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COVERING CODES OF A GRAPH ASSOCIATED TO A FINITE VECTOR SPACE

КОДИ ПОКРИТТЯ ГРАФА, ЩО ПОВ'ЯЗАНИЙ ЗІ СКІНЧЕННИМ ВЕКТОРНИМ ПРОСТОРОМ

In this paper, we investigate the problem of covering the vertices of a graph associated to a finite vector space as introduced by Das [Commun. Algebra, **44**, 3918 – 3926 (2016)], such that we can uniquely identify any vertex by examining the vertices that cover it. We use locating-dominating sets and identifying codes, which are closely related concepts for this purpose. We find the location-domination number and the identifying number of the graph and study the exchange property for locating-dominating sets and identifying codes.

Досліджується задача покриття вершин графа, що пов'язаний із скінченним векторним простором, як це визначено у A. Das [Commun. Algebra, **44**, 3918 – 3926 (2016)], так що ми можемо однозначно ідентифікувати будь-яку вершину за вершинами, що її накривають. У цій роботі використовуються розміщені домінуючі множини, а також ідентифікаційні коди, які у даному випадку є дуже близькими поняттями. Знайдено число розміщеного домінування й ідентифікаційне число для графа, а також вивчено властивість обміну для розміщених домінуючих та ідентифікаційних кодів.

1. Preliminaries. The association of graphs to algebraic structures has become an interesting research topic for the past few decades. See for instance: commuting graphs for groups [2, 6, 21], power graphs for groups and semigroups [8, 10, 26], zero divisor graph associated to a commutative ring [1, 3]. The association of a graph and vector space has history back in 1958 by Gould [18]. Later, Chen [13] investigated on vector spaces associated with a graph. Carvalho [9] studied vector space and the Petersen graph. In the recent past, Manjula [25] used vector spaces and made it possible to use techniques of linear algebra in studying the graph. Intersection graphs assigned to vector space were studied [22, 32]. Das [15] introduced a new graph structure, called non-zero component graph on finite dimensional vector spaces. He showed that the graph is connected and found its domination number and independence number [16]. He characterized the maximal cliques in the graph and found the exact clique number, for some particular cases [16]. Das has also given some results on size, edge-connectivity and the chromatic number of the graph [16]. For more work on the non-zero component graph, see [27, 33].

The covering code problem for a given graph involves finding a set of vertices with the minimum cardinality whose neighborhoods uniquely overlap at any given graph vertex. The problem has demonstrated its fundamental nature through a wide variety of applications. Locating-dominating sets were introduced by Slater [29, 31] and identifying codes by Karpovsky et al. [23]. Locating-dominating sets are very similar to identifying codes with the subtle difference that only the vertices not in the locating-dominating set are required to have unique identifying sets. The decision problem for locating-dominating sets for directed graphs has been shown to be an NP-complete problem [11]. A considerable literature has been developed in this field (see [5, 12, 14, 17, 20, 28 – 30, 34]). In [7], it was pointed out that each locating-dominating set is both locating and dominating set. However, a set that is both locating and dominating is not necessarily a locating-dominating set.

The initial application of locating-dominating sets and identifying codes was fault-diagnosis in the maintenance of multiprocessor systems [23]. More recently, identifying codes and locating-dominating sets were extended to applications for joint monitoring and routing in wireless sensor networks [24] and environmental monitoring [4].

A natural question arises in reader's mind that how can we distinguish the need of identifying codes or locating-dominating sets for a system? A system, in which processors or sensors are able to send the information about themselves and their neighbors, an identifying code is necessary. However, the systems where the sensors work without failure or if their only task is to test their neighborhoods (not themselves) then we shall search for locating-dominating sets. Moreover, the existence of identifying codes is not always guaranteed in a graph (as we shall see in our later discussion) and then a locating-dominating set is the next best alternative.

In this paper, we study the locating-dominating sets and identifying codes for the graph associated to finite vector space as defined in [15]. Also, we find location-domination number and identifying number of the graph and study the exchange property of the graph for these graph invariants.

