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ON A CLASS OF DUAL RICKART MODULES
ПРО КЛАС ДУАЛЬНИХ МОДУЛІВ РІКАРТА

Let R be a ring and let Ω_R be the set of maximal right ideals of R. An R-module M is called an sd-Rickart module if for every nonzero endomorphism f of M, Imf is a fully invariant direct summand of M. We obtain a characterization for an arbitrary direct sum of sd-Rickart modules to be sd-Rickart. We also obtain a decomposition of an sd-Rickart R-module M, provided R is a commutative noetherian ring and $\mathrm{Ass}(M) \cap \Omega_R$ is a finite set. In addition, we introduce and study a generalization of sd-Rickart modules.

Нехай R — кільце, а Ω_R — множина максимальних правих ідеалів R. R-модуль M називається sd-модулем Рікарта, якщо для будь-якого ненульового ендоморфізму f на M множина Imf є повністю інваріантним прямим доданком M. Отримано характеризацію умов, за яких довільна пряма сума sd-модулів Рікарта є sd-модулем Рікарта. Також отримано розклад R-модуля M, що є sd-модулем Рікарта, за умови, що R — комутативне нетерове кільце та $\mathrm{Ass}(M) \cap \Omega_R$ є скінченною множиною. На додаток визначаються та вивчаються узагальнення sd-модулів Рікарта.

1. Introduction. Throughout this article, all rings are associative and have an identity. All modules are unital right modules. R is a ring, Ω_R is the set of its maximal right ideals and M is an R-module. We use $J(R)$, $\mathrm{End}_R(M)$, and $E(M)$ to denote the Jacobson radical of R, the endomorphism ring of M, and the injective hull of M, respectively. If R is a commutative Noetherian ring, then $\mathrm{Ass}(M)$ will denote the set of all prime ideals associated to M. By $\mathbb{Q}$ and $\mathbb{Z}$ we denote the ring of rational and integer numbers, respectively. If p is a prime number, then $\mathbb{Z}_{p^\infty}$ will denote the Prüfer p-group. Recently, the notion of *dual Baer* modules was introduced and studied in [8]. A module M is said to be *dual Baer* if for every submodule N of M, there exists an idempotent $e \in S = \mathrm{End}_R(M)$ such that $D(N) = eS$, where $D(N) = \{\varphi \in S \mid \mathrm{Im}\,\varphi \subseteq N\}$. In [9], Lee, Rizvi and Roman provided some motivations to study the concept of a *dual Rickart module* which is a related concept to that of a dual Baer module. A module M is called *dual Rickart* (or *d-Rickart*) if $\mathrm{Im}\, f$ is a direct summand of M for every $f \in \mathrm{End}_R(M)$. A submodule N of M is called *fully invariant* if $f(N)$ is contained in N for every R-endomorphism f of M. In [3], Călugăreanu and Schultz introduced and studied the notion of *stable modules*. A module M is said to be *stable* if all endomorphic images are fully invariant. Clearly, if M is an R-module such that $\mathrm{End}_R(M)$ is a commutative ring, then M is a stable module. Abelian groups whose endomorphism ring is commutative was characterized in [13, 14]. It is of interest to investigate the intersection $\mathcal{C}$ of the class of dual Rickart modules and the class of stable modules. Any element in $\mathcal{C}$ will be called a *strongly dual Rickart module*. So, an R-module M is strongly dual Rickart if and only if the image of any single element of $\mathrm{End}_R(M)$ is a fully invariant direct summand of M.

A module M is said to have the (D_2) *condition* if for every submodule N of M for which M/N is isomorphic to a direct summand of M, N is a direct summand of M. In Section 2, we study the notion of sd-Rickart modules. We show that this notion is distinct from that of d-Rickart modules. It is shown that the class of sd-Rickart modules is precisely the class of d-Rickart modules M for which every direct summand of M is fully invariant in M. Also, we characterize an sd-Rickart module having the (D_2) condition as the one which has a strongly regular ring as its endomorphism ring. In particular, the R-module R_R is sd-Rickart if and only if R is a strongly regular ring. Then we study the question of when is the direct sum of sd-Rickart modules, sd-Rickart? We first show that the class of sd-Rickart modules is not closed under finite direct sums. Then we obtain a characterization for an arbitrary direct sum of sd-Rickart modules to be sd-Rickart. We also prove that if R is a commutative Noetherian ring and M is an R-module with $\operatorname{Ass}(M)\cap\Omega_R$ is a finite set, then M is an sd-Rickart module if and only if $M \cong N\oplus\big(\oplus_{i=1}^{k} R/P_i\big)$ for some distinct maximal ideals $P_1,\ldots,P_k$ of R and an sd-Rickart module N such that $\operatorname{Rad}(N)=N$ and $\operatorname{Hom}_R(\oplus_{i=1}^{k} R/P_i, N)=0$.

