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ON THE CARDINALITY OF UNIQUE RANGE SETS WITH WEIGHT ONE

ПРО ПОТУЖНІСТЬ УНІКАЛЬНОГО НАБОРУ МНОЖИН З ОДИНИЧНОЮ ВАГОЮ

Two meromorphic functions f and g are said to share the set $S \subset \mathbb{C} \cup \{\infty\}$ with weight $l \in \mathbb{N} \cup \{0\} \cup \{\infty\}$, if $E_f(S,l) = E_g(S,l)$, where

$$E_f(S,l) = \bigcup_{a \in S} \big\{(z,t) \in \mathbb{C} \times \mathbb{N} \mid f(z) = a \text{ with multiplicity } p\big\},$$

where $t = p$ if $p \leq l$ and $t = p+1$ if $p > l$.

In this paper, we improve and supplement the result of L. W. Liao and C. C. Yang [Indian J. Pure and Appl. Math., **31**, № 4, 431 – 440 (2000)] by showing that there exist a finite set S with 13 elements such that $E_f(S,1) = E_g(S,1)$ implies $f \equiv g$.

Дві мероморфні функції f і g сумісно мають множину $S \subset \mathbb{C} \cup \{\infty\}$ з одиничною вагою $l \in \mathbb{N} \cup \{0\} \cup \{\infty\}$, якщо $E_f(S,l) = E_g(S,l)$, де

$$E_f(S,l) = \bigcup_{a \in S} \big\{(z,t) \in \mathbb{C} \times \mathbb{N} \mid f(z) = a \text{ з кратністю } p\big\},$$

за умови, що $t = p$, якщо $p \leq l$, і $t = p+1$, якщо $p > l$.

У цій роботі ми вдосконалюємо та доповнюємо результат L. W. Liao і C. C. Yang [Indian J. Pure and Appl. Math., **31**, № 4, 431 – 440 (2000)], пропонуючи доведення того, що існує скінченна множина S, що складається з 13 елементів, така, що з $E_f(S,1) = E_g(S,1)$ випливає $f \equiv g$.

1. Introduction and definitions. By $\mathbb{C}$ and $\mathbb{N}$, we mean the set of complex numbers and set of natural numbers, respectively. By meromorphic function, we mean an analytic function defined on $\mathbb{C}$ except possibly at isolated singularities, each of which is a pole.

The tool we used in this paper is Nevanlinna theory. For the standard notations of the Nevanlinna theory, one can go through the Hayman's monograph [8].

It will be convenient to let that E be denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For any nonconstant meromorphic function $h(z)$, we denote by $S(r,h)$, any quantity satisfying

$$S(r,h) = o(T(r,h)), \qquad r \longrightarrow \infty, \quad r \notin E.$$

Suppose f and g be two nonconstant meromorphic functions and $a \in \mathbb{C}$. We say that f and g share the value a-CM (counting multiplicities), provided that $f - a$ and $g - a$ have the same zeros with the same multiplicities. Similarly, we say that f and g share the value a-IM (ignoring multiplicities), provided that $f - a$ and $g - a$ have the same set of zeros, where the multiplicities are not taken into account. Moreover, we say that f and g share ∞-CM (resp., IM), if $1/f$ and $1/g$ share 0-CM (resp., IM).

In the course of studying the factorization of meromorphic functions, F. Gross [6] first generalized the idea of value sharing by introducing the concept of a unique range set. Before going to the details of the paper, we first recall the definition of set sharing.

Definition 1.1 [15]. *For a nonconstant meromorphic function f and any set $S \subset \mathbb{C} \cup \{\infty\}$, we define*

$$E_f(S) = \bigcup_{a \in S} \{(z,p) \in \mathbb{C} \times \mathbb{N} \mid f(z) = a \ \textit{with multiplicity} \ p\},$$

$$\overline{E}_f(S) = \bigcup_{a \in S} \{(z,1) \in \mathbb{C} \times \mathbb{N} \mid f(z) = a \ \textit{with multiplicity} \ p\}.$$

If $E_f(S) = E_g(S)$ $\big($resp., $\overline{E}_f(S) = \overline{E}_g(S)\big)$, then it is said that f and g share the set S counting multiplicities or in short CM (resp., ignoring multiplicities or in short IM).