Now, we recall some definitions of graph theory which are necessary for this article. We use Γ to denote a connected graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. The *degree* of the vertex v in Γ, denoted by $deg(v)$, is the number of edges to which v belongs. The *open neighborhood* of the vertex u of Γ is $N(u) = \{v \in V(\Gamma) \colon uv \in E(\Gamma)\}$ and the *closed neighborhood* of u is $N[u] = N(u) \cup \{u\}$.

Formally, we define a locating-dominating set as: A set L_D of vertices of Γ is called a locating-dominating set for Γ if for every two distinct vertices $u, v \in V(\Gamma) \setminus L_D$, we have $\varnothing \neq N(u) \cap L_D \neq \neq N(v) \cap L_D \neq \varnothing$. The *location-domination number*, denoted by $\lambda(\Gamma)$, is the minimum cardinality of a locating-dominating set of Γ.

An identifying code is a subset of vertices in a graph with the property that the neighborhood of every vertex has a unique intersection with the code. Formally it is defined as: A set I_D is called an *identifying code* for the graph Γ if $N[u] \cap I_D \neq N[v] \cap I_D$ for all $u, v \in V(\Gamma)$. The minimum cardinality of an identifying code is called the *identifying number* of Γ and we denote it by $I(\Gamma)$.

Unlike identifying codes, every graph has a trivial locating-dominating set, the entire set of vertices. On the other hand, a graph may not have an identifying code, because if $N[u] = N[v]$ for some $u, v \in V(\Gamma)$, then clearly $V(\Gamma)$ is not an identifying code. Since an identifying code is also a locating-dominating set, therefore

$$\lambda(\Gamma(\mathbb{V})) \leq I(\Gamma(\mathbb{V})). \tag{1}$$

Two vertices u, v are *adjacent twins* if $N[u] = N[v]$ and *non-adjacent twins* if $N(u) = N(v)$. If u, v are adjacent or non-adjacent twins, then u, v are *twins*. A set of vertices T is called a *twin-set* if any two of its vertices are twins [19]. By definition of twin vertices and twin-set, we have the following straightforward results.

Proposition 1.1. *Suppose that u, v are twins in a connected graph Γ and L_D is a locating-dominating set of Γ, then either u or v is in L_D. Moreover, if $u \in L_D$ and $v \notin L_D$, then $(L_D \setminus \{u\}) \cup \{v\}$ is a locating-dominating set of Γ.*

Proposition 1.2. *Let T be a twin-set of order $m \geq 2$ in a connected graph Γ. Then every locating-dominating set L_D of Γ contains at least $m - 1$ vertices of T.*

1.1. Non-zero component graph. Let $\mathbb{V}$ be a vector space over a field $\mathbb{F}$ with a basis $\{b_1, b_2, \ldots, b_n\}$. A vector $v \in \mathbb{V}$ can be expressed uniquely as a linear combination of the form $v = c_1 b_1 + c_2 b_2 + \ldots + c_n b_n$. A *non-zero component graph*, denoted by $\Gamma(\mathbb{V})$, can be associated with a finite dimensional vector space in the following way: the vertex set of the graph $\Gamma(\mathbb{V})$ consists of the non-zero vectors and two vertices are joined by an edge if they share at least one b_i with non-zero coefficient in their unique linear combination with respect to $\{b_1, b_2, \ldots, b_n\}$ [15]. It is proved in [15] that $\Gamma(\mathbb{V})$ is independent of the choice of basis, i.e., isomorphic non-zero component graphs are obtained with two different bases.

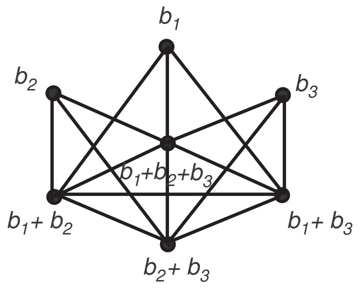

Fig. 1. $\dim(\mathbb{V}) = 3$, $\mathbb{F} = Z_2 = \{\bar{0}, \bar{1}\}$.