In Section 3, we introduce the concept of a generalized strongly dual Rickart module. A module M is called a *generalized strongly dual Rickart* (*gsd-Rickart* for short) module if for every nonzero endomorphism f of M, $\operatorname{Im} f$ contains a nonzero fully invariant direct summand of M. A result (Proposition 3.1) and an example (Example 3.1) are provided to show that the concept of a gsd-Rickart module is distinct from that of an sd-Rickart module. Then we show that there are many similarities between the two concepts. For example, we show that if a module M is a direct sum of indecomposable submodules, then M is an sd-Rickart module if and only if M is a gsd-Rickart module. Also, we determine the structure of injective gsd-Rickart modules over commutative Noetherian hereditary rings. We conclude the paper by describing the structure of sd-Rickart modules and gsd-Rickart modules over discrete valuation rings.

2. Strongly dual Rickart modules.

Definition 2.1. *A module M is called a strongly dual Rickart* (*or an sd-Rickart*) *module if for every nonzero endomorphism f of M, $\operatorname{Im} f$ is a fully invariant direct summand of M.*

A ring R is said to be *abelian* (or *normal*) if all its idempotents are central elements (see [4] (15.31) and [15, p. 37]). Recall that a ring R is called *strongly regular* if for each $x\in R$, there exists $y\in R$ such that $x^2y=x$. This is equivalent to the condition that R is an abelian von Neumann regular ring (see [7], Theorem 3.5).

The next result will be of interest.

Proposition 2.1. *Every module which has a strongly regular endomorphism ring*, *is sd-Rickart.*

Proof. Let M be an R-module such that $S=\operatorname{End}_R(M)$ is a strongly regular ring. Let $\varphi\in S$. By [15] (Proposition 7.9), there exists $0\neq e^2=e\in S$ such that $\operatorname{Im}\varphi=e(M)$. Since the idempotent e is a central element in S, $e(M)$ is a fully invariant submodule of M.

Example **2.1.** (i) It follows from Proposition 2.1 that every indecomposable nonsingular injective module is sd-Rickart (see [1], Lemma 25.4, and [7], Corollary 1.23). In particular, for any nonsingular uniform module M, $E(M)$ is an sd-Rickart module. So, the $\mathbb{Z}$-module $\mathbb{Q}$ is an sd-Rickart $\mathbb{Z}$-module.

(ii) It easy to see that every module, for which all nonzero endomorphisms are epimorphisms, is sd-Rickart. Thus every indecomposable d-Rickart module is sd-Rickart (see [8], Corollary 2.2, and [9], Corollary 4.3). In particular, for any prime integer p, the $\mathbb{Z}$-modules $\mathbb{Z}_{p^\infty}$ and $\mathbb{Z}/p\mathbb{Z}$ are sd-Rickart.

Let $e=e^2\in R$. Then e is called a *left semicentral idempotent* if $xe=exe$ for all $x\in R$. Equivalently, eR is a two-sided ideal of R (see [2], Lemma 2.1). Recall that a module M is called *weak duo* if every direct summand of M is fully invariant (see [11]). The next result presents other ways of stating an "sd-Rickart module".

Proposition 2.2. *The following statements are equivalent for a module M*:

(i) *M is an sd-Rickart module*;

(ii) *M is a d-Rickart weak duo module*;

(iii) *M is a d-Rickart module and every idempotent of* $\operatorname{End}_R(M)$ *is left semicentral.*

Proof. (i) $\Rightarrow$ (ii) Clearly, M is a d-Rickart module. Let K be a direct summand of M and let $\pi : M \to K$ be the projection map. By hypothesis, $\operatorname{Im}\pi = K$ is fully invariant in M. Therefore M is a weak duo module.

(ii) $\Rightarrow$ (i) This is immediate.

(ii) $\Leftrightarrow$ (iii) This follows from [1] (Corollary 5.8) and [2] (Lemma 1.9(ii)).

***Corollary* 2.1.** *Every direct summand of an sd-Rickart module M is sd-Rickart.*

Proof. This follows easily from Proposition 2.2, Propositions 2.8 [9] and 1.8 [11].

Consider the following condition:

(D_2) If N is a submodule of M such that M/N is isomorphic to a direct summand of M, then N is a direct summand of M.

It is well known that every projective module has the condition (D_2). The following example shows that, in general, an sd-Rickart module need not have the condition (D_2).

***Example* 2.2.** Consider the $\mathbb{Z}$-module $M = \mathbb{Z}_{p^\infty}$, where p is a prime number. Let L be any nonzero proper submodule of M. Then $M/L \cong M$, but L is not a direct summand of M. Therefore, M does not satisfy (D_2). On the other hand, M is an sd-Rickart module.

The next proposition characterizes sd-Rickart modules satisfying (D_2).

Proposition 2.3. *The following statements are equivalent for an R-module M*:

(i) *M is an sd-Rickart module having the (D_2) condition*;

(ii) *$\operatorname{End}_R(M)$ is a strongly regular ring.*

Proof. (i) $\Rightarrow$ (ii) By [9] (Theorem 3.8), $S = \operatorname{End}_R(M)$ is a von Neumann regular ring. Let e be an idempotent of S. By Proposition 2.2, e is left semicentral. Therefore eS is a two-sided ideal of S by [2] (Lemma 2.1). So e is central by [7] (Lemma 3.1). Therefore, S is a strongly regular ring.

(ii) $\Rightarrow$ (i) This follows from Proposition 2.1 and Theorem 3.8 [9].

The next result characterizes the class of rings R for which the R-module R_R is sd-Rickart.

