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QUATERNIONIC FRACTIONAL FOURIER TRANSFORM FOR BOEHMIANS
КВАТЕРНІОННЕ ДРОБОВЕ ПЕРЕТВОРЕННЯ ФУР'Є ДЛЯ БЬОМІАНІВ

We construct a Boehmian space of quaternion valued functions using the quaternionic fractional convolution. Applying the convolution theorem, the quaternionic fractional Fourier transform is extended to the context of Boehmians and its properties are established.

За допомогою кватерніонної дробової згортки побудовано бьоміанів простір функцій із значеннями у кватерніонах. Застосовуючи теорему про згортку, ми поширюємо кватерніонне дробове перетворення Фур'є на бьоміанів простір та встановлюємо його властивості.

1. Introduction. It is well-known that the classical Fourier transform on square integrable functions is of order 4. To introduce a generalization of Fourier transform with fractional order, the fractional Fourier transform was introduced by Namias [26] and the works on fractional Fourier transform have been developed with different objectives on pure and applied mathematics. In particular, in view of classical analysis, properties, applications and generalizations of the fractional Fourier transform are discussed (see [7, 17, 20 – 22, 25, 26, 29, 30, 35, 39, 40]). Following the introduction of the Fourier transform of quaternion valued functions.

The fractional Fourier transform is discussed on quaternion valued functions on $\mathbb{R}^2$ and their properties including inversion formula and Parseval's identity are derived in [13, 38]. Recently the fractional Fourier transform on quaternion valued functions on $\mathbb{R}$ is introduced in [32] and all of its properties including the inversion formula, Parsevel's identity, convolution and product theorems are proved. As the convolution theorem for quaternionic fractional Fourier transform in [32] is quite analogous to that of fractional Fourier transform of complex valued functions in [39], in this paper, following the techniques employed in [40], we extend the quaternionic fractional Fourier transform to a suitable Boehmian space of quaternion valued functions. It is obvious to observe that the fractional Fourier transform on Boehmians of quaternion valued functions is simultaneously generalizing the theory of fractional Fourier transform on quaternion valued L^2-functions [32] and the fractional Fourier transform on Boehmians of complex valued functions [40].

To facilitate the reader, we recall the division algebra of quaternions, L^p-spaces of quaternion valued functions, the theory of fractional Fourier transform in Section 2. The general theory of Boehmians and the construction of two suitable Boehmian spaces are discussed in Section 3. The last section is devoted to the definition and properties of the extended quaternionic fractional Fourier transform.

2. Preliminaries. As usual, we denote by $\mathbb{R}$ and $\mathbb{C}$, the sets of all real and complex numbers, respectively. The set of all quaternions is defined by is defined by $\mathbb{H} = \big\{q_1 + jq_2 : q_1, q_2 \in \mathbb{C}\big\}$, where the j is an imaginary number other than the imaginary complex number i satisfying the properties $j^2 = -1$ and $jz = \overline{z}j$ for all $z \in \mathbb{C}$, where $\overline{z}$ is the complex conjugate of z. The addition and the multiplication of two quaternions are explicitly given by

$$(p_1 + jp_2) + (q_1 + jq_2) = (p_1 + q_1) + j(p_2 + q_2),$$

$$(p_1 + jp_2)(q_1 + jq_2) = (p_1 q_1 - \bar{p}_2 q_2) + j(\bar{p}_1 q_2 + p_2 q_1).$$

It is well-known that $\mathbb{H}$ is a skew-field but not a field with respect to the addition and multiplication defined above. The conjugate and absolute value of a quaternion $q = q_1 + jq_2 \in \mathbb{H}$ are given by $q^c = \bar{q}_1 - jq_2$ and $|q| = \sqrt{|q_1|^2 + |q_2|^2}$, respectively, where $|q_k|$ is the absolute value of the complex number q_k. The conjugate operator and absolute value operator on quaternions satisfy the following properties:

$$(p+q)^c = p^c + q^c, \qquad (pq)^c = q^c p^c, \qquad (q^c)^c = q \qquad \forall p,\, q \in \mathbb{H},$$

and

$$qq^c = |q|^2, \qquad |q| = |q^c|, \qquad |p+q| \le |p| + |q|, \qquad |pq| = |p|\,|q| \quad \forall p,\, q \in \mathbb{H}.$$

