as \quad m \to \infty, \tag{2.1}$$

$$\sum_{n=1}^{m} n X_n |\Delta^2 \lambda_n| = O(1) \quad as \quad m \to \infty, \tag{2.2}$$

$$\sum_{n=1}^{m} \frac{P_n}{n} = O(P_m), \tag{2.3}$$

$$\sum_{n=1}^{m} \frac{p_n}{P_n} \frac{|t_n|^k}{X_n^{k-1}} = O(X_m) \quad as \quad m \to \infty, \tag{2.4}$$

$$\sum_{n=1}^{m} \frac{|t_n|^k}{n X_n^{k-1}} = O(X_m) \quad as \quad m \to \infty, \tag{2.5}$$

then the series $\sum a_n \lambda_n$ *is summable* $|\bar{N}, p_n|_k$, $k \geq 1$.

3. An application of absolute matrix summability to infinite series. Let $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Then A defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $As = (A_n(s))$, where

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v, \quad n = 0, 1, \ldots .$$

Definition 3.1. *Let* (θ_n) *be any sequence of positive real numbers. The series* $\sum a_n$ *is said to be summable* $|A, \theta_n|_k,\ k \geq 1,$ *if* (*see* [14, 15])

$$\sum_{n=1}^{\infty} \theta_n^{k-1} \left|\bar{\Delta} A_n(s)\right|^k < \infty,$$

where

$$\bar{\Delta} A_n(s) = A_n(s) - A_{n-1}(s).$$

If we take $\theta_n = \dfrac{P_n}{p_n}$, then we obtain $|A, p_n|_k$ summability (see [16]), and if we take $\theta_n = n$, then we have $|A|_k$ summability (see [18]). Also, if we take $\theta_n = \dfrac{P_n}{p_n}$ and $a_{nv} = \dfrac{p_v}{P_n}$, then we have $|\bar{N}, p_n|_k$ summability. Furthermore, if we take $\theta_n = n,\ a_{nv} = \dfrac{p_v}{P_n}$ and $p_n = 1$ for all values of n, then $|A, \theta_n|_k$ summability reduces to $|C, 1|_k$ summability (see [10]). Finally, if we take $\theta_n = n$ and $a_{nv} = \dfrac{p_v}{P_n}$, then we obtain $|R, p_n|_k$ summability (see [3]).

4. Main results. The Fourier series play an important role in many areas of applied mathematics and mechanics. Recently some papers have been done concerning absolute matrix summability of infinite series and Fourier series (see [5, 6, 19–21]). The aim of this paper is to generalize Theorem 2.1 for $|A, \theta_n|_k$ summability method for these series.

Given a normal matrix $A = (a_{nv})$, we associate two lower semimatrices $\bar{A} = (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ as follows:

$$\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n, v = 0, 1, \ldots,$$

and

$$\hat{a}_{00} = \bar{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \quad n = 1, 2, \ldots.$$

It may be noted that $\bar{A}$ and $\hat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then we have

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v = \sum_{v=0}^{n} \bar{a}_{nv} a_v \tag{4.1}$$

and

$$\bar{\Delta} A_n(s) = \sum_{v=0}^{n} \hat{a}_{nv} a_v. \tag{4.2}$$

Using this notation we have the following theorem.

Theorem 4.1. *Let* $k \geq 1$ *and* $A = (a_{nv})$ *be a positive normal matrix such that*

$$\bar{a}_{n0} = 1, \quad n = 0, 1, \ldots, \tag{4.3}$$

$$a_{n-1,v} \geq a_{nv} \quad for \quad n \geq v+1, \tag{4.4}$$

$$\sum_{v=1}^{n-1} \frac{1}{v} \hat{a}_{n,v+1} = O(a_{nn}). \tag{4.5}$$

Let (X_n) be a quasi-σ-power increasing sequence and let $(\theta_n a_{nn})$ be a non increasing sequence. If the sequences (X_n), (λ_n) and (p_n) satisfy the conditions (2.1)–(2.3) *of Theorem* 2.1, *and*

$$\sum_{n=1}^{m} \theta_n^{k-1} a_{nn}^k \frac{|t_n|^k}{X_n^{k-1}} = O(X_m) \quad as \quad m \to \infty, \tag{4.6}$$

$$\sum_{n=1}^{m} (\theta_n a_{nn})^{k-1} \frac{|t_n|^k}{n X_n^{k-1}} = O(X_m) \quad as \quad m \to \infty, \tag{4.7}$$

then the series $\sum a_n \lambda_n$ is summable $|A, \theta_n|_k,\ k \geq 1$.