Proposition 2.4. *The following conditions are equivalent for a ring R*:

(i) *R_R is an sd-Rickart R-module*;

(ii) *for every two-sided ideal I of R, I_R is an sd-Rickart R-module*;

(iii) *R has a free sd-Rickart R-module F*;

(iv) *R is a strongly regular ring.*

Proof. (i) $\Rightarrow$ (iv) This follows easily from Proposition 2.3 since the R-module R_R satisfies (D_2).

(iv) $\Rightarrow$ (ii) Let I be a two-sided ideal of R. By [7] (Theorem 3.7), $\operatorname{End}_R(I_R)$ is a strongly regular ring. Therefore, I_R is an sd-Rickart R-module by Proposition 2.3.

(ii) $\Rightarrow$ (iii) This is obvious.

(iii) $\Rightarrow$ (i) This follows from Corollary 2.1.

It is clear that every sd-Rickart module is d-Rickart. Next, we exhibit an example of a module for which the converse is not true.

***Example* 2.3.** Let R be a von Neumann regular ring which is not strongly regular. Then the R-module R_R is d-Rickart by [9] (Remark 2.2). On the other hand, the R-module R_R is not sd-Rickart by Proposition 2.4. As an example of a ring satisfying these conditions, we can consider the endomorphism ring of a nonzero vector space V over a division ring such that the dimension of V is different from 1.

The next result is taken from [11] (Lemma 1.9).

Lemma 2.1. *Let a module $M = M_1 \oplus M_2$ be a direct sum of submodules M_1 and M_2. Then M_1 is a fully invariant submodule of M if and only if $\operatorname{Hom}_R(M_1, M_2) = 0$.*

Next, we will be concerned with the study of direct sums of sd-Rickart modules. We begin with a result which shows that a direct sum of sd-Rickart modules may not be sd-Rickart.

Proposition 2.5. *For any nonzero module M, the module $M \oplus M$ is not sd-Rickart.*

Proof. This is a direct consequence of Proposition 2.2 and Lemma 2.1.

***Remark* 2.1.** Let (P) be a property of modules which is closed under finite direct sums (e.g., being injective or projective). Proposition 2.5 shows that it is not possible to characterize the class of rings R for which every R-module satisfying (P) is an sd-Rickart R-module.

The next result is a characterization for an arbitrary direct sum of sd-Rickart modules to be sd-Rickart.

Theorem 2.1. *Let a module $M = \oplus_{i \in I} M_i$ be a direct sum of submodules M_i, $i \in I$. Then the following statements are equivalent*:

(i) *M is an sd-Rickart module*;

(ii) *M_i is an sd-Rickart submodule of M for every $i \in I$ and $\mathrm{Hom}_R(M_i, M_j) = 0$ for all distinct $i, j \in I$.*

Proof. (i) $\Rightarrow$ (ii) This follows from Proposition 2.2, Corollary 2.1 and Lemma 2.1.

(ii) $\Rightarrow$ (i) By Lemma 2.1, M_i is a fully invariant submodule of M for every $i \in I$. Therefore M is a d-Rickart module by [9] (Proposition 5.14). Let N be a direct summand of M. Then there exists an epimorphism $\psi \in \mathrm{End}_R(M)$ such that $N = \psi(M)$. Therefore, $N = \sum_{i \in I} \psi(M_i)$. Since $\psi(M_i) \subseteq M_i$ for all $i \in I$, we have $\psi(M_i) = N \cap M_i$ for all $i \in I$. This implies that $N = \oplus_{i \in I}(N \cap M_i)$. Note that the modules M_i, $i \in I$, are weak duo modules by Proposition 2.2. Applying [11] (Theorem 2.7), we see that M is a weak duo module. Consequently, M is an sd-Rickart module (see Proposition 2.2).

It is well-known that every semisimple module is a d-Rickart module. The next result that is a direct consequence of Theorem 2.1 and Lemma 2.1 provides more examples of d-Rickart modules which are not sd-Rickart.

***Corollary* 2.2.** *The following statements are equivalent for a nonzero semisimple module M*:

(i) *M is an sd-Rickart module*;

(ii) *$M = \oplus_{i \in I} S_i$ is a direct sum of simple submodules S_i, $i \in I$, such that S_i is not isomorphic to S_j for all $i \neq j$ in I.*

Lemma 2.2. *Let a module $M = M_1 \oplus M_2$ be a direct sum of submodules M_1 and M_2 such that* $\mathrm{Rad}(M_1) = M_1$ *and* $\mathrm{Rad}(M_2) = 0$. *If* $\mathrm{Hom}_R(M_2, M_1) = 0$, *then M_1 and M_2 are fully invariant submodules of M.*

Proof. It is clear that $M_1 = \mathrm{Rad}(M)$. Hence M_1 is fully invariant in M by [1] (Proposition 9.14). Also, since $\mathrm{Hom}_R(M_2, M_1) = 0$, M_2 is fully invariant in M by Lemma 2.1.

Proposition 2.6. *Let a module $M = M_1 \oplus M_2$ be a direct sum of submodules M_1 and M_2 such that* $\mathrm{Rad}(M_1) = M_1$ *and* $\mathrm{Rad}(M_2) = 0$. *Then the following statements are equivalent*:

(i) *M is an sd-Rickart module*;

(ii) *M_1 and M_2 are sd-Rickart modules with* $\mathrm{Hom}_R(M_2, M_1) = 0$.