Thus, if S is singleton, then it coincides with the usual definition of value sharing.

In 1977, F. Gross [6] proposed the following problem which has later became popular as "*Gross problem*". The problem was as follows:

Question **A.** Does there exist a finite set S such that any two nonconstant entire functions f and g sharing the set S must be $f \equiv g$?

In 1982, F. Gross and C. C. Yang [7] gave the affirmative answer to the above question as follows:

Theorem A [7]. *Let $S = \{z \in \mathbb{C} : e^z + z = 0\}$. If two entire functions f, g satisfy $E_f(S) = E_g(S)$, then $f \equiv g$.*

In paper [7], they first introduced the terminology unique range set for entire function (in short URSE). Later the analogous definition for meromorphic functions was also introduced in literature.

Definition 1.2 [15]. *Let $S \subset \mathbb{C} \cup \{\infty\}$ and f and g be two nonconstant meromorphic* (*resp., entire*) *functions. If $E_f(S) = E_g(S)$ implies $f \equiv g$, then S is called a unique range set for meromorphic* (*resp., entire*) *functions or in brief URSM* (*resp., URSE*).

In 1997, H. X. Yi [18] introduced the analogous definition for reduced unique range sets.

Definition 1.3 [18]. *A set $S \subset \mathbb{C} \cup \{\infty\}$ is said to be a unique range set for meromorphic* (*resp., entire*) *functions in ignoring multiplicity, in short URSM-IM* (*resp., URSE-IM*) *or a reduced unique range set for meromorphic* (*resp., entire*) *functions, in short RURSM* (*resp., RURSE*) *if $\overline{E}_f(S) = \overline{E}_g(S)$ implies $f \equiv g$ for any pair of nonconstant meromorphic* (*resp., entire*) *functions.*

During the last few years the notion of unique range sets as well as reduced unique range sets have been generating an increasing interest among the researchers. For the literature, one can also go through the research monograph written by C. C. Yang and H. X. Yi [15].

Next we recall the following notations:

$$\lambda_M = \inf\{\sharp(S) \mid S \text{ is an URSM}\} \text{ and } \lambda_E = \inf\{\sharp(S) \mid S \text{ is an URSE}\},$$

where $\sharp(S)$ is the cardinality of the set S.

In 1996, P. Li [11] showed that $\lambda_E \geq 5$ whereas C. C. Yang and H. X Yi [15] established that $\lambda_M \geq 6$. Also, G. Frank and M. Reinders [3] estimated that $\lambda_M \leq 11$. And for entire functions corresponding estimation is $\lambda_E \leq 7$. Till date these estimations are the best.

In course of time, researchers are also paying their attention to find the lowest cardinality of URSM-IM as well as URSE-IM. In 1997, H. X. Yi [18] gave the existence of URSM-IM with 19 elements. After one year, in 1998, H. X. Yi [19] further improved his result [8] and obtained URSM-IM of 17 elements. Also in 1998, M. L. Fang and H. Guo [2] and in 1999, S. Bartels [1] independently gave the existence of URSM-IM with 17 elements. In connection to our discussions, the following question is natural:

***Question* 1.1.** Can one further reduced the lower bound of the unique range sets by relaxing the sharing notations?

As an attempt to reduce the cardinalities of unique range sets, L. W. Liao and C. C. Yang [13] introduced the following notation:

Definition 1.4. *Let f be a nonconstant meromorphic function and $S \subset \mathbb{C} \cup \{\infty\}$. We define*

$$E_{1)}(S, f) = \bigcup_{a \in S} E_{1)}(a, f),$$

where $E_{1)}(a, f)$ is the set of all simple a-points of f.