If $f = f_1 + jf_2$ is a quaternion valued function, we define

$$\check{f} = \check{f}_1 + j\check{f}_2 \quad \text{and} \quad \tilde{f}(x) = \bar{f}_1 - j\check{f}_2(x),$$

where $\bar{f}_1(x)$ is the complex conjugate of $f_1(x)$ and $\check{f}_1(x) = f_1(-x)$ for all $x \in \mathbb{R}$. Let $L^p(\mathbb{R},\mathbb{H}) = \{f_1 + jf_2 \colon f_1, f_2 \in L^p(\mathbb{R},\mathbb{C})\}$, where $L^p(\mathbb{R},\mathbb{C})$ is the Banach space of all complex valued functions φ, satisfying $\int_{\mathbb{R}} |\varphi(x)|^p\, dx < +\infty$, $p = 1, 2$. As in the case of complex valued p-integrable functions, the point-wise addition and point-wise scalar multiplication on $L^p(\mathbb{R},\mathbb{H})$ are defined by

$$(f+g)(x) = f(x) + g(x) \quad \text{and} \quad (qf)(x) = qf(x) \quad \forall x \in \mathbb{R}.$$

As $\mathbb{H}$ is not a field, we can say that $L^p(\mathbb{R},\mathbb{H})$ is a left $\mathbb{H}$-module, and it is equipped with the *norm* defined as follows:

$$\|f\|_p = \|f_1 + jf_2\|_p = \left(\int_{\mathbb{R}} |f_1(x)|^p + |f_2(x)|^p dx \right)^{\frac{1}{p}}, \quad p = 1, 2.$$

It is obvious that

$$f_n \to 0 \quad \text{in} \quad L^p(\mathbb{R},\mathbb{H}) \quad \text{iff} \quad f_{n,1} \to 0, \qquad f_{n,2} \to 0 \quad \text{in} \quad L^p(\mathbb{R},\mathbb{C}), \tag{1}$$

where $f_n = f_{n,1} + jf_{n,2}$ for all $n \in \mathbb{N}$, and, hence, $L^p(\mathbb{R},\mathbb{H})$ is also complete with respect to the metric induced by the norm. Furthermore, the norm on $L^2(\mathbb{R},\mathbb{H})$ is also obtained as $\|f\|_2 = \sqrt{\langle f, f\rangle}$, where

$$\langle f, g\rangle = \int_{\mathbb{R}} f(x)(g(x))^c dx \quad \forall f,\, g \in L^2(\mathbb{R},\mathbb{H}).$$

Therefore, we say that $L^2(\mathbb{R},\mathbb{H})$ is an example of a left $\mathbb{H}$-Hilbert space, as per the following definition.

Definition 1. *A nonempty set $\mathcal{H}$ is called a quaternion left Hilbert space if it is a left $\mathbb{H}$-module and there exists a function $\langle\cdot,\cdot\rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{H}$ with the following properties*:

(1) $\langle u, v\rangle = \big(\langle v, u\rangle\big)^c$ $\forall u, v \in \mathcal{H}$;

(2) $\langle pu + qv, w\rangle = p\langle u, w\rangle + q\langle v, w\rangle$ $\forall p, q \in \mathbb{H}$, $\forall u, v, w \in \mathcal{H}$;

(3) *for each* $u \in \mathcal{H}$, $\langle u, u \rangle$ *is real and nonnegative*;

(4) *for* $u \in \mathcal{H}$, $\langle u, u \rangle = 0$ *iff* $u = 0$;

(5) *every Cauchy sequence in the normed space* $\big(\mathcal{H}, \|\cdot\|_2\big)$ *converges in* $\mathcal{H}$, *where* $\|u\|_2 = \sqrt{\langle u, u \rangle}$ *for all* $u \in \mathcal{H}$.

Upto the author's knowledge, the first paper in English, mentioning the definition of quaternionic Hilbert space is [37].

In 1980's the qunaternionic Fourier transform was introduced and applied independently by Sommen, Earnest et al., and Delsuc. Then another modified version of the quaternioninc Fourier transform was introduced by T. Ell in 1992, which is commonly used by many researchers in this field of research. For more details on the history of quaternionic Fourier transform, refer the reader to [9]. After this various research papers on quternionic Fourier transforms were published. To mention a few works, we refer to [10, 15–17].