Proof. (i) $\Rightarrow$ (ii) This follows from Theorem 2.1.

(ii) $\Rightarrow$ (i) Let $\varphi \colon M_1 \to M_2$ be a homomorphism. Since $\mathrm{Im}\,\varphi \cong M_1/\mathrm{Ker}\varphi$, $\mathrm{Rad}(\mathrm{Im}\,\varphi) = \mathrm{Im}\,\varphi$. But $\mathrm{Rad}(M_2) = 0$. Then $\varphi = 0$. Therefore, $\mathrm{Hom}_R(M_1, M_2) = 0$. The result follows from Theorem 2.1 and Lemma 2.2.

Proposition 2.7. *Let a module $M = M_1 \oplus M_2$ be a direct sum of submodules M_1 and M_2 such that* $\mathrm{Rad}(M_1) = M_1$ *and M_2 is semisimple. Then M is a d-Rickart module if and only if M_1 is a d-Rickart module and* $\mathrm{Hom}_R(M_2, M_1) = 0$.

Proof. The sufficiency follows from Lemma 2.2 and [9] (Proposition 5.14). Conversely, suppose that M is a d-Rickart module. By [9] (Proposition 2.8), M_1 is a d-Rickart module. Now let $f: M_2 \to M_1$ be a homomorphism. Let $\pi_2: M \to M_2$ denote the projection map and let $\mu_1: M_1 \to M$ denote the inclusion map. Then $\mu_1 f \pi_2$ is an endomorphism of M. So $\mathrm{Im}(\mu_1 f \pi_2) = \mathrm{Im}\, f$ is a direct summand of M_1. This gives $\mathrm{Rad}(\mathrm{Im}\, f) = \mathrm{Im}\, f$. But $\mathrm{Im}\, f$ is a semisimple submodule of M_1. Then $f = 0$. It follows that $\mathrm{Hom}_R(M_2, M_1) = 0$.

Remark **2.2.** It is obvious from [9] (Theorem 2.29) that every indecomposable injective module over a right hereditary ring, is an sd-Rickart module. In particular, for every prime ideal P of a Dedekind domain R, $E(R/P)$ is an sd-Rickart R-module.

Proposition 2.8. *Let R be a Dedekind domain. The following statements are equivalent for a nonzero torsion R-module M*:

(i) *M is an sd-Rickart R-module*;

(ii) *there exist distinct maximal ideals P_i, $i \in I$, of R and submodules M_i, $i \in I$, of M such that $M = \oplus_{i \in I} M_i$ and for each $i \in I$, either $M_i \cong E(R/P_i)$ or $M_i \cong R/P_i$.*

Proof. (i) $\Rightarrow$ (ii) By Proposition 2.2 and [11] (Theorem 3.10), there exist distinct maximal ideals P_i, $i \in I$, of R and submodules M_i, $i \in I$, of M such that $M = \oplus_{i \in I} M_i$, and for each $i \in I$, either $M_i \cong E(R/P_i)$ or $M_i \cong R/P_i^{n_i}$ for some positive integer n_i. Since each M_i is an sd-Rickart module (see Corollary 2.1), we have $M_i \cong E(R/P_i)$ or $M_i \cong R/P_i$ by [9] (Proposition 4.13).

(ii) $\Rightarrow$ (i) Clearly, each M_i is the P_i-primary component of M. So each M_i is a fully invariant submodule of M. In addition, we have $\mathrm{Hom}_R(M_i, M_j) = 0$ for all distinct i, j in I. Also, note that each M_i is an sd-Rickart module (see Remark 2.2). Therefore, M is an sd-Rickart module by Theorem 2.1.

Lemma 2.3. *Let I be a finitely generated ideal of a commutative ring R. If M is a d-Rickart R-module*, *then MI is a direct summand of M.*

Proof. By assumption, we have $I = \alpha_1 R + \ldots + \alpha_k R$ for some positive integer k and elements α_i, $1 \leq i \leq k$, of R. Fix $1 \leq i \leq k$ and consider the homomorphism $f_i: M \to M$ defined by $f_i(x) = x\alpha_i$ for all $x \in M$. It is clear that $\mathrm{Im}\, f_i = M\alpha_i$. Since M is a d-Rickart module, $M\alpha_i$ is a direct summand of M. In addition, M has the SSP by [9] (Proposition 2.11). Then $MI = \sum_{i=1}^{k} M\alpha_i$ is a direct summand of M.

A module M is said to be *radical* if $\mathrm{Rad}(M) = M$. The sum of all radical submodules of a module M will be denoted by $P(M)$. A module M is said to be *reduced* if $P(M) = 0$.