For a positive integers $n \geq 3$ and $c \neq 0, 1$, we shall denote by $P(z)$, the Frank – Reinders polynomial [3] as

$$P(z) = \frac{(n-1)(n-2)}{2} z^n - n(n-2) z^{n-1} + \frac{n(n-1)}{2} z^{n-2} - c. \tag{1.1}$$

Clearly the restrictions on c implies that $P(z)$ has only simple zeros. Using the methods of Frank – Reinders [3], in connection to the Question 1.1, L. W. Liao and C. C. Yang [13] proved following theorem.

Theorem B [13]. *Suppose that $n(\geq 1)$ be a positive integer. Further suppose that $S = \{z : P(z) = 0\}$, where the polynomial $P(z)$ of degree n defined by* (1.1). *Let f and g be two nonconstant meromorphic functions satisfying $E_{1)}(S, f) = E_{1)}(S, g)$. If $n \geq 15$, then $f \equiv g$.*

The motivation of this paper is to improve and supplement Theorem B utilizing the method of Frank – Reinders [3]. Before going to state our main result, we recall some well-known definitions which will be useful for the proof of the main result of this paper.

Definition 1.5 [10]. *For a nonconstant meromorphic function f and any set $S \subset \mathbb{C} \cup \{\infty\}$, $l \in \mathbb{N} \cup \{0\} \cup \{\infty\}$, we define*

$$E_f(S, l) = \bigcup_{a \in S} \{(z, t) \in \mathbb{C} \times \mathbb{N} \mid f(z) = a \ with \ multiplicity \ p\},$$

where $t = p$ if $p \leq l$ and $t = p + 1$ if $p > l$.

Two meromorphic functions f and g are said to share the set S with weight l, if $E_f(S, l) = E_g(S, l)$.

Clearly $E_f(S) = E_f(S, \infty)$ and $\overline{E}_f(S) = E_f(S, 0)$.

Definition 1.6. *Suppose $a \in \mathbb{C} \cup \{\infty\}$ and $m \in \mathbb{N}$.*

i) *We denote by $N(r, a; f \mid = 1)$, the counting function of simple a-points of f.*

ii) *By $N(r, a; f \mid \leq m)$ $(resp., N(r, a; f \mid \geq m)$, we denote the counting function of those a-points of f whose multiplicities are not greater* (*resp., less*) *than m where each a-point is counted according to its multiplicity.*

Similarly, one can define $\overline{N}(r, a; f \mid \leq m)$ and $\overline{N}(r, a; f \mid \geq m)$ as the reduced counting function of $N(r, a; f \mid \leq m)$ and $N(r, a; f \mid \geq m)$, respectively.

Analogously, $N(r, a; f \mid < m)$, $N(r, a; f \mid > m)$, $\overline{N}(r, a; f \mid < m)$ and $\overline{N}(r, a; f \mid > m)$ are defined.

Definition 1.7. *Suppose that f and g be two nonconstant meromorphic functions such that f and g share $(a, 0)$. Further, suppose that z_0 be an a-point of f with multiplicity p, an a-point of g with multiplicity q.*

i) *We denote by* $\overline{N}_L(r,a;f)$, *the reduced counting function of those* a*-points of* f *and* g *where* $p>q$.

ii) *By* $N_E^{1)}(r,a;f)$, *the counting function of those* a*-points of* f *and* g *where* $p=q=1$.

iii) *By* $\overline{N}_E^{(2}(r,a;f)$, *the reduced counting function of those* a*-points of* f *and* g *where* $p=q\geq 2$. *Similarly, we can define* $\overline{N}_L(r,a;g)$, $N_E^{1)}(r,a;g)$, $\overline{N}_E^{(2}(r,a;g)$.

If f and g share (a,m), $m\geq 1$, then $N_E^{1)}(r,a;f)=N(r,a;f\mid=1)$.