Theorem 1.1 [16]. *If $\mathbb{V}$ be an n-dimensional vector space over a finite field $\mathbb{F}$ with q elements, then the order of $\Gamma(\mathbb{V})$ is $q^n - 1$ and the size of $\Gamma(\mathbb{V})$ is*

$$\frac{q^{2n} - q^n + 1 - (2q-1)^n}{2}.$$

Theorem 1.2 [15]. *Let $\mathbb{V}$ be an n-dimensional vector space over a finite field $\mathbb{F}$ with q elements and $\Gamma(\mathbb{V})$ be its associated graph with respect to a basis $\{b_1, b_2, ..., b_n\}$, then a vertex having s non-zero coefficients in its unique linear combination of basis vector has degree $(q^s - 1)q^{n-s} - 1$.*

2. Locating-dominating sets and identifying codes of non-zero component graph. In this section, we study the location-domination number of non-zero component graph $\Gamma(\mathbb{V})$.

We partition the vertex set of $\Gamma(\mathbb{V})$ into n classes T_i, where $T_i = \{v \in \mathbb{V} \colon v$ is a linear combination of basis vectors with i non-zero coefficients$\}$. For example, if $n = 3$ and $q = 2$, then $T_2 = \{b_1 + b_2, b_2 + b_3, b_1 + b_3\}$ (see Fig. 1).

Lemma 2.1. *Let $\mathbb{V}$ be a vector space of dimension n over a field $\mathbb{F}$ of 2 elements. If $v \in T_s$ for s, $1 \le s \le n$, then, for an r, $1 \le r \le n$,*

$$|N(v) \cap T_r| = \begin{cases} \binom{n}{r} - \binom{n-s}{r} - 1 & \textit{if} \quad r \le n-s \ \textit{and} \ r = s, \\ \binom{n}{r} - \binom{n-s}{r} & \textit{if} \quad r \le n-s \ \textit{and} \ r \ne s, \\ \binom{n}{r} - 1 & \textit{if} \quad n-s < r \le n \ \textit{and} \ r = s, \\ \binom{n}{r} & \textit{if} \quad n-s < r \le n \ \textit{and} \ r \ne s. \end{cases}$$

Proof. We consider the following cases for r:

1. If $r \leq n-s$, then $\binom{n-s}{r}$ elements of T_r have s zero coefficients in their unique linear combination of basis vectors for those s basis vectors which have the non-zero coefficients in the unique linear combination of v, and hence these elements of T_r are not adjacent to v. Thus, $|N(v) \cap \cap T_r| = \binom{n}{r} - \binom{n-s}{r}$ or $\binom{n}{r} - \binom{n-s}{r} - 1$ according as $r \neq s$ or $r = s$, respectively.

2. If $r > n-s$, then each element of T_r will have at least one non-zero coefficient in its unique linear combination of basis vectors for those s basis vectors which have the non-zero coefficients in the unique linear combination of v, and hence v is adjacent to all elements of T_r. Thus, $|N(v) \cap T_r| = = \binom{n}{r}$ or $\binom{n}{r} - 1$ according as $r \neq s$ or $r = s$, respectively.

Lemma 2.1 is proved.

Let $v \in T_s$, then it can be seen from Lemma 2.1 that

$$\deg(v) = \left[\sum_{r=1}^{n} |N(v) \cap T_r|\right] - 1 =$$

$$= \sum_{r=1}^{n-s} \left[\binom{n}{r} - \binom{n-s}{r}\right] + \sum_{r=n-s+1}^{n} \binom{n}{r} - 1 = (2^s - 1)2^{n-s} - 1$$

which is consistent with Theorem 1.2 for $q = 2$.