It may be remarked that if we take $A = (\bar{N}, p_n)$ and $\theta_n = \dfrac{P_n}{p_n}$, then the conditions (4.6), (4.7) are reduced to (2.4), (2.5). Also, the condition (4.5) satisfied by condition (2.3). Therefore, we have Theorem 2.1.

We need the following lemmas for the proof of our theorem.

Lemma 4.1 [17]. *From the conditions* (4.3) *and* (4.4) *of Theorem* 4.1, *we have*

$$\sum_{v=0}^{n-1} |\bar{\Delta} a_{nv}| \leq a_{nn},$$

$$\hat{a}_{n,v+1} \geq 0,$$

$$\sum_{n=v+1}^{m+1} \hat{a}_{n,v+1} = O(1).$$

Lemma 4.2 [4]. *Under the conditions of Theorem* 2.1 *we have that*

$$n X_n |\Delta \lambda_n| = O(1) \quad as \quad n \to \infty,$$

$$\sum_{n=1}^{\infty} X_n |\Delta \lambda_n| < \infty.$$

***Proof of Theorem* 4.1.** Let (I_n) denotes the A-transform of the series $\sum_{n=1}^{\infty} a_n \lambda_n$. Then, by (4.1) and (4.2), we have

$$\bar{\Delta} I_n = \sum_{v=1}^{n} \hat{a}_{nv} a_v \lambda_v.$$

Applying Abel's transformation to this sum, we obtain

$$\bar{\Delta} I_n = \sum_{v=1}^{n} \hat{a}_{nv} a_v \lambda_v \frac{v}{v} = \sum_{v=1}^{n-1} \Delta \left(\frac{\hat{a}_{nv} \lambda_v}{v} \right) \sum_{r=1}^{v} r a_r + \frac{\hat{a}_{nn} \lambda_n}{n} \sum_{r=1}^{n} r a_r =$$

$$=\sum_{v=1}^{n-1}\Delta\left(\frac{\hat{a}_{nv}\lambda_v}{v}\right)(v+1)t_v+\hat{a}_{nn}\lambda_n\frac{n+1}{n}t_n=$$

$$=\sum_{v=1}^{n-1}\bar{\Delta}a_{nv}\lambda_v t_v\frac{v+1}{v}+\sum_{v=1}^{n-1}\hat{a}_{n,v+1}\Delta\lambda_v t_v\frac{v+1}{v}+\sum_{v=1}^{n-1}\hat{a}_{n,v+1}\lambda_{v+1}\frac{t_v}{v}+a_{nn}\lambda_n t_n\frac{n+1}{n}=$$

$$=I_{n,1}+I_{n,2}+I_{n,3}+I_{n,4}.$$

To complete the proof of Theorem 4.1, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty}\theta_n^{k-1}|I_{n,r}|^k<\infty \quad \text{for} \quad r=1,2,3,4.$$

First, by applying Hölder's inequality with indices k and k', where $k>1$ and $\frac{1}{k}+\frac{1}{k'}=1$, we get

$$\sum_{n=2}^{m+1}\theta_n^{k-1}|I_{n,1}|^k\le\sum_{n=2}^{m+1}\theta_n^{k-1}\left\{\sum_{v=1}^{n-1}\left|\frac{v+1}{v}\right|\left|\bar{\Delta}a_{nv}\right||\lambda_v||t_v|\right\}^k=$$

$$=O(1)\sum_{n=2}^{m+1}\theta_n^{k-1}\sum_{v=1}^{n-1}\left|\bar{\Delta}a_{nv}\right||\lambda_v|^k|t_v|^k\left\{\sum_{v=1}^{n-1}\left|\bar{\Delta}a_{nv}\right|\right\}^{k-1}.$$