Definition 1.8. *We denote by* $\overline{N}(r,a;f\mid=k)$, *the reduced counting function of those* a*-points of* f *whose multiplicities is exactly* k, *where* $k\geq 2$ *is an integer.*

Definition 1.9 [10]. *Let* f, g *share a value* a *IM. We denote by* $\overline{N}_*(r,a;f,g)$, *the reduced counting function of those* a*-points of* f *whose multiplicities differ from the multiplicities of the corresponding* a*-points of* g. *Clearly*

$$\overline{N}_*(r,a;f,g)\equiv\overline{N}_*(r,a;g,f \qquad \textit{and} \qquad \overline{N}_*(r,a;f,g)=\overline{N}_L(r,a;f)+\overline{N}_L(r,a;g).$$

2. Main results.

Theorem 2.1. *Suppose that* $n(\geq 1)$ *be a positive integer. Further suppose that* $S=\{z: P(z)=0\}$ *where the polynomial* $P(z)$ *of degree* n *is defined by* (1.1). *Let* f *and* g *be two nonconstant meromorphic functions satisfying* $E_f(S,1)=E_g(S,1)$. *If* $n\geq 13$, *then* $f\equiv g$.

***Corollary* 2.1.** *Suppose that* $n(\geq 1)$ *be a positive integer. Further suppose that* $S=\{z:P(z)=$ $=0\}$ *where the polynomial* $P(z)$ *of degree* n *is defined by* (1.1). *Let* f *and* g *be two nonconstant entire functions satisfying* $E_f(S,1)=E_g(S,1)$. *If* $n\geq 8$, *then* $f\equiv g$.

3. Lemmas. We define for any two nonconstant meromorphic functions f and g

$$Q(z)=\frac{P(z)+c}{c}, \qquad F=Q(f), \qquad G=Q(g).$$

Henceforth, we shall denote by H the following function:

$$H=\left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)-\left(\frac{G''}{G'}-\frac{2G'}{G-1}\right).$$

Lemma 3.1 [14]. *Let* f *be a nonconstant meromorphic function and*

$$R(f)=\frac{\sum_{k=0}^{n}a_kf^k}{\sum_{j=0}^{m}b_jf^j}$$

be an irreducible rational function in f *with constant coefficients* $\{a_k\}$ *and* $\{b_j\}$, *where* $a_n\neq 0$ *and* $b_m\neq 0$. *Then*

$$T(r,R(f))=dT(r,f)+S(r,f),$$

where $d=\max\{n,m\}$.

Lemma 3.2 [15]. *For a nonconstant meromorphic function* f,

$$T\left(r,\frac{1}{f}\right)=T(r,f)+O(1),$$

where $O(1)$ *is a bounded quantity.*

Lemma 3.3 [15]. *For a nonconstant meromorphic function f and for a complex number $a \in \in \mathbb{C} \cup \{\infty\}$*

$$T\left(r, \frac{1}{f-a}\right) = T(r,f) + O(1),$$

where $O(1)$ is a bounded quantity depending on a.

Lemma 3.4 [15, p. 23]. *Suppose that f is a nonconstant meromorphic function in the complex plane and $a_1, a_2, \ldots, a_q$ are $q(\geq 3)$ distinct values in $\mathbb{C} \cup \{\infty\}$. Then*

$$(q-2)T(r,f) < \sum_{j=1}^{q} \overline{N}(r, a_j; f) + S(r,f),$$

where $S(r,f)$ is a quantity such that $\dfrac{S(r,f)}{T(r,f)} \to 0$ as $r \to +\infty$ out side of a set E in $(0,\infty)$ with finite linear measure.

A polynomial $P(z)$ over $\mathbb{C}$ is called a *uniqueness polynomial* for meromorphic (resp., entire) functions if for any two nonconstant meromorphic (resp., entire) functions f and g, $P(f) \equiv P(g)$ implies $f \equiv g$.

In 2000, H. Fujimoto [4] first discovered a special property of a polynomial which was later termed as critical injection property. A polynomial $P(z)$ is said to satisfy *critical injection property* if $P(\alpha) \neq P(\beta)$ for any two distinct zeros α, β of the derivative $P'(z)$.

Let $P(z)$ be a monic polynomial without multiple zero whose derivatives has mutually distinct k-zeros given by $d_1, d_2, \ldots, d_k$ with multiplicities $q_1, q_2, \ldots, q_k$ respectively. The following theorem of Fujimoto helps us to find many uniqueness polynomials.