Next, we recall the fractional Fourier transform on complex valued functions and quaternionic fractional Fourier transform, respectively, from [6, 26, 32]. The fractional Fourier transform $\mathcal{F}_\alpha(f)$ of a suitable complex valued function f on $\mathbb{R}$ is defined by

$$\mathcal{F}_\alpha(f)(u) = \int\limits_{\mathbb{R}} f(t) K_\alpha(t,u)\, dt \quad \forall u \in \mathbb{R},$$

where

$$K_\alpha(t,u) = \begin{cases} \sqrt{\dfrac{c(\alpha)}{2\pi}} e^{ia(\alpha)(t^2+u^2) - iutb(\alpha)}, & \alpha \notin \pi\mathbb{Z}, \\ \delta(t-u), & \alpha \in 2\pi\mathbb{Z}, \\ \delta(t+u), & \alpha + \pi \in 2\pi\mathbb{Z}, \end{cases}$$

$a(\alpha) = \dfrac{\cot\alpha}{2}$, $b(\alpha) = \sec\alpha$ and $c(\alpha) = \sqrt{1 - i\cot\alpha}$ and $\alpha \in \mathbb{R} \setminus \pi\mathbb{Z}$. Following the definition of quaternionic Fourier transform in [14], the fractional Fourier transform of quaternion valued function $f \in L^1(\mathbb{R}, \mathbb{H})$ is defined in [32] by

$$\mathcal{F}_\alpha(f) = \mathcal{F}_\alpha(f_1) + j\mathcal{F}_\alpha(f_2), \quad \text{where} \quad f = f_1 + jf_2. \tag{2}$$

Then $\mathcal{F}_\alpha$ is extended to $L^2(\mathbb{R}, \mathbb{H})$ as a Hilbert space isomorphism, as in the case of classical Fourier transform. Further, we also have $\mathcal{F}_\alpha \circ \mathcal{F}_\beta = \mathcal{F}_{\alpha+\beta}$ on $L^2(\mathbb{R}, \mathbb{H})$ and $\mathcal{F}_\alpha$ is the identity operator if $\alpha = 0$. Thus, $\mathcal{F}_\alpha^{-1} = \mathcal{F}_{-\alpha}$.

Definition 2. *For* $f \in L^2(\mathbb{R}, \mathbb{H})$ *and* $g \in L^1(\mathbb{R}, \mathbb{H})$, *define*

$$(f \star_\alpha g)(x) = \big(f_1 \star g_1 - \mathcal{F}_{-2\alpha}\overline{f_2} \star g_2\big) + j\big(\mathcal{F}_{-2\alpha}\overline{f_1} \star g_2 + f_2 \star g_1\big),$$

where $\star$ *is the convolution defined in* [40] (Definition 1) *as follows*:

$$(f \star g)(x) = \frac{c(\alpha)}{\sqrt{2\pi}} e^{ia(\alpha)x^2} (\tilde{f} * \tilde{g})(x) \quad \textit{and} \quad \tilde{f}(x) = e^{ia(\alpha)x^2} f(x). \tag{3}$$

Theorem 1 (convolution theorem, [32])**.** *For* $f \in L^2(\mathbb{R}, \mathbb{H})$ *and* $g \in L^1(\mathbb{R}, \mathbb{H})$, *we have* $\mathcal{F}_\alpha(f \star_\alpha g)(u) = \mathcal{F}_\alpha(f)(u)\mathcal{F}_\alpha(g)(u)e^{-ia(\alpha)u^2}$ *for all* $u \in \mathbb{R}$.

3. Quaternionic fractional fourier transform on Boehmians. A class of generalized functions, called Boehmian space with two convergences, was introduced by P. Mikusiński [23]. Since most of the Boehmian spaces of functions are larger than dual spaces of suitable function spaces, a number of papers on extensions of integral transforms in the context of Boehmian spaces have been published. To mention a few recent papers, we refer to [8, 11, 12, 19, 27, 28, 31, 33, 34, 36]. A complete list of papers on Boehmians is available in the following link http://mikusinski.cos.ucf.edu/boehmians.pdf.

In particular, Boehmians of quaternion valued functions were constructed, for the purpose of extending quaternionic integral transforms, for example, quaternionic wavelet transform [1], quaternionic ridgelet transform [2], quaternionic Gabor transform [3], quaternionic Stockwell transform [4] and quaternionic curvelet transform [5]. In this line, we provide an extension of the quaternionic fractional Fourier transform to a Boehmian space of quaternion valued functions.

Let us recall the abstract construction of Boehmian space from the literature [23, 24]. Let G be a complex topological vector space, $(S,\cdot)$ be a commutative semigroup, $\circ\colon G\times S\to G$ be satisfying the following conditions:

1) $(a+b)\circ s = a\circ s + b\circ s\ \forall a,b\in G,\ \forall s\in S$;
2) $(\kappa a)\circ s = \kappa(a\circ s)\ \forall\kappa\in\mathbb{C}, \forall a\in G,\ \forall s\in S$;
3) $a\circ(s\cdot t) = (a\circ s)\circ t\ \forall a\in G,\ \forall s,t\in S$;
4) if $a_n\to a$ as $n\to\infty$ in G and $s\in S$, then $a_n\circ s\to a\circ s$ as $n\to\infty$,

and Δ be a collection of sequences from S with the following properties:

1) if $(s_n),(t_n)\in\Delta$, then $(s_n\cdot t_n)\in\Delta$;
2) if $a_n\to a$ in G as $n\to\infty$ and $(s_n)\in\Delta$, then $a_n\circ s_n\to a$ as $n\to\infty$ in G.

