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ON SKOROKHOD DIFFERENTIABLE MEASURES*

ПРО МІРИ, ДИФЕРЕНЦІЙОВНІ ЗА СКОРОХОДОМ

This paper is a survey of Skorohod** differentiability of measures on linear spaces, which also gives new proofs of some key results in this area along with some new observations.

Наведено огляд диференційовності мір за Скороходом на лінійних просторах, який містить також нові доведення деяких ключових результатів у цій області разом із низкою нових спостережень.

1. Introduction. Among the major contributions to measure theory of Anatolii Vladimirovich Skorohod, one of the most eminent probabilists of the XX century, it is customary to cite several results of especial significance: the Skorohod representation of weakly convergent sequences of measures by almost surely convergent sequences of mappings, his approach to stochastic equations where weak solutions are interpreted as measures on path spaces, the Skorohod space and his fundamental results on weak convergence of measures on path spaces, the proof of exponential integrability of norms of Gaussian vectors (prior to Fernique's celebrated theorem), and Skorohod's differentiability of measures. All these seminal achievements — particularly the last one, the most analytic in this list — have a clear probabilistic flavor, and definitely belong to measure theory and all have been very well presented in many surveys and monographs. In particular, the first topic is discussed in detail in my books [11] and [16] and the last one is thoroughly covered in [12]. Nevertheless, I find Skorohod's treatment of differentiation on the space of measures the most appropriate subject for this memorial issue, and there are two reasons for this. One is a personal reason and the other one is connected with recent activities in the study of BV spaces on spaces with measures and surface measures in infinite dimensions.

Thus, this paper is a brief survey of Skorohod differentiability of measures, which also gives new proofs of some key results in this area along with some new observations.

Skorohod differentiability of measures is one of several of the most natural options for differentiability of families of measures. It is a classical situation in probability theory and mathematical statistics that one has a family of measures μ_t on a measurable space $(\Omega, \mathcal{B})$ depending on a parameter t from an interval T or a more general parametric space with a certain differentiable structure. What does it mean that $t \mapsto \mu_t$ is differentiable? In many applications, a quite common setting is this: the measures μ_t are given by densities f_t with respect to a fixed measure λ on Ω, which reduces the question to differentiable mappings on T with values in a suitable space of functions, such as $L^1(\lambda)$ or $L^2(\lambda)$. Of course, one can also speak of the usual differentiability of the function $t \mapsto f_t(x)$ for fixed x. However, the whole space of bounded measures $\mathcal{M}_{\mathcal{B}}$ on $(\Omega, \mathcal{B})$ possesses a

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** (Editor's remark) We have preserved the author's spelling "A. V. Skorohod" across the paper. This spelling was chosen for the English translation of the book "Интегрирование в гильбертовом пространстве", where the notation of (Skorohod) differentiability of measures was introduced (see [49]).

number of different important topologies which do not reduce to topologies on function spaces. First of all, $\mathcal{M}_{\mathcal{B}}$ is a Banach space with respect to the total variation norm $\|\cdot\|$, which leads to a natural differentiation of mappings with values in this space. Next, there are two natural locally convex topologies on $\mathcal{M}_{\mathcal{B}}$. One is the topology of setwise convergence generated by the seminorms

$$\mu \mapsto \big|\mu(B)\big|, \quad B \in \mathcal{B},$$

and the other one is the topology of duality with the space of bounded $\mathcal{B}$-measurable functions, in which generating seminorms have the form

$$\mu \mapsto \left|\int_X f\, d\mu\right|,$$

where f is some bounded $\mathcal{B}$-measurable function. The latter is somewhat stronger than the former. Finally, if X is a completely regular space and $\mathcal{B}$ is its Baire or Borel σ-algebra, then there is a yet weaker topology given by duality with the space $C_b(X)$ of bounded continuous functions on X, i.e., the topology induced by the seminorms

$$\mu \mapsto \left|\int_X f\, d\mu\right|, \quad f \in C_b(X).$$

This is precisely the topology relevant for our discussion of Skorohod's differentiability. Note that this topology is not comparable on the whole space of measures with the topology of setwise convergence, but on subsets bounded in variation and on the set of nonnegative measures it is weaker than the topology of setwise convergence.

All these topologies can be used for analysis of arbitrary families of measures depending on a parameter (see [12, 50, 51]), but the Skorohod definition (as well as the Fomin definition) deals with a quite specific situation where the family of measures consists of shifts of a single measure along a fixed vector, that is, has the form μ_{th}, where

$$\mu_h(B) := \mu(B+h)$$

for a measure μ defined on the Baire σ-algebra $\mathcal{B}a(X)$ of a locally convex space X (the σ-algebra generated by all continuous functions on X) or on the broader Borel σ-algebra $\mathcal{B}(X)$ (the one generated by all open sets). In terms of integrals, we have

$$\int_X f(x)\, \mu_h(dx) = \int_X f(x-h)\, \mu(dx).$$

It should be noted that differentiability of such families had already been considered by Pitcher [44] for distributions of diffusion processes (with constant diffusion coefficients) in the path space. However, Pitcher was dealing with a stronger kind of differentiability, which was a particular case of Fomin's differentiability introduced in 1966 in [34] (see also [5, 32, 35]). Precise definitions are given in the next section.