Theorem 2.2. *Let R be a commutative Noetherian ring and let Ω_R be the set of maximal ideals of R. Let M be an R-module such that $\mathrm{Ass}(M) \cap \Omega_R$ is a finite set. Then the following conditions are equivalent*:

(i) *M is a d-Rickart R-module*;

(ii) *$M = M_1 \oplus M_2$ is a direct sum of a d-Rickart submodule M_1 and a semisimple submodule M_2 such that $M_1 = \mathrm{Rad}(M_1) = \mathrm{Rad}(M) = P(M)$ and $\mathrm{Hom}_R(M_2, M_1) = 0$.*

Proof. (i) $\Rightarrow$ (ii) Without loss of generality we can assume that $\mathrm{Rad}(M) \neq M$. Note that $\mathrm{Rad}(M) = \bigcap_{m \in \Omega_R} Mm$ by [5] (Lemma 3). Then there exists a maximal ideal m_1 of R such that $Mm_1 \neq M$. Since R is Noetherian, the ideal m_1 is finitely generated. Therefore, Mm_1 is a direct summand of M by Lemma 2.3. Let N_1 be a submodule of M such that $M = Mm_1 \oplus N_1$. Clearly, $N_1 m_1 = 0$. So N_1 is semisimple. As $N_1 \neq 0$, $N_1 \cong (R/m_1)^{\Lambda_1}$ for some index set Λ_1. Assume that $\mathrm{Rad}(Mm_1) \neq Mm_1$. Then there exists a maximal ideal m_2 of R such that $Mm_1m_2 \neq Mm_1$ (see [5], Lemma 3). Note that $m_1 \neq m_2$ since $Mm_1^2 = Mm_1$. Moreover,

Mm_1 is a d-Rickart module by [9] (Proposition 2.8). By the same method as before, we conclude that $Mm_1 \cong Mm_1m_2 \oplus (R/m_2)^{\Lambda_2}$ for some index set Λ_2. Thus $M \cong Mm_1m_2 \oplus (R/m_2)^{\Lambda_2} \oplus$ $\oplus\ (R/m_1)^{\Lambda_1}$. We continue in this way. Since $\mathrm{Ass}(M) \cap \Omega_R$ is a finite set, this process must terminate. So there exist submodules M_1 and M_2 of M such that $\mathrm{Rad}(M_1) = M_1$ and M_2 is semisimple. By Proposition 2.7, M_1 is a d-Rickart module and $\mathrm{Hom}_R(M_2, M_1) = 0$. It is easily seen that $M_1 = \mathrm{Rad}(M) = P(M)$.

(ii) $\Rightarrow$ (i) This is immediate by Proposition 2.7.

***Remark* 2.3.** From the proof of Theorem 2.2, it follows that if M is a nonzero d-Rickart module over a commutative Noetherian ring with $\mathrm{Rad}(M) \neq M$, then M has a simple direct summand.

Combining Corollary 2.2, Proposition 2.6 and Theorem 2.2, we obtain the following result.

***Corollary* 2.3.** *Let R be a commutative Noetherian ring and Ω_R the set of maximal ideals of R. Let M be a nonzero R-module such that $\mathrm{Ass}(M) \cap \Omega_R$ is a finite set. Then the following conditions are equivalent*:

(i) *M is an sd-Rickart R-module*;

(ii) *$M \cong N \oplus (\oplus_{i=1}^k R/P_i)$ for some distinct maximal ideals $P_1, \ldots, P_k$ of R and an sd-Rickart module N such that $\mathrm{Rad}(N) = N$ and $\mathrm{Hom}_R(\oplus_{i=1}^k R/P_i, N) = 0$.*

The next example shows that Theorem 2.2 and Corollary 2.3 are not true if the condition "$\mathrm{Ass}(M) \cap \Omega_R$ is a finite set" is deleted from their hypotheses.

***Example* 2.4.** Consider the reduced $\mathbb{Z}$-module $M = \prod_{p \text{ prime}} \mathbb{Z}/p\mathbb{Z}$. As

$$\mathrm{End}_{\mathbb{Z}}(M) \cong \prod_{p \text{ prime}} \mathbb{Z}/p\mathbb{Z}$$

is a strongly regular ring, M is an sd-Rickart module (see Proposition 2.1). On the other hand, it is clear that the module M is not semisimple.

***Corollary* 2.4.** *Assume that R is a Dedekind domain with quotient field K. Let M be a nonzero R-module such that $\mathrm{Ass}(M)$ is a finite set. Then the following statements are equivalent*:

(i) *M is an sd-Rickart R-module*;

(ii) *$M \cong K \oplus (\oplus_{i=1}^k R/P_i)$ for some distinct maximal ideals $P_1, \ldots, P_k$ of R, or $M \cong \big(\oplus_{i=1}^n E(R/P_i)\big) \oplus (\oplus_{j=n+1}^m R/P_j)$, where P_i, $1 \leq i \leq m$, are distinct maximal ideals of R.*

Proof. This follows from Theorem 2.1, Proposition 2.8 and Corollary 2.3 and the fact that an R-module M is injective if and only if $\mathrm{Rad}(M) = M$.

Next, we show that torsion-free sd-Rickart modules over a commutative domain are injective.