A pair of sequences $\big((a_n),(s_n)\big)$ with $a_n\in G$ for all $n\in\mathbb{N}$ and $(s_n)\in\Delta$ is called a quotient if $a_n\circ s_m = a_m\circ s_n$ for all $m,n\in\mathbb{N}$, and it is denoted by $\dfrac{(a_n)}{(s_n)}$. An equivalence relation $\sim$ is defined on the collection of all quotients by

$$\frac{(a_n)}{(s_n)}\sim\frac{(b_n)}{(t_n)}\quad\text{if}\quad a_n\circ t_m = b_m\circ s_n\quad \forall m,n\in\mathbb{N}.$$

Every equivalence class induced by $\sim$ is called a Boehmian and the collection of all Boehmians $\mathcal{B}=\mathcal{B}(G,(S,\cdot),\circ,\Delta)$ is a vector space over $\mathbb{C}$ with respect to the addition and scalar multiplication defined as follows:

$$\left[\frac{(a_n)}{(s_n)}\right]+\left[\frac{(b_n)}{(t_n)}\right]=\left[\frac{(a_n\circ t_n+b_n\circ s_n)}{(s_n\cdot t_n)}\right],\qquad c\left[\frac{(a_n)}{(s_n)}\right]=\left[\frac{(sa_n)}{(s_n)}\right].$$

Every element $a\in G$ can be uniquely identified as a member of $\mathcal{B}$ by $\left[\dfrac{(a\circ t_n)}{(t_n)}\right]$, where $(t_n)\in\Delta$ is arbitrary and the operation $\circ$ is also extended as $\left[\dfrac{(a_n)}{(s_n)}\right]\circ v=\left[\dfrac{(a_n\circ v)}{(s_n)}\right]$ and $\left[\dfrac{(a_n)}{(s_n)}\right]\circ\left[\dfrac{(b_n)}{(t_n)}\right]=$ $=\left[\dfrac{(a_n\circ t_n+b_n\circ s_n)}{(s_n\cdot t_n)}\right]$, whenever $v,b_n\in S$ for all $n\in N$.

Definition 3 (δ-convergence)**.** *A sequence (B_n) of Boehmians is said to δ-converge to B in $\mathcal{B}$ if there exist $a_{n,k},a_k\in G,\ n,k\in\mathbb{N}$ and $(s_k)\in\Delta$ such that $B_n=\left[\dfrac{(a_{n,k})}{(s_k)}\right]$, $B=\left[\dfrac{(a_k)}{(s_k)}\right]$ and for each $k\in\mathbb{N}, a_{n,k}\to a_k$ as $n\to\infty$ in G. In this case, we write $B_n\overset{\delta}{\to}B$ in $\mathcal{B}$ as $n\to\infty$.*

Definition 4 (Δ-convergence)**.** *A sequence (B_n) of Boehmians is said to Δ-converge to B in $\mathcal{B}$ if there exist $a_n \in G$ for all $n \in \mathbb{N}$ and $(s_n) \in \Delta$ such that $(B_n - B) \circ s_n = \left[\dfrac{(a_n \circ s_k)}{(s_k)}\right]$ and $a_n \to 0$ as $n \to \infty$ in G. In this case, we write $B_n \overset{\Delta}{\to} B$ in $\mathcal{B}$ as $n \to \infty$.*

It should be noted that although G is mentioned as a complex vector space in the abstract construction of the Boehmian space $\mathcal{B}(G, (S, \cdot), \circ, \Delta)$, this constraint could be relaxed by taking G as a left-module over quaternions.