By using

$$\Delta\hat{a}_{nv}=\hat{a}_{nv}-\hat{a}_{n,v+1}=\bar{a}_{nv}-\bar{a}_{n-1,v}-\bar{a}_{n,v+1}+\bar{a}_{n-1,v+1}=a_{nv}-a_{n-1,v},$$

and (4.3) and (4.4), we have

$$\sum_{v=1}^{n-1}|\bar{\Delta}a_{nv}|=\sum_{v=1}^{n-1}|a_{nv}-a_{n-1,v}|=\sum_{v=1}^{n-1}(a_{n-1,v}-a_{nv})=$$

$$=\sum_{v=0}^{n-1}a_{n-1,v}-a_{n-1,0}-\sum_{v=0}^{n}a_{nv}+a_{n0}+a_{nn}=$$

$$=1-a_{n-1,0}-1+a_{n0}+a_{nn}\le a_{nn}.$$

By using $\sum_{n=v+1}^{m+1}|\bar{\Delta}a_{nv}|\le a_{vv}$, we obtain

$$\sum_{n=2}^{m+1}\theta_n^{k-1}|I_{n,1}|^k=O(1)\sum_{n=2}^{m+1}\theta_n^{k-1}a_{nn}^{k-1}\left\{\sum_{v=1}^{n-1}|\bar{\Delta}a_{nv}||\lambda_v|^k|t_v|^k\right\}=$$

$$=O(1)\sum_{v=1}^{m}|\lambda_v|^{k-1}|\lambda_v||t_v|^k\sum_{n=v+1}^{m+1}(\theta_n a_{nn})^{k-1}|\bar{\Delta}a_{nv}|=$$

$$=O(1)\sum_{v=1}^{m}(\theta_v a_{vv})^{k-1}|\lambda_v|^{k-1}|\lambda_v||t_v|^k\sum_{n=v+1}^{m+1}|\bar{\Delta}a_{nv}|=$$

$$= O(1)\sum_{v=1}^{m}(\theta_v a_{vv})^{k-1}\frac{1}{X_v^{k-1}}|\lambda_v||t_v|^k a_{vv} =$$

$$= O(1)\sum_{v=1}^{m-1}\Delta|\lambda_v|\sum_{r=1}^{v}\theta_r^{k-1}a_{rr}^k\frac{|t_r|^k}{X_r^{k-1}} + O(1)|\lambda_m|\sum_{v=1}^{m}\theta_v^{k-1}a_{vv}^k\frac{|t_v|^k}{X_v^{k-1}} =$$

$$= O(1)\sum_{v=1}^{m-1}|\Delta\lambda_v|X_v + O(1)|\lambda_m|X_m = O(1) \quad \text{as} \quad m\to\infty,$$

by virtue of the hypotheses of Theorem 4.1, Lemmas 4.1 and 4.2. Also, we have

$$\sum_{n=2}^{m+1}\theta_n^{k-1}|I_{n,2}|^k \le \sum_{n=2}^{m+1}\theta_n^{k-1}\left\{\sum_{v=1}^{n-1}|\frac{v+1}{v}||\hat{a}_{n,v+1}||\Delta\lambda_v||t_v|\right\}^k =$$

$$= O(1)\sum_{n=2}^{m+1}\theta_n^{k-1}\left\{\sum_{v=1}^{n-1}\hat{a}_{n,v+1}|\Delta\lambda_v||t_v|\frac{X_v}{X_v}\right\}^k =$$

$$= O(1)\sum_{n=2}^{m+1}\theta_n^{k-1}\left\{\sum_{v=1}^{n-1}\hat{a}_{n,v+1}|\Delta\lambda_v|X_v\frac{1}{X_v^k}|t_v|^k\right\}\left\{\sum_{v=1}^{n-1}\hat{a}_{n,v+1}|\Delta\lambda_v|X_v\right\}^{k-1} =$$

$$= O(1)\sum_{n=2}^{m+1}\theta_n^{k-1}a_{nn}^{k-1}\left\{\sum_{v=1}^{n-1}\hat{a}_{n,v+1}|\Delta\lambda_v|X_v\frac{1}{X_v^k}|t_v|^k\right\}\left\{\sum_{v=1}^{m-1}|\Delta\lambda_v|X_v\right\}^{k-1} =$$