Lemma 3.5 [5]. *Suppose that $P(z)$ satisfy critical injection property. Then $P(z)$ will be a uniqueness polynomial if and only if*

$$\sum_{1 \leq l < m \leq k} q_l q_m > \sum_{l=1}^{k} q_l.$$

In particular, the above inequality is always satisfied whenever $k \geq 4$. If $k = 3$ and $\max\{q_1, q_2, q_3\} \geq \geq 2$ or $k = 2$, $\min\{q_1, q_2\} \geq 2$ and $q_1 + q_2 \geq 5$, then also the above inequality holds.

4. Proof of Theorem 2.1. By the assumption, it is clear that F and G shares $(1,1)$. Now we consider two cases.

***Case* 1.** First we assume that $H \not\equiv 0$. Then by simple calculations, we have

$$N(r,\infty;H) \leq \overline{N}(r,0;F| \geq 2) + \overline{N}(r,0;G| \geq 2) + \overline{N}(r,\infty;F) +$$
$$+ \overline{N}(r,\infty;G) + \overline{N}_*(r,1;F,G) + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G'),$$

where $\overline{N}_0(r,0;F')$ is the reduced counting function of zeros of F' which is not zeros of $F(F-1)$. Similarly $\overline{N}(r,0;G')$ is defined. Thus,

$$N(r,\infty;H) \leq \overline{N}(r,0;f) + \overline{N}(r,0;g) + \overline{N}(r,1;f) +$$
$$+ \overline{N}(r,1;g) + \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) +$$
$$+ \overline{N}_*(r,1;F,G) + \overline{N}_\star(r,0;f') + \overline{N}_\star(r,0;g'), \tag{4.1}$$

where $\overline{N}_{\star}(r,0;f')$ is the reduced counting function of zeros of f' which is not zeros of $f(f-1)$ and $(F-1)$. We denote by $\overline{N}_{*}(r,1;F,G)$ the reduced counting function of those 1-points of F whose multiplicities differ from the multiplicities of the corresponding 1-points of G. Clearly

$$\overline{N}(r,1;F|=1)=\overline{N}(r,1;G|=1)\leq N(r,\infty;H), \tag{4.2}$$

where $\overline{N}(r,1;F|=1)$ is the counting function of those simple 1-points of F which are also simple 1-points of G.

Now using the Second Fundamental Theorem, equations (4.1) and (4.2), we get

$$\begin{gathered}(n+1)(T(r,f)+T(r,g))\leq\\ \leq\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;f)+\overline{N}(r,0;g)+\\ +\overline{N}(r,1;f)+\overline{N}(r,1;g)+\overline{N}(r,1;F)+\overline{N}(r,1;G)-\\ -N_{\star}(r,0,f')-N_{\star}(r,0,g')+S(r,f)+S(r,g)\leq\\ \leq 2\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;f)+\overline{N}(r,0;g)+\\ +\overline{N}(r,1;f)+\overline{N}(r,1;g)\}+\{\overline{N}(r,1;F)+\overline{N}(r,1;G)-\\ -\overline{N}(r,1;F|=1)\}+\overline{N}_{*}(r,1;F,G)+S(r,f)+S(r,g).\end{gathered} \tag{4.3}$$

Again

$$\begin{gathered}\overline{N}(r,1;F)+\overline{N}(r,1;G)-\overline{N}(r,1;F|=1)+\overline{N}_{*}(r,1;F,G)\leq\\ \leq\frac{1}{2}\{N(r,1;F)+N(r,1;G)\}+\frac{1}{2}\{N(r,1;F|\geq 2)+N(r,1;,G|\geq 2)\}\leq\\ \leq\frac{n}{2}\left(T(r,f)+T(r,g)\right)+\frac{1}{2}\{N(r,0;f'\mid f\neq 0)+N(r,0;g'\mid g\neq 0)\}.\end{gathered} \tag{4.4}$$