In a problem of extending an integral transform to the context of Boehmians, the crux is proving the convolution theorem for the integral transform, and constructing the suitable Boehmian space(s) by proving the auxiliary results of their constructions. If this part is done properly, the definition of the extended integral transform and its properties will be obtained by straightforward arguments. In this paper, we shall first construct the Boehmian space

$$\mathcal{B}^2_{\star_\alpha} = \mathcal{B}\big(L^2(\mathbb{R}, \mathbb{H}), (L^1(\mathbb{R}, \mathbb{C}), \star)\star_\alpha, \Delta_\alpha\big),$$

where Δ_α is the collection of all sequences (δ_n) from $L^1(\mathbb{R}, \mathbb{C})$, called delta sequences, satisfying the following conditions:

1) $\displaystyle\int_{\mathbb{R}} \delta_n(t) e^{-ia(\alpha)t^2}\, dt = 1 \;\; \forall n \in \mathbb{N};$

2) $\displaystyle\int_{\mathbb{R}} |\delta_n(t)| dt \le M \;\; \forall n \in \mathbb{N}$, for some $M > 0$;

3) support of $\delta_n(t) \to \{0\}$ as $n \to \infty$; that is, given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that support of $\delta_n \subseteq (-\varepsilon, \varepsilon)$ $\forall n \ge N$,

and $\star_\alpha$, $\star$ are as defined in Definition 2. We observe that $\star_\alpha$ is a binary operation on $L^1(\mathbb{R}, \mathbb{H})$, and it is not commutative, as the multiplication on $\mathbb{H}$-is not commutative. So, we choose $(L^1(\mathbb{R}, \mathbb{C}), \star)$ as the commutative semigroup, from which the delta sequences are to be taken. It is interesting to note that $\star_\alpha$ coincides with $\star$, whenever f and g are complex valued functions such that $f \star g$ exists. Now we prove the auxiliary results required to construct the Boehmian space $\mathcal{B}^2_{\star_\alpha}$.

Lemma 1. *If $f \in L^2(\mathbb{R}, \mathbb{H})$ and $g \in L^1(\mathbb{R}, \mathbb{H})$, then $f *_\alpha g \in L^2(\mathbb{R}, \mathbb{H})$.*

Proof. If $f = f_1 + jf_2$, $g = g_1 + jg_2$, then $f_1, f_2 \in L^2(\mathbb{R}, \mathbb{C})$, $g_1, g_2 \in L^1(\mathbb{R}, \mathbb{C})$ and, hence, $\mathcal{F}_{-2\alpha}\overline{f_1}, \mathcal{F}_{-2\alpha}\overline{f_2} \in L^2(\mathbb{R}, \mathbb{C})$. Therefore, by using the fact that

$$\|\mu \star \lambda\|_2 = \|\mu * \lambda\|_2 \le \|\mu\|_2 \|\lambda\|_1, \quad \text{whenever} \quad \mu \in L^2(\mathbb{R}, \mathbb{C}) \quad \text{and} \quad \lambda \in L^1(\mathbb{R}, \mathbb{C}),$$

we get $f_1 \star g_1, \mathcal{F}_{-2\alpha}\overline{f_2} \star g_2, f_2 \star g_1, \mathcal{F}_{-2\alpha}\overline{f_1} \star g_1 \in L^2(\mathbb{R}, \mathbb{C})$. Thus, we have $f *_\alpha g \in L^2(\mathbb{R}, \mathbb{H})$.

Lemma 2. *If $f, g \in L^2(\mathbb{R}, \mathbb{H})$, $h \in L^1(\mathbb{R}, \mathbb{H})$, and $q \in \mathbb{H}$, then $(f + g) \star_\alpha h = f \star_\alpha h + g \star_\alpha h$ and $(qf) \star_\alpha h = q(f \star_\alpha h)$.*

Proof. We observe that using $(\mu + \nu) * \lambda = \mu * \lambda + \nu * \lambda$, we can prove that

$$(\mu + \nu) \star \lambda = \mu \star \lambda + \nu \star \lambda \quad \forall \mu, \nu \in L^2(\mathbb{R}, \mathbb{C}),\ \lambda \in L^1(\mathbb{R}, \mathbb{C}).$$

Therefore, for $f = f_1 + jf_2$, $g = g_1 + jg_2$, $h = h_1 + jh_2$, then

$$(f + g) \star_\alpha h = [f_1 + g_1] \star h_1 - \mathcal{F}_{-2\alpha}\overline{[f_2 + g_2]} \star h_2 +$$

$$+ j(\mathcal{F}_{-2\alpha}\overline{[f_1 + g_1]} \star h_2 + [f_2 + g_2] \star h_1) =$$

$$= [f_1 + g_1] \star h_1 - [\mathcal{F}_{-2\alpha}\overline{f_2} + \mathcal{F}_{-2\alpha}\overline{g_2}] \star h_2 +$$

$$+ j\big([\mathcal{F}_{-2\alpha}\overline{f_1} + \mathcal{F}_{-2\alpha}\overline{g_1}] \star h_2 + [f_2 + g_2] \star h_1\big) =$$