***Remark* 2.1.** Let $\mathbb{V}$ be a vector space of dimension n over a field $\mathbb{F}$ of 2 elements. If $v \in T_s$ for s, $1 \leq s \leq n$, then $\deg(v) = (2^s - 1)2^{n-s} - 1$.

Lemma 2.2. *Let $\mathbb{V}$ be a vector space of dimension $n \geq 4$ over a field $\mathbb{F}$ of 2 elements. If $u, v \in V(\Gamma(\mathbb{V})) \setminus T_{n-1}$, then $N(u) \cap T_2 \neq N(v) \cap T_2$.*

Proof. Since $u \in T_r$ and $v \in T_s$ for some $1 \leq r, s \leq n$, $r, s \neq n-1$, therefore u has r non-zero coefficients in its unique linear combination of basis vectors $B = \{b_1, b_2, \ldots, b_n\}$. Let $B_u \subseteq B$ and $B_v \subseteq B$ be the sets of those basis vectors which have non-zero coefficients in the unique linear combination of basis vectors for u and v, respectively. Then u is not adjacent to $\binom{n-r}{2}$ elements of T_2 which have exactly two non-zero coefficients of basis vectors in B_u and zero coefficients of basis vectors in $B \setminus B_u$. Since $u \notin T_{n-1}$, therefore such elements exist in T_2 which have exactly two non-zero coefficients of basis vectors of B_u. Thus, $N(u) \cap T_2 = T_2 \setminus \{$two element sum of basis vectors in $B \setminus B_r\}$. Since $u \neq v$, therefore $B_u \neq B_v$ and, hence, $N(u) \cap T_2 \neq N(v) \cap T_2$.

Lemma 2.2 is proved.

An immediate consequence of Lemma 2.2 is that the set $T_2 \cup T_{n-1}$ forms a locating-dominating set for $\Gamma(\mathbb{V})$ for a vector space $\mathbb{V}$ of dimension $n \geq 4$ over a field of 2 elements.

Since the elements of T_{n-1} have non-zero coefficients for $n-1$ basis vectors, therefore we use the notation $u_j = \sum_{i=1}^{n} b_i - b_j$ in the proof of Lemma 2.3 for the elements of T_{n-1} which have zero coefficient for the basis vector b_j. Also, $N[u_j] = V(\Gamma(\mathbb{V})) \setminus \{b_j\}$, therefore two elements $u_i, u_j \in T_{n-1}$ have same neighbors in $\Gamma(\mathbb{V})$ except the elements b_i and b_j of T_1.

Lemma 2.3. *Let $\mathbb{V}$ be a vector space of dimension $n \geq 3$ over a field $\mathbb{F}$ of 2 elements. Let L_D be a locating-dominating set for $\Gamma(\mathbb{V})$ and $|L_D \cap T_1| = s$.*

(a) *If $0 \leq s \leq n-2$, then $|L_D \cap T_{n-1}| \geq n-s$.*

(b) *If* $s = n-1$, *then* $|L_D \cap \{T_n \cup T_{n-1}\}| \geq 1$.

Proof. Without loss of generality assume that $L_D \cap T_1 = \{b_1, b_2, \ldots, b_s\}$.

(a) Let $u_i, u_j \in T_{n-1}$ for $s+1 \leq i \neq j \leq n$ be two distinct elements of T_{n-1}, then $N(u_i) \cap \{L_D \cap T_1\} = N(u_j) \cap \{L_D \cap T_1\} = \varnothing$. Since u_i and u_j have different neighbors only in $\{b_{s+1}, b_{s+2}, \ldots, b_n\} \subseteq T_1$ which is not subset of L_D, therefore these $n-s$ elements of T_{n-1} must belong to L_D. Hence, $|L_D \cap T_{n-1}| \geq n-s$.

(b) Let $u_n \in T_{n-1}$ and $v \in T_n$, then $N(u_n) \cap \{L_D \cap T_1\} = N(v) \cap \{L_D \cap T_1\} = \varnothing$. Since u_n and v have only one different neighbor $b_n \in T_1$ which is not in L_D, therefore either u_n or v must belong to L_D.