Proposition 2.9. *Let M be a torsion-free module over a commutative domain R. If M is an sd-Rickart R-module, then M is injective.*

Proof. Assume that M is a d-Rickart R-module. We only need to show that M is a divisible R-module (see [12], Proposition 2.7). Suppose that there exists a nonzero element $r \in R$ such that $Mr \neq M$. Consider the homomorphism $f\colon M \to M$ defined by $f(x) = xr$ for all $x \in M$. Since M is d-Rickart, it follows that $\mathrm{Im}\, f = Mr$ is a direct summand of M. Let N be a nonzero submodule of M such that $Mr \oplus N = M$. Then clearly, we have $Nr = 0$, a contradiction.

***Example* 2.5.** (i) From Corollary 2.4, it follows that the $\mathbb{Z}$-module $\mathbb{Q}$ is the only torsion-free sd-Rickart $\mathbb{Z}$-module. So the $\mathbb{Z}$-module $M = \mathbb{Q} \oplus \mathbb{Q}$ is a torsion-free injective d-Rickart $\mathbb{Z}$-module which is not sd-Rickart (see [9], Theorem 2.29).

The torsion submodule of an sd-Rickart R-module M over a commutative domain R is not, in general, a direct summand of M as shown below.

***Remark* 2.4.** Consider again the sd-Rickart $\mathbb{Z}$-module $M = \prod_{p \text{ prime}} \mathbb{Z}/p\mathbb{Z}$ (see Example 2.4). It is easy to see that the torsion submodule of M is $T(M) = \oplus_{p \text{ prime}} \mathbb{Z}/p\mathbb{Z}$ and $T(M)$ is not a direct summand of M.

3. Strongly dual Rickart modules versus gsd-Rickart modules. In this section, we introduce a generalization of sd-Rickart modules and then we compare the two notions.

Definition 3.1. *A module M is called a generalized strongly dual Rickart* (*or a gsd-Rickart*) *module if for every nonzero endomorphism f of M, $\operatorname{Im} f$ contains a nonzero fully invariant direct summand of M.*

Recall that a ring R is called *semipotent* (or an I_0*-ring*) if every right ideal that is not contained in $J(R)$ contains a nonzero idempotent (see [10, p. 257; 15, p. 131, 132]). A ring without nonzero nilpotent elements is called a *reduced* ring.

Proposition 3.1. *The following conditions are equivalent for a ring R*:

(i) *R_R is a gsd-Rickart R-module*;

(ii) *R is a reduced semipotent ring with $J(R) = 0$*;

(iii) *R is an abelian semipotent ring with $J(R) = 0$.*

Proof. (i) $\Rightarrow$ (ii) For every $r \in R$, let $\varphi_r : R \to R$ denote the R-endomorphism of the right R-module R_R defined by $\varphi_r(x) = rx$ for all $x \in R$. By (i), it follows that for every $0 \neq r \in R$, $rR = \operatorname{Im} \varphi_r$ contains a nonzero direct summand of R_R. Therefore, R is a semipotent ring with $J(R) = 0$ (see also [15], Remark 15.3(3)). Now let $0 \neq a \in R$. Then $\operatorname{Im} \varphi_a = aR$. By assumption, there is $b \in R$ such that $0 \neq ab = (ab)^2$ and $(ab)R$ is a fully invariant submodule of R_R (see [1], Proposition 7.1). It is easily seen that $(ab)R$ is a two-sided ideal of R. By [7] (Lemma 3.1), it follows that ab is a central idempotent of R. Therefore, R is a reduced semipotent ring by [15] (Remark 15.3(4)).

(ii) $\Rightarrow$ (i) This follows from [15] (Remark 15.3(4)) and the fact that every endomorphism of R_R is a left multiplication by an element of R.

(ii) $\Leftrightarrow$ (iii) This follows from [15] (Remark 15.3(4)).

It is clear that every sd-Rickart module is gsd-Rickart. The next example shows that the class of gsd-Rickart modules contains strictly the class of sd-Rickart modules.

***Example* 3.1.** (i) From Propositions 2.4 and 3.1, it follows that if R is a commutative semipotent ring such that $J(R) = 0$ and R is not von Neumann regular, then R_R is a gsd-Rickart R-module which is not sd-Rickart.

(ii) Consider the ring $D = \prod_{i=1}^{\infty} A_i$, where each $A_i = \mathbb{Q}$ the field of rational numbers. Let R be the subring of D generated by the ideal $\oplus_{i=1}^{\infty} A_i$ and by the subring $\{(x, x, x, \ldots) \mid x \in \mathbb{Z}\}$. By [15] (Example 15.7(9)), the ring R is a commutative semipotent ring such that $J(R) = 0$ and R is not von Neumann regular. From (i), we conclude that R_R is a gsd-Rickart R-module, but is not an sd-Rickart R-module.

Note that if R is the ring given in Example 3.1(ii), then the R-module R_R is gsd-Rickart, while R_R is not a d-Rickart R-module by [9] (Remark 2.2). The next example exhibits a d-Rickart module which is not gsd-Rickart.

***Example* 3.2.** Let R be a von Neumann regular ring which is not strongly regular (see Example 2.3). By [7] (Theorem 3.5), R is not an Abelian ring. So R_R is not a gsd-Rickart R-module by Proposition 3.1. However, it is not hard to see that R_R is a d-Rickart R-module (see [9], Remark 2.2).