$$= O(1)\sum_{v=1}^{m}v|\Delta\lambda_v|\frac{1}{X_v^{k-1}}\frac{1}{v}|t_v|^k\sum_{n=v+1}^{m+1}(\theta_n a_{nn})^{k-1}\hat{a}_{n,v+1} =$$

$$= O(1)\sum_{v=1}^{m}(\theta_v a_{vv})^{k-1}v|\Delta\lambda_v|\frac{1}{X_v^{k-1}}\frac{1}{v}|t_v|^k\sum_{n=v+1}^{m+1}\hat{a}_{n,v+1} =$$

$$= O(1)\sum_{v=1}^{m}v(\theta_v a_{vv})^{k-1}|\Delta\lambda_v|\frac{1}{vX_v^{k-1}}|t_v|^k =$$

$$= O(1)\sum_{v=1}^{m-1}\Delta(v|\Delta\lambda_v|)\sum_{r=1}^{v}(\theta_r a_{rr})^{k-1}\frac{|t_r|^k}{rX_r^{k-1}} + O(1)m|\Delta\lambda_m|\sum_{r=1}^{m}(\theta_r a_{rr})^{k-1}\frac{|t_r|^k}{rX_r^{k-1}} =$$

$$= O(1)\sum_{v=1}^{m-1}|\Delta(v|\Delta\lambda_v|)|X_v + O(1)m|\Delta\lambda_m|X_m =$$

$$= O(1)\sum_{v=1}^{m-1}vX_v|\Delta^2\lambda_v| + O(1)\sum_{v=1}^{m-1}X_v|\Delta\lambda_v| + O(1)m|\Delta\lambda_m|X_m =$$

$$= O(1) \quad \text{as} \quad m\to\infty,$$

by virtue of the hypotheses of Theorem 4.1, Lemmas 4.1 and 4.2.

Furthermore, as in $I_{n,1}$, we get

$$\sum_{n=2}^{m+1} \theta_n^{k-1} |I_{n,3}|^k \leq \sum_{n=2}^{m+1} \theta_n^{k-1} \left\{ \sum_{v=1}^{n-1} |\hat{a}_{n,v+1}||\lambda_{v+1}| \frac{|t_v|}{v} \right\}^k =$$

$$= O(1) \sum_{n=2}^{m+1} \theta_n^{k-1} \left\{ \sum_{v=1}^{n-1} |\hat{a}_{n,v+1}||\lambda_{v+1}|^k \frac{|t_v|^k}{v} \right\} \left\{ \sum_{v=1}^{n-1} \frac{1}{v} \hat{a}_{n,v+1} \right\}^{k-1} =$$

$$= O(1) \sum_{n=2}^{m+1} \theta_n^{k-1} a_{nn}^{k-1} \sum_{v=1}^{n-1} |\lambda_{v+1}||\lambda_{v+1}|^{k-1} \frac{|t_v|^k}{v} \hat{a}_{n,v+1} =$$

$$= O(1) \sum_{v=1}^{m} \frac{|t_v|^k}{v} \frac{1}{X_v^{k-1}} |\lambda_{v+1}| \sum_{n=v+1}^{m+1} (\theta_n a_{nn})^{k-1} \hat{a}_{n,v+1} =$$

$$= O(1) \sum_{v=1}^{m} (\theta_v a_{vv})^{k-1} \frac{|t_v|^k}{v} \frac{1}{X_v^{k-1}} |\lambda_{v+1}| \sum_{n=v+1}^{m+1} \hat{a}_{n,v+1} =$$

$$= O(1) \sum_{v=1}^{m} (\theta_v a_{vv})^{k-1} \frac{1}{X_v^{k-1}} |\lambda_{v+1}| \frac{|t_v|^k}{v} =$$

$$= O(1) \quad \text{as} \quad m \to \infty,$$

by virtue of the hypotheses of Theorem 4.1, Lemmas 4.1 and 4.2.