Lemma 2.3 is proved.

***Corollary* 2.1.** *Let* $\mathbb{V}$ *be a vector space of dimension* $n \geq 3$ *over a field* $\mathbb{F}$ *of* 2 *elements. Let* L_D *be a locating-dominating set for* $\Gamma(\mathbb{V})$, *then* $|L_D| \geq n$.

Proof. If $0 \leq s \leq n-2$, then $|L_D \cap \{T_1 \cup T_{n-1}\}| \geq s + n - s = n$ by Lemma 2.3(a). If $s = n-1$, then $|L_D \cap \{T_1 \cup T_{n-1} \cup T_n\}| \geq n - 1 + 1 = n$ by Lemma 2.3(b). If $s = n$, then clearly $|L_D \cap T_1| = n$.

Since $\lambda(P_3) = 2$, where P_3 is the path graph of order 3, therefore we have the following proposition.

Proposition 2.1. *Let* $\mathbb{V}$ *be a vector space of dimension* 2 *over a field* $\mathbb{F}$ *of* 2 *elements, then* $\lambda(\Gamma(\mathbb{V})) = 2$.

Let $\mathbb{V}$ be a vector space of dimension n and $q \geq 3$. Then class T_i for each i, $1 \leq i \leq n$, has $\binom{n}{i}$ twin subsets of vertices of $\Gamma(\mathbb{V})$ and each of these twin subsets has the cardinality $(q-1)^i$. We use the notation T_{i_k} where $1 \leq k \leq \binom{n}{i}$ to denote the kth twin set in the class T_i.

Theorem 2.1. *Let* $\mathbb{V}$ *be a vector space over a field* $\mathbb{F}$ *of* q *elements with* $\{b_1, b_2, \ldots, b_n\}$ *as basis*:

(a) *If* $q = 2$ *and* $n \geq 3$, *then* $\lambda(\Gamma(\mathbb{V})) = n$.

(b) *If* $q \geq 3$, *then* $\lambda(\Gamma(\mathbb{V})) = \sum_{i=1}^{n} \binom{n}{i}((q-1)^i - 1)$.

Proof. (a) For $q = 2$ and $n \geq 3$, we first prove that T_1 is a locating-dominating set for $\Gamma(\mathbb{V})$. Let $u, v \in V(\Gamma(\mathbb{V})) \setminus T_1$. If $u, v \in T_s$ for some s when $2 \leq s \leq n-1$, then both u and v have s non-zero coefficients in their linear combinations of basis vectors. Since $u \neq v$ and $s < n$, therefore $\varnothing \neq N(u) \cap T_1 \neq N(v) \cap T_1 \neq \varnothing$. If $u \in T_r$ and $v \in T_s$ for some r, s $2 \leq r \neq s \leq n$, then $|N(u) \cap T_1| \neq |N(v) \cap T_1|$ by Lemma 2.1, and, hence, $\varnothing \neq N(u) \cap T_1 \neq N(v) \cap T_1 \neq \varnothing$. Thus, T_1 is a locating dominating set for $\Gamma(\mathbb{V})$. Hence, $\lambda(\Gamma(\mathbb{V})) \leq n$. Also, $\lambda(\Gamma(\mathbb{V})) \geq n$ by Corollary 2.1.

(b) If $q \geq 3$, then from Proposition 1.2, a minimal locating-dominating set of $\Gamma(\mathbb{V})$ contains at least $(q-1)^i - 1$ vertices from T_{i_k} for each i, $1 \leq i \leq n$ and each k, $1 \leq k \leq \binom{n}{i}$, and hence $\lambda(\Gamma(\mathbb{V})) \geq \sum_{i=1}^{n} \binom{n}{i}((q-1)^i - 1)$. Moreover, a subset of $\Gamma(\mathbb{V})$ of cardinality greater than $\sum_{i=1}^{n} \binom{n}{i}((q-1)^i - 1)$ has all the vertices of at least one twin subset T_{i_k}. Thus, from Proposition

1.1, a locating-dominating set of cardinality greater than $\sum_{i=1}^{n}\binom{n}{i}[(q-1)^i-1]$ is not a minimal locating-dominating set, and hence, $\lambda(\Gamma(\mathbb{V}))\leq\sum_{i=1}^{n}\binom{n}{i}[(q-1)^i-1]$.