Using (4.3) and (4.4), we get

$$\begin{gathered}\left(\frac{n}{2}-3\right)(T(r,f)+T(r,g))\leq\\ \leq 2\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\}+\frac{1}{2}\left\{N\left(r,0;\frac{f'}{f}\right)+N\left(r,0;\frac{g'}{g}\right)\right\}+\\ +S(r,f)+S(r,g)\leq\\ \leq 2\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\}+\frac{1}{2}\left\{N\left(r,\infty;\frac{f'}{f}\right)+N\left(r,\infty;\frac{g'}{g}\right)\right\}+\\ +S(r,f)+S(r,g)\leq\\ \leq\frac{5}{2}\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\}+\frac{1}{2}\{\overline{N}(r,0;f)+\overline{N}(r,0;g)\}+S(r,f)+S(r,g).\end{gathered}$$

That is

$$(n-7)(T(r,f)+T(r,g))\leq 5\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\}+S(r,f)+S(r,g),$$

which is a contradiction if $n\geq 13$ (resp., 8) for meromorphic (resp., entire) case.

***Case* 2.** Next we assume that $H \equiv 0$. Now on integration two times, we have

$$F \equiv \frac{AG + B}{CG + D}, \tag{4.5}$$

where A, B, C, D are constant satisfying $AD - BC \neq 0$.

Thus, applying Lemma 3.1 in equation (4.5), we get

$$T(r, f) = T(r, g) + O(1). \tag{4.6}$$

Next we consider the following two cases.

***Subcase* 2.1.** First we assume that $AC \neq 0$. Then equation (4.5) can be written as

$$F - \frac{A}{C} = \frac{BC - AD}{C(CG + D)}.$$

Thus,

$$\overline{N}\left(r, \frac{A}{C}; F\right) = \overline{N}(r, \infty; G).$$

Now applying the Second Fundamental Theorem and equation (4.6), we obtain

$$nT(r, f) + O(1) = T(r, F) \leq$$

$$\leq \overline{N}(r, \infty; F) + \overline{N}(r, 0; F) + \overline{N}\left(r, \frac{A}{C}; F\right) + S(r, F) \leq$$

$$\leq \overline{N}(r, \infty; f) + \overline{N}(r, 0; f) + 2T(r, f) + \overline{N}(r, \infty; g) + S(r, f) \leq 5T(r, f) + S(r, f)$$

which is impossible as $n > 5$.

***Subcase* 2.2.** Thus we consider $AC = 0$. Since $AD - BC \neq 0$, so $A = C = 0$ never occur. Thus the following two subcases are obvious:

2.2.1. First we assume that $A = 0$ and $C \neq 0$. Then obviously $B \neq 0$. Hence, equation (4.5) can be written as

$$F \equiv \frac{1}{\gamma G + \delta}, \tag{4.7}$$

where $\gamma = \dfrac{C}{B}$ and $\delta = \dfrac{D}{B}$. Now we assume that there exist no z_0 such that $F(z_0) = 1$. Then applying the Second Fundamental Theorem and using the equation (4.6), we have

$$T(r, F) \leq \overline{N}(r, \infty; F) + \overline{N}(r, 0; F) + \overline{N}(r, 1; F) + S(r, F) \leq$$

$$\leq \overline{N}(r, \infty; f) + \overline{N}(r, 0; f) + 2T(r, f) + S(r, f) \leq \frac{4}{n} T(r, F) + S(r, F)$$

which is impossible as $n > 4$.

Thus, there exist atleast one z_0 such that $F(z_0) = 1$. Since F and G share 1, hence, $\gamma + \delta = 1$ with $\gamma \neq 0$. Thus, equation (4.7) becomes

$$F \equiv \frac{1}{\gamma G + 1 - \gamma},$$

and

$$\overline{N}\left(r, 0; G + \frac{1-\gamma}{\gamma}\right) = \overline{N}(r, \infty; F).$$