(since $\mathcal{F}_{-2\alpha}$ and complex conjugation are linear)

$$= f_1 \star h_1 + g_1 \star h_1 - \mathcal{F}_{-2\alpha}\overline{f_2} \star h_2 - \mathcal{F}_{-2\alpha}\overline{g_2} \star h_2 +$$

$$+ j(\mathcal{F}_{-2\alpha}\overline{f_1} \star h_1 + \mathcal{F}_{-2\alpha}\overline{g_1} \star h_2 + f_2 \star h_1 + g_2 \star h_1) =$$

$$= \big[f_1 \star h_1 - \mathcal{F}_{-2\alpha}\overline{f_2} \star h_2 + j(\mathcal{F}_{-2\alpha}\overline{f_1} \star h_2 + f_2 \star h_1)\big] +$$

$$+ \big[g_1 \star h_1 - \mathcal{F}_{-2\alpha}\overline{g_2} \star h_2 + j(\mathcal{F}_{-2\alpha}\overline{g_1} \star h_2 + g_2 \star h_1)\big] =$$

$$= f \star_\alpha h + g \star_\alpha h.$$

If $\tau \in \mathbb{C}$, then using $(\tau\mu) * \nu = \tau(\mu * \nu)$, we can prove that

$$(\tau\mu) \star_\alpha \nu = \tau(\mu \star_\alpha \nu). \tag{4}$$

Next, by a direct computation, we have

$$j(f \star_\alpha h) = j[f_1 \star h_1 - \mathcal{F}_{-2\alpha}\overline{f_2} \star h_2] - (\mathcal{F}_{-2\alpha}\overline{f_1} \star h_2 + f_2 \star h_1) =$$

$$= j[f_1 \star h_1] - j[\mathcal{F}_{-2\alpha}\overline{f_2} \star h_2] - \mathcal{F}_{-2\alpha}\overline{f_1} \star h_2 - f_2 \star h_1$$

and

$$(jf) \star_\alpha h = (-f_2 + jf_1) \star_\alpha (h_1 + jh_2) =$$

$$= -f_2 \star h_1 - \mathcal{F}_{-2\alpha}\overline{f_2} \star h_2 - \mathcal{F}_{-2\alpha}\overline{f_1} \star h_2 + j(f_1 \star h_1),$$

which implies that

$$j(f \star_\alpha h) = (jf) \star h. \tag{5}$$

Finally, for $q = q_1 + jq_2 \in \mathbb{H}$,

$$(qf) \star_\alpha h = (q_1 f + jq_2 f) \star_\alpha h =$$

(since multiplication is distributive over addition in $\mathbb{H}$)

$$= [(q_1 f) \star_\alpha h] + [(jq_2 f) \star_\alpha h] \quad \text{(by the first assertion of this lemma)} =$$

$$= q_1(f \star_\alpha h) + jq_2(f \star_\alpha h) \quad \text{(by (4) and (5))} =$$

$$= (q_1 + jq_2)(f \star_\alpha h) = q(f \star_\alpha h).$$

Lemma 2 is proved.

Lemma 3. *If* $f_n \to f$ *in* $L^2(\mathbb{R},\mathbb{H})$ *as* $n\to\infty$ *and* $h\in L^1(\mathbb{R},\mathbb{H})$, *then* $f_n \star_\alpha h \to f\star_\alpha h$ *in* $L^2(\mathbb{R},\mathbb{H})$ *as* $n\to\infty$.

Proof. First, we observe that

$$\|\mu\star\nu\|_2 = \|\mu * \nu\|_2 \le \|\mu\|_2\|\nu\|_1 \quad \forall\mu\in L^2(\mathbb{R},\mathbb{C}), \quad \nu\in L^1(\mathbb{R},\mathbb{C}),$$

which implies that

$$\mu_n\star\nu \to \mu\star\nu \text{ in } L^2(\mathbb{R},\mathbb{C}) \quad \text{whenever} \quad \mu_n\to\mu \quad \text{in} \quad L^2(\mathbb{R},\mathbb{C}) \quad \text{as} \quad n\to\infty. \tag{6}$$