Theorem 2.1 is proved.

Since $I(P_3)=2$, therefore we have the following proposition.

Proposition 2.2. *Let $\mathbb{V}$ be a vector space of dimension 2 over a field $\mathbb{F}$ of 2 elements, then $I(\Gamma(\mathbb{V}))=2$.*

The following theorem gives the identifying number of $\Gamma(\mathbb{V})$.

Theorem 2.2. *Let $\mathbb{V}$ be a finite vector space over a field $\mathbb{F}$ of 2 elements, then $I(\Gamma(\mathbb{V}))=n$.*

Proof. For $n\geq 3$ and $q=2$, by Theorem 2.1(a) and inequality (1), $I(\Gamma(\mathbb{V})\geq n$. Note that T_1 is an identifying code for $\Gamma(\mathbb{V})$ because for each vertex say $u\in V(\Gamma(\mathbb{V}))$, $N[u]\cap T_1$ is the set of all those elements of T_1 which have non-zero coefficients in the unique linear combination of basis vectors for u. Thus, for any two distinct elements $u,v\in V(\Gamma(\mathbb{V}))$, $N[u]\cap T_1$ and $N[v]\cap T_1$ are distinct. Hence, $I(\Gamma(\mathbb{V}))\leq n$.

Theorem 2.2 is proved.

Let $\mathbb{V}$ be a finite vector space and $q\geq 3$, then $\Gamma(\mathbb{V})$ has twin sets T_{i_k}, $1\leq i\leq n$, $1\leq k\leq\binom{n}{i}$ and each of these twin subset has adjacent twins, therefore identifying code for $\Gamma(\mathbb{V})$ does not exist. Thus, we have the following remark.

***Remark* 2.2.** Let $\mathbb{V}$ be a vector space of dimension $n\geq 3$ and $q\geq 3$, then identifying code for $\Gamma(\mathbb{V})$ does not exist.

Lemma 2.4. *Let $\mathbb{V}$ be a vector space of dimension $n\geq 3$ and $q=2$, then T_1 is the only minimal identifying code for $\Gamma(\mathbb{V})$.*

Proof. Suppose on contrary I'_D be another minimal identifying code of $\Gamma(\mathbb{V})$, then there exist at least one element say $b_r\in T_1$ such that $b_r\notin I'_D$ (because otherwise $T_1\subset I'_D$). Take two elements $u_r\in T_{n-1}$ (using same notation as in proof of Lemma 2.3) and $w\in T_n$. Since $N[w]=V(\Gamma(\mathbb{V}))$ and $N[u_r]=V(\Gamma(\mathbb{V}))\setminus\{b_r\}$, therefore $N[w]\cap I'_D=N[u_r]\cap I'_D\neq\varnothing$, a contradiction.

Lemma 2.4 is proved.

2.1. Exchange property. Locating-dominating sets are said to have *the exchange property* in a graph Γ if for every two distinct minimal locating-dominating sets L_{D_1}, L_{D_2} and $u_1\in L_{D_1}$, then there exists $u_2\in L_{D_2}$ so that $(L_{D_2}\setminus\{u_2\})\cup\{u_1\}$ is also a minimal locating-dominating set. If locating-dominating sets has the exchange property in a graph Γ, then all minimal locating-dominating sets of Γ has same number of elements. To show that the exchange property does not hold in a graph, it is sufficient to show that there exist two minimal locating-dominating sets of different cardinalities. However, the condition is not necessary.