If $\gamma \neq 1$, then the Second Fundamental Theorem and equation (4.6), imply

$$T(r, G) \leq \overline{N}(r, \infty; G) + \overline{N}(r, 0; G) + \overline{N}\left(r, 0; G + \frac{1-\gamma}{\gamma}\right) + S(r, G) \leq$$

$$\leq \overline{N}(r, \infty; g) + \overline{N}(r, 0; g) + 2T(r, g) + \overline{N}(r, \infty; f) + S(r, g) \leq \frac{5}{n}T(r, F) + S(r, F),$$

which is again impossible as $n > 5$. Hence, $\gamma = 1$, i.e., $FG \equiv 1$. Then

$$f^{n-2}\prod_{i=1}^{2}(f - \gamma_i)\; g^{n-2}\prod_{i=1}^{2}(g - \gamma_i) \equiv \frac{4c^2}{(n-1)^2(n-2)^2}, \tag{4.8}$$

where γ_i, $i = 1, 2$, are the roots of the equation $z^2 - \dfrac{2n}{n-1}z + \dfrac{n}{n-2} = 0$.

Let z_0 be a γ_i-point of f of order p. Then z_0 must be a pole of g (say of order q). Then $p = nq \geq n$. So,

$$\overline{N}(r, \gamma_i; f) \leq \frac{1}{n}N(r, \gamma_i; f).$$

Again, let z_0 be a zero of f of order t. Then z_0 must be a pole of g (say of order s). Then $(n-2)t = ns$. Thus, $t > s$. Now $2s = (n-2)(t-s) \geq (n-2)$. Consequently $(n-2)t = ns$ gives $t \geq \dfrac{n}{2}$. So,

$$\overline{N}(r, 0; f) \leq \frac{2}{n}N(r, 0; f).$$

Similar calculations are valid for g also. Again

$$\overline{N}(r, \infty; f) \leq \overline{N}(r, 0; g) + \sum_{i=0}^{2}\overline{N}(r, \gamma_i; g) \leq$$

$$\leq \frac{2}{n}N(r, 0; g) + \frac{1}{n}\sum_{i=0}^{2}N(r, \gamma_i; g) \leq \frac{4}{n}T(r, g) + O(1).$$

Next we apply the Second Fundamental Theorem for the identity (4.8) and we get

$$2T(r, f) \leq \overline{N}(r, \infty; f) + \overline{N}(r, 0; f) + \sum_{i=0}^{2}\overline{N}(r, \gamma_i; f) + S(r, f) \leq$$

$$\leq \frac{4}{n}T(r, f) + \frac{2}{n}T(r, f) + \frac{2}{n}T(r, f) + S(r, f)$$

which is a contradiction as $n \geq 5$.

2.2.2. Thus we consider that $A \neq 0$ and $C = 0$. Then obviously $D \neq 0$ and the equation (4.5) can be written as

$$F \equiv \lambda G + \mu,$$

where $\lambda = \dfrac{A}{D}$ and $\mu = \dfrac{B}{D}$.

Then obviously there exist atleast one z_0 such that $F(z_0) = 1$, otherwise we arrived at a contradiction like previous subcase. Thus, $\lambda + \mu = 1$ with $\lambda \neq 0$. Hence,

$$\overline{N}\left(r,0;G+\frac{1-\lambda}{\lambda}\right)=\overline{N}(r,0;F).$$

If $\lambda\neq 1$, then using the Second Fundamental Theorem and equation (4.6), we obtain

$$T(r,G)\leq\overline{N}(r,\infty;G)+\overline{N}(r,0;G)+\overline{N}\left(r,0;G+\frac{1-\lambda}{\lambda}\right)+S(r,G)\leq$$

$$\leq\overline{N}(r,\infty;g)+\overline{N}(r,0;g)+2T(r,g)+\overline{N}(r,0;f)+2T(r,f)+S(r,g)\leq\frac{7}{n}T(r,G)+S(r,G)$$

which is a contradiction as $n>7$. Thus, $\lambda=1$, i.e., $F\equiv G$, that is, $P(f)\equiv P(g)$.

Since

$$P'(z)=\frac{n(n-1)(n-2)}{2}z^{n-3}(z-1)^2$$

and $P(0)\neq P(1)$, So $P(z)$ satisfy critical injection property. Thus, in view of Lemma 3.5, $P(z)$ is a uniqueness polynomial. Hence, $f\equiv g$.

Theorem 2.1 is proved.

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