Proposition 3.2. *If a module M is gsd-Rickart*, *then so are its direct summands.*

Proof. Let $M = N \oplus L$ and let φ be an endomorphism of N. Consider the endomorphism $\psi = \mu\varphi\pi$ of M, where $\pi : M \to N$ is the projection map and $\mu : N \to M$ is the inclusion map. As M is gsd-Rickart, there exists a fully invariant direct summand K of M such that $K \subseteq \operatorname{Im}\psi$. Note that $\operatorname{Im}\psi = \operatorname{Im}\varphi$. It follows that K is a fully invariant direct summand of N with $K \subseteq \operatorname{Im}\varphi$.

Proposition 3.3. *The following statements are equivalent for an indecomposable module* M:

(i) M *is an sd-Rickart module*;

(ii) M *is a d-Rickart module*;

(iii) M *is a gsd-Rickart module*;

(iv) *every endomorphism of* M *is an epimorphism.*

Proof. The proof is immediate.

Proposition 3.4. *Let* M *be a gsd-Rickart module. Then*:

(i) *Every indecomposable direct summand* N *of* M *is fully invariant in* M.

(ii) *If* $M = N \oplus L$ *is a direct sum of an indecomposable submodule* N *and a submodule* L, *then* $\operatorname{Hom}_R(N, L) = 0$.

(iii) *If* N *and* K *are indecomposable direct summands of* M *such that* $N \cap K = 0$ *and* $N \oplus K$ *is a direct summand of* M, *then* $\operatorname{Hom}_R(N, K) = \operatorname{Hom}_R(K, N) = 0$.

Proof. (i) Consider the projection map $\pi : M \to N \leq M$. Since M is gsd-Rickart, $N = \operatorname{Im}\pi$ contains a nonzero fully invariant direct summand A of M. Clearly, $A = N$.

(ii) This follows from (i) and Lemma 2.1.

(iii) This follows from (ii).

The next corollary is a direct consequence of Proposition 3.4. This result and Example 3.3 show that the gsd-Rickart property does not go to direct sums of gsd-Rickart modules.

Corollary **3.1.** *For any nonzero indecomposable module* M, *the module* $M \oplus M$ *is not gsd-Rickart.*

Example **3.3.** Let p be a prime integer. It is clear that the $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$ is gsd-Rickart, while the $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{p^\infty}$ is not gsd-Rickart by Corollary 3.1.

Remark **3.1.** (i) Every ring R has an R-module which is not gsd-Rickart. In fact, if S is any simple R-module, then the R-module $S \oplus S$ is not gsd-Rickart (see Corollary 3.1).

(ii) Let (P) be a property of modules which is closed under finite direct sums. Assume that there is at least one indecomposable module which satisfies the property (P). Corollary 3.1 shows that it is not possible to characterize the class of rings R for which every R-module satisfying (P) is a gsd-Rickart module.

Proposition 3.5. *Let a module* $M = \oplus_{i \in I} M_i$ *be a direct sum of indecomposable submodules* M_i, $i \in I$. *Then the following conditions are equivalent*:

(i) M *is an sd-Rickart module*;

(ii) M *is a gsd-Rickart module*;

(iii) M_i *is an sd-Rickart submodule of* M *for all* $i \in I$, *and* $\operatorname{Hom}_R(M_i, M_j) = 0$ *for all distinct* $i, j \in I$.

Proof. (i) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (iii) follows from Propositions 3.2, 3.3 and 3.4.

(iii) $\Rightarrow$ (i) by Theorem 2.1.

Combining Proposition 3.5 with Corollary 2.2, we obtain the following result.

Corollary **3.2.** *The following conditions are equivalent for a nonzero semisimple module* M:

(i) M *is sd-Rickart*;

(ii) M *is gsd-Rickart*;

(iii) $M = \oplus_{i \in I} S_i$ *is a direct sum of simple submodules* S_i, $i \in I$, *such that* S_i *is not isomorphic to* S_j *whenever* $i \neq j$.

***Corollary* 3.3.** *Let* M *be a nonzero module that has either the ascending or the descending chain condition on direct summands. Then* M *is an sd-Rickart module if and only if* M *is gsd-Rickart.*

Proof. This follows from Proposition 3.5 and [1] (Proposition 10.14).

Recall that a module M is said to have *finite uniform dimension* if M does not contain an infinite independent set of submodules (see [4] (5.1) or [6, p. 79]).

A module M is said to have *finite hollow dimension* if for some $n \in \mathbb{N}$, there exists an epimorphism from M to a direct sum of n nonzero modules but no epimorphism from M to a direct sum of more than n nonzero modules (see [4] (5.2)).

***Corollary* 3.4.** *Let* M *be a nonzero module. If* M *satisfies one of the following conditions*:

(i) M *is Noetherian*;

(ii) M *is Artinian*;

(iii) M *has finite hollow dimension*;

(iv) M *has finite uniform dimension.*

Then M *is an sd-Rickart module if and only if* M *is a gsd-Rickart module.*

Proof. This follows from Corollary 3.3, [4] (5.3) and [6] (Theorem 3.14).

***Corollary* 3.5.** *Let* M *be a nonzero Noetherian module over a commutative ring* R. *Then the following statements are equivalent*:

(i) M *is an sd-Rickart* R*-module*;

(ii) M *is a gsd-Rickart* R*-module*;

(iii) $M \cong \oplus_{i=1}^{k} R/P_i$ *for some positive integer* k *and distinct maximal ideals* P_i, $1 \leq i \leq k$, *of* R.