Lemma 2.5. *For $q=2$ and $n>3$, the exchange property does not hold for locating-dominating sets in graph $\Gamma(\mathbb{V})$.*

Proof. For $n=4$, the exchange property does not hold because T_1 and $\{b_1+b_4,b_2+b_4,b_3+$ $+\,b_4\}\cup T_3$ are minimal locating-dominating sets of different cardinalities.

For $n\geq 5$, T_1 and $T_2\cup T_{n-1}$ are two locating-dominating sets of cardinalities n and $\binom{n}{2}+n$ by Lemma 2.2. For notational convenience, we use $A=T_2\cup T_{n-1}$. We will prove that A is a minimal locating-dominating set of $\Gamma(\mathbb{V})$. Let $u\in A$ and $w\in T_n$. There are two possible cases for u.

1. If $u\in T_2$, then u has exactly two non-zero coefficients in its unique linear combination of basis vectors, say these vectors set as B_u. Choose an element in $v\in T_{n-2}$ such that v has

exactly $n-2$ non-zero coefficients in the unique linear combination of basis vectors in $B \setminus B_u$. Then $N(v) \cap A \setminus \{u\} = N(w) \cap A \setminus \{u\}$. Thus, $A \setminus \{u\}$ is not locating-dominating set.

2. If $u \in T_{n-1}$, then $N(u) \cap A \setminus \{u\} = N(w) \cap A \setminus \{u\}$. Thus, $T_2 \cup T_{n-1}$ is a minimal locating-dominating set. Hence, exchange property does not hold for locating-dominating sets in graph $\Gamma(\mathbb{V})$.

Lemma 2.5 is proved.

In the proof of the following lemma, we use the same notation T_{i_k} for the kth twin set of class T_i as we have used in the proof of Theorem 2.1(b).

Lemma 2.6. *For $q \geq 3$, the exchange property holds for locating-dominating sets in graph $\Gamma(\mathbb{V})$.*

Proof. Since there are $(q-1)^i$ choices for removing one vertex from a twin set T_{i_k} of cardinality $(q-1)^i$, therefore there are $\prod_{i=1}^{n} \binom{n}{i} (q-1)^i$ minimal locating-dominating sets in $\Gamma(\mathbb{V})$. Let $L_{D_1} \neq L_{D_2}$ be two such minimal locating-dominating sets. Let $u_1 \in L_{D_1}$, we further assume that $u_1 \notin L_{D_2}$ (for otherwise $(L_{D_2} \setminus \{u_1\}) \cup \{u_1\}$ is obviously a minimal locating-dominating set of $\Gamma(\mathbb{V})$. Also, $u_1 \in T_{i_k}$ for some i, $1 \leq i \leq n$, and some k, $1 \leq k \leq \binom{n}{i}$. Since $u_1 \in$ $\in \{L_{D_1} \cap T_{i_k}\} \setminus \{L_{D_2} \cap T_{i_k}\}$ and L_{D_1}, L_{D_2} are minimal, therefore there exists an element $u_2 \in$ $\in \{L_{D_2} \cap T_{i_k}\} \setminus \{L_{D_1} \cap T_{i_k}\}$. Since both u_1 and u_2 belong to the same twin set T_{i_k}, therefore by Proposition 1.1 $(L_{D_2} \setminus \{u_2\}) \cup \{u_1\}$ is a minimal locating-dominating set of $\Gamma(\mathbb{V})$. Hence, the exchange property holds in $\Gamma(\mathbb{V})$.

Lemma 2.6 is proved.

From Lemma 2.4 we have the following remark.

***Remark* 2.3.** Let $\mathbb{V}$ be a vector space of dimension $n \geq 3$ and $q = 2$, then identifying code have the exchange property in $\Gamma(\mathbb{V})$.

From Lemma 2.5 we have the following remark.

***Remark* 2.4.** In general, the locating-dominating sets does not have the exchange property for all graphs.

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