Again, as in $I_{n,1}$, we obtain

$$\sum_{n=1}^{m} \theta_n^{k-1} |I_{n,4}|^k = O(1) \sum_{n=1}^{m} \theta_n^{k-1} a_{nn}^k |\lambda_n|^k |t_n|^k = O(1) \sum_{n=1}^{m} \theta_n^{k-1} a_{nn}^k |\lambda_n|^{k-1} |\lambda_n||t_n|^k =$$

$$= O(1) \sum_{n=1}^{m} \theta_n^{k-1} a_{nn}^k \frac{1}{X_n^{k-1}} |\lambda_n||t_n|^k = O(1) \quad \text{as} \quad m \to \infty,$$

by virtue of hypotheses of the Theorem 4.1, Lemmas 4.1 and 4.2.

Theorem 4.1 is proved.

5. An application of absolute matrix summability to Fourier series. Let f be a periodic function with period 2π and integrable (L) over $(-\pi, \pi)$. Without any loss of generality the constant term in the Fourier series of f can be taken to be zero, so that

$$f(x) \sim \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \sum_{n=1}^{\infty} C_n(x),$$

where

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx, \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx.$$

We write

$$\phi(t) = \frac{1}{2}\left\{f(x+t) + f(x-t)\right\},$$

$$\phi_\alpha(t) = \frac{\alpha}{t^\alpha}\int\limits_0^t (t-u)^{\alpha-1}\phi(u)\,du, \quad \alpha > 0.$$

It is well-known that if $\phi_1(t) \in \mathcal{BV}(0,\pi)$, then $t_n(x) = O(1)$, where $t_n(x)$ is the $(C,1)$ mean of the sequence $(nC_n(x))$ (see [9]).

Using this fact, Bor has obtained the following main result dealing with the trigonometric Fourier series.

Theorem 5.1 [7]. *Let (X_n) be a quasi-σ-power increasing sequence. If $\phi_1(t) \in \mathcal{BV}(0,\pi)$ and the sequences (p_n), (λ_n), and (X_n) satisfy the conditions of Theorem* 2.1, *then the series $\sum C_n(x)\lambda_n$ is summable $|\bar{N}, p_n|_k$, $k \geq 1$.*

By using the above theorem, we have obtained the following result for $|A,\theta_n|_k$ summability.

Theorem 5.2. *Let A be a positive normal matrix satisfying the conditions of Theorem* 4.1. *Let (X_n) be a quasi-σ-power increasing sequence. If $\phi_1(t) \in \mathcal{BV}(0,\pi)$ and the sequences (p_n), (λ_n), and (X_n) satisfy the conditions of Theorem* 4.1, *then the series $\sum C_n(x)\lambda_n$ is summable $|A,\theta_n|_k$, $k \geq 1$.*

6. Applications. We can apply Theorems 4.1 and 5.2 to the weighted mean in which $A = (a_{nv})$ is defined as $a_{nv} = \dfrac{p_v}{P_n}$ when $0 \leq v \leq n$, where $P_n = p_0 + p_1 + \ldots + p_n$. We have

$$\bar{a}_{nv} = \frac{P_n - P_{v-1}}{P_n} \quad \text{and} \quad \hat{a}_{n,v+1} = \frac{p_n P_v}{P_n P_{n-1}}.$$

So, the following results can be easily verified.

7. Conclusions.

1. If we take $\theta_n = \dfrac{P_n}{p_n}$ in Theorems 4.1 and 5.2, then we have a result dealing with $|A,p_n|_k$ summability.

2. If we take $\theta_n = n$ in Theorems 4.1 and 5.2, then we have a result dealing with $|A|_k$ summability.

3. If we take $\theta_n = \dfrac{P_n}{p_n}$ and $a_{nv} = \dfrac{p_v}{P_n}$ in Theorems 4.1 and 5.2, then we have Theorems 2.1 and 5.1, respectively.

4. If we take $\theta_n = n$, $a_{nv} = \dfrac{p_v}{P_n}$ and $p_n = 1$ for all values of n in Theorems 4.1 and 5.2, then we have a new result concerning $|C,1|_k$ summability.

5. If we take $\theta_n = n$ and $a_{nv} = \dfrac{p_v}{P_n}$ in Theorems 4.1 and 5.2, then we have $|R,p_n|_k$ summability.

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