Proof. (i) $\Leftrightarrow$ (ii) follows from Corollary 3.4.

(i) $\Leftrightarrow$ (iii) This is a consequence of [9] (Proposition 4.13) and Corollary 3.2.

Applying Corollary 3.2 to semisimple modules over local rings, we get the following example.

***Example* 3.4.** (i) Let R be a local ring with maximal right ideal m. If M is a semisimple R-module, then M is sd-Rickart if and only if M is gsd-Rickart if and only if $M = 0$ or $M \cong R_R/m$.

(ii) If R is a division ring, then an R-module M is sd-Rickart if and only if M is gsd-Rickart if and only if $M = 0$ or $M \cong R_R$.

Proposition 3.6. *Assume that* R *is a commutative Noetherian hereditary ring* (*e.g.,* R *is a Dedekind domain*). *Then the following conditions are equivalent for a nonzero injective* R*-module* M:

(i) M *is an sd-Rickart* R*-module*;

(ii) M *is a gsd-Rickart* R*-module*;

(iii) *there exist distinct prime ideals* P_i, $i \in I$, *of* R *and submodules* M_i, $i \in I$, *of* M *such that*:

(a) $M = \oplus_{i \in I} M_i$,

(b) *for each* $i \in I$, $M_i \cong E(R/P_i)$,

and

(c) $P_i \nsubseteq P_j$ *whenever* $i \neq j$.

Proof. (i) $\Leftrightarrow$ (ii) follows from [1] (Theorem 25.6) and Proposition 3.5.

(i) $\Rightarrow$ (iii) follows from [1] (Theorem 25.6), [12] (Corollary on page 53 and Proposition 4.21) and Proposition 3.5.

(iii) $\Rightarrow$ (i) Note that each $M_i,\ i \in I,$ is an sd-Rickart R-module by [12, p. 53] (Corollary) and Remark 2.2. Moreover, applying [12] (Proposition 4.21), we conclude that $\mathrm{Hom}_R(M_i, M_j) = 0$ for all distinct $i, j \in I$. Using Theorem 2.1, we see that M is an sd-Rickart R-module.

The next consequence of Proposition 3.6 describes the structure of injective sd-Rickart $\mathbb{Z}$-modules.

***Corollary* 3.6.** *The following statements are equivalent for a nonzero injective $\mathbb{Z}$-module M*:

(i) *M is an sd-Rickart $\mathbb{Z}$-module*;

(ii) *M is a gsd-Rickart $\mathbb{Z}$-module*;

(iii) *$M \cong \mathbb{Q}$ or $M = \oplus_{i \in I} \mathbb{Z}_{p_i^\infty}$, where $p_i,\ i \in I$, are distinct prime integers.*

Proposition 3.7. *Assume that R is a commutative Noetherian local ring with maximal ideal m. Then every nonzero gsd-Rickart reduced R-module is a simple R-module.*

Proof. Let M be a nonzero gsd-Rickart reduced R-module. Then $Mm \neq M$ as $\mathrm{Rad}(M) = Mm$. Assume that $Mm \neq 0$. So there exists a nonzero element $a \in m$ such that $Ma \neq 0$. Consider the endomorphism φ of M defined $\varphi(x) = xa$ for all $x \in M$. Since M is gsd-Rickart, there exists a nonzero direct summand A of M such that $A \subseteq Ma \subseteq Mm$. Let B be a submodule of M such that $A \oplus B = M$. Then $Am \oplus Bm = Mm$ and $A = A \cap Mm = Am \oplus (Bm \cap A) = Am$. Hence $\mathrm{Rad}(A) = Am = A$, a contradiction. It follows that $Mm = 0$ and M is semisimple. Therefore, $M \cong R/m$ (see Example 3.4).

We conclude this paper by describing the structure of sd-Rickart modules and gsd-Rickart modules over discrete valuation rings.

Proposition 3.8. *Let R be a discrete valuation ring with quotient field Q and maximal ideal m. Then the following statements are equivalent for an R-module M*:

(i) *M is an sd-Rickart module*;

(ii) *M is a gsd-Rickart module*;

(iii) *$M = 0$ or $M \cong R/m$ or $M \cong Q$ or $M \cong Q/R$ or $M \cong Q \oplus R/m$.*

Proof. (i) $\Rightarrow$ (ii) This is clear.

(ii) $\Rightarrow$ (iii) Let M_1 be the largest divisible submodule of M. Then there exists a reduced submodule M_2 of M such that $M = M_1 \oplus M_2$. By Propositions 3.2 and 3.6, $M_1 = 0$ or $M_1 \cong Q$ or $M_1 \cong Q/R$. Now applying Propositions 3.2 and 3.7, we conclude that $M_2 = 0$ or $M_2 \cong R/m$. Therefore, $M = 0$ or $M \cong R/m$ or $M \cong Q$ or $M \cong Q/R$ or $M \cong Q \oplus R/m$ by Proposition 3.5.

(iii) $\Rightarrow$ (i) This follows from Proposition 2.6 and Remark 2.2.

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