If $f_n = f_{n,1}+jf_{n,2}$ for all $\forall n\in\mathbb{N}$ and $f=f_1+jf_2$, then by (1), we have $f_{n,r}-f_r\to 0$ as $n\to\infty$ for $r\in\{1,2\}$. Therefore, by (6), we get that $(f_{n,r}-f_r)\star h_s \to 0$ as $n\to\infty$ for $r,s\in\{1,2\}$. By using (6) and continuity of $\mathcal{F}_{-2\alpha}$, we obtain that

$$(f_{n,1}-f_1)\star h_1\to 0 \quad \text{as} \quad n\to\infty,$$

$$\mathcal{F}_{-2\alpha}(\overline{f_{n,2}-f_2})\star h_2\to 0 \quad \text{as} \quad n\to\infty,$$

$$(\mathcal{F}_{-2\alpha}(\overline{f_{n,1}-f_1})\star h_2\to 0 \quad \text{as} \quad n\to\infty,$$

$$(f_{n,2}-f_2)\star h_1\to 0 \quad \text{as} \quad n\to\infty,$$

and, hence, again using (1), we have

$$\left[(f_{n,1}-f_1)\star h_1 - \mathcal{F}_{-2\alpha}(\overline{f_{n,2}-f_2})\star h_2\right] + j\left[(\mathcal{F}_{-2\alpha}(\overline{f_{n,1}-f_1})\star h_2 + (f_{n,2}-f_2)\star h_1\right]\to 0$$

as $n\to\infty$. Thus, $f_n\star_\alpha h\to f\star_\alpha h$ in $L^2(\mathbb{R},\mathbb{H})$ as $n\to\infty$.

Lemma 4. *If* $f\in L^2(\mathbb{R},\mathbb{H})$ *and* $(\delta_n)\in\Delta_\alpha$, *then* $f\star_\alpha\delta_n\to f$ *in* $L^2(\mathbb{R},\mathbb{H})$ *as* $n\to\infty$. *Further, if* $f_n\to f$ *in* $L^2(\mathbb{R},\mathbb{H})$ *as* $n\to\infty$, *then* $f_n\star_\alpha\delta_n\to f$ *in* $L^2(\mathbb{R},\mathbb{H})$ *as* $n\to\infty$.

Proof. From [40], it is well-known that the above lemma is true if f and f_n are complex valued functions. Assuming this fact, and using the continuity of $\mathcal{F}_{-2\alpha}$, one can prove the above lemma, as in the proof of the previous lemma.

In the next section, we shall define the extended quaternionic fractional Fourier transform on $\mathcal{B}^2_{\star_\alpha}$. As the codomain of the extended quaternionic fractional Fourier transform, we introduce another Boehmian space

$$\mathcal{B}^2_{\odot_\alpha} = \mathcal{B}(L^2(\mathbb{R},\mathbb{H}), L^1(\mathbb{R},\mathbb{C}), \odot_\alpha, \hat{\Delta}_\alpha),$$

where $(f\odot_\alpha g)(x) = f(x)g(x)e^{ia(\alpha)x^2}$ for all $x\in\mathbb{R}$, and $\hat{\Delta}_\alpha = \left\{(\mathcal{F}_\alpha(\delta_n))\colon (\delta_n)\in\Delta_\alpha\right\}$. By the convolution theorem for quaternionic fractional Fourier transform (Theorem 1), it is clear that $\mathcal{F}_\alpha(f\star_\alpha g) = \mathcal{F}_\alpha(f)\odot_\alpha\mathcal{F}_\alpha(g)$. Therefore, all the auxiliary results for constructing this Boehmian space could be obtained by applying the convolution theorem for quaternionic fractional Fourier transform in the corresponding results for the construction of $\mathcal{B}^2_{\star_\alpha}$.

4. Extended quaternionic fractional Fourier transform. For a given Boehmian $B\in\mathcal{B}^2_{\star_\alpha}$, we define the extended quaternionic fractional Fourier transform of B by the Boehmian $\left[\dfrac{(\mathcal{F}_\alpha(f_n))}{(\mathcal{F}_\alpha(\delta_n))}\right]\in$ $\in\mathcal{B}^2_{\odot_\alpha}$, where $\dfrac{(f_n)}{(\delta_n)}$ is an arbitrary representative of B. Since

$$f_n \star_\alpha \delta_m = f_m \star_\alpha \delta_n \quad \forall m, n \in \mathbb{N},$$

applying the convolution theorem for quaternionic fractional Fourier transform, we get

$$\mathcal{F}_\alpha(f_n) \odot_\alpha \mathcal{F}_\alpha(\delta_m) = \mathcal{F}_\alpha(f_m) \odot_\alpha \mathcal{F}_\alpha(\delta_n) \quad \forall m, n \in \mathbb{N}.$$

Thus, $\dfrac{(\mathcal{F}_\alpha(f_n))}{(\mathcal{F}_\alpha(\delta_n))}$ represents a Boehmian in $\mathcal{B}^2_{\odot_\alpha}$. Further, if $\dfrac{(g_n)}{(\varepsilon_n)}$ is another representative of B, by applying the convolution theorem, we get that $\dfrac{(\mathcal{F}_\alpha(f_n))}{(\mathcal{F}_\alpha(\delta_n))}$ is equivalent to $\dfrac{(\mathcal{F}_\alpha(g_n))}{(\mathcal{F}_\alpha(\varepsilon_n))}$, and, hence, the extended quaternionic fractional Fourier transform $\mathfrak{F}_\alpha : \mathcal{B}^2_{\star_\alpha} \to \mathcal{B}^2_{\odot_\alpha}$ is well defined. As the proofs of the following theorems are similar to that of any integral transform on Boehmians satisfying the convolution theorem as in Theorem 1, we prefer to omit the details. For example, we refer to [18].

Theorem 2. *The extended quaternionic fractional Fourier transform $\mathfrak{F}_\alpha$ on $\mathcal{B}^2_{\star_\alpha}$ is*

1) *consistent with the quaternionic fractional Fourier transform on $L^2(\mathbb{R}, \mathbb{H})$ as defined in* (2);
2) *a $\mathbb{H}$-linear map*;
3) *an injective map*;
4) *a continuous map with respect to δ-convergence as well as Δ-convergence*;
5) $\mathfrak{F}_\alpha(B * C) = (\mathfrak{F}_\alpha B) \odot_\alpha (\mathfrak{F}_\alpha C)$ $\forall B, C \in \mathcal{B}^2_{\star_\alpha}$ *with* $C = \left[\dfrac{(g_n)}{(\varepsilon_n)}\right]$ *and* $g_n \in L^1(\mathbb{R}, \mathbb{C})$ $\forall n \in \mathbb{N}$.

Theorem 3. *Let* $X = \left[\dfrac{(\Phi_n)}{(\mathcal{F}_\alpha(\delta_n))}\right] \in \mathcal{B}^2_{\odot_\alpha}$. *Then X is in the range of $\mathfrak{F}_\alpha : \mathcal{B}^2_{\star_\alpha} \to \mathcal{B}^2_{\odot_\alpha}$ iff there Φ_n belongs to the range of $\mathcal{F}_\alpha : L^2(\mathbb{R}, \mathbb{H}) \to L^2(\mathbb{R}, \mathbb{H})$ for all $n \in \mathbb{N}$.*

5. Conclusion. In this paper, we extended the one-dimensional quaternionic fractional Fourier transform to a suitable space of Boemians and obtained its properties consistency, continuity, linearity, etc. The quaternionic linear canonical transform [17] generalizes the fractional Fourier transform and quaternionic Fourier transform simultaneously by providing a suitable generalized kernel. Likewise, the present work is also generalizing the quaternionic fractional Fourier transform of functions and the Fourier transform on Boehmians simultaneously.

In the application point of view, playing the role of identity for the usual convolution, the Dirac's delta distribution δ is useful to find the system waiting function g of a filter, in signal processing. The output of an input signal f passing through a filter with system waiting function g can be simply calculated by $f * g$. This tool is useful for the signals represented in frequency domain by the classical Fourier transform. If a signal is quaternion valued and it is represented in frequency domain by the fractional Fourier transform, then we need we need a suitable identity for $\star_\alpha$ to find the system waiting function of a filter.

Let $B = \left[\dfrac{(f_n)}{(\delta_n)}\right] \in \mathcal{B}^2_{\star_\alpha}$ and $(\varepsilon_n), (\lambda_n) \in \Delta_\alpha$ be arbitrary. From the definition of Boehmians, one can easily observe that

$$\left[\frac{(\varepsilon_n)}{(\varepsilon_n)}\right] = \left[\frac{(\lambda_n)}{(\lambda_n)}\right] \in \mathcal{B}^2_{\star_\alpha} \quad \text{and} \quad B = \left[\frac{(f_n)}{(\delta_n)}\right] = \left[\frac{(f_n \star_\alpha \varepsilon_n)}{(\delta_n \star \varepsilon_n)}\right] = B \star_\alpha \left[\frac{(\varepsilon_n)}{(\varepsilon_n)}\right].$$

Therefore, if we denote $I = \left[\dfrac{(\varepsilon_n)}{(\varepsilon_n)}\right]$, then $B \star_\alpha I = B$ is anlogous to $f * \delta = f$. So this work may be helpful for the people working on signal processing those who need a theory of quaternionic fractional Fourier transform which is applicable on the "*identity*" for $\star_\alpha$.

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