2. Notation and terminology. Throughout the paper X will stand for a real locally convex space. The space of bounded continuous functions on X with its sup-norm is denoted by $C_b(X)$. We recall some basic concepts connected with Radon measures on locally convex spaces (see [11] or [26] for details), but the essence of the problems we discuss is such that without significant loss the reader can assume that we deal with separable Hilbert spaces, as it was in the original construction of Skorohod. The general framework of locally convex spaces is just more natural for Skorohod's differentiability, because it actually simplifies matters making them "coordinate-free".

A measure on a σ-algebra $\mathcal{A}$ in a space X will always mean a real (finite) countably additive measure. A signed measure μ is the difference of two mutually singular nonnegative measures μ^+ and μ^-, called its positive and negative parts. The measure $|\mu| = \mu^+ + \mu^-$ is called the total variation of μ and $\|\mu\| = |\mu|(X)$ is called the variation norm of μ. Convergence in this norm is called convergence in variation. An equivalent norm is given by $\sup_{A\in\mathcal{A}} |\mu(A)|$.

A measure ν on $\mathcal{A}$ is called absolutely continuous with respect to a measure μ on $\mathcal{A}$ if $\nu(A) = 0$ when $|\mu|(A) = 0$. This is denoted by $\nu \ll \mu$, and in this case there is a function $\varrho \in L^1(|\mu|)$, called the Radon – Nikodym derivative of ν with respect to μ, such that ν is given by density ϱ with respect to μ, which is written as $\nu = \varrho \cdot \mu$.

The Borel σ-algebra $\mathcal{B}(X)$ of a topological space X is generated by all open sets. The Baire σ-algebra $\mathcal{B}a(X)$ is the smallest σ-algebra with respect to which all continuous functions are measurable, i.e., it is generated by all sets $\{f > 0\}$, where $f \in C_b(X)$. If X is a metric space, then $\mathcal{B}(X) = \mathcal{B}a(X)$, but on general topological spaces the Baire σ-algebra can be strictly smaller, for example, this happens if X is an uncountable power of the real line (in other words, the space of all real functions on an uncountable set with the topology of pointwise convergence).

Borel and Baire measures are measures on Borel and Baire sets, respectively. As noted above, there is no difference between these two classes of measures on a metric space.

A Borel measure on a topological space X is called Radon if, for every Borel set B and every $\varepsilon > 0$, there is a compact set $K \subset B$ such that $|\mu|(B\backslash K) < \varepsilon$. On a complete separable metric space, all Borel measures are Radon.

A family $\mathcal{M}$ of Radon measures on X is called uniformly tight if, for every $\varepsilon > 0$, there is a compact set K such that $|\mu|(X\backslash K) < \varepsilon$ for all $\mu \in \mathcal{M}$. If $\mathcal{M}$ is bounded in variation, then uniform tightness implies that $\mathcal{M}$ has compact closure in the weak topology. The converse is true for complete separable metric spaces, but not for arbitrary spaces (even simple ones such as the space of rational numbers).

The Sobolev space $W^{1,1}(\mathbb{R}^n)$ consists of integrable functions f such that their generalized partial derivatives $\partial_{x_i} f$ are also integrable functions. The space $BV(\mathbb{R}^n)$ of functions of bounded variation consists of integrable functions f such that their generalized partial derivatives $\partial_{x_i} f$ are bounded measures (see [53]). For example, the indicator function of $[0, 1]$ belongs to $BV(\mathbb{R})$, but not to $W^{1,1}(\mathbb{R})$, its generalized derivative is the difference of Dirac's measures at 0 and 1.

Definition 2.1. *Let X be a linear space, let $\mathcal{A}$ be a σ-algebra of subsets of X, and let μ be a measure on $\mathcal{A}$. The measure μ is said to be Fomin differentiable along a vector $h \in X$ such that $A + th \in \mathcal{A}$ for all $A \in \mathcal{A}$ and $t \in \mathbb{R}$ if there is a finite limit*

$$d_h\mu(A) := \lim_{t\to 0} \frac{\mu(A + th) - \mu(A)}{t} \quad \forall A \in \mathcal{A}.$$

It follows by the Nikodym theorem (which asserts that a sequence of countably additive measures on a σ-algebra bounded on every set is bounded in variation and that if this sequence converges on every set, then the limit is a countably additive measure, see, e.g., [11], § 4.6) such that $A \mapsto d_h\mu(A)$ is automatically a bounded measure on $\mathcal{A}$ absolutely continuous with respect to μ. It is called the Fomin derivative of μ along h. The Radon – Nikodym density β_h^μ of $d_h\mu$ with respect to μ is called the logarithmic derivative of μ along h.

The terminology is explained by the fact that if μ is a measure on the real line, then it is Fomin differentiable along 1 if and only if it has an absolutely continuous density ϱ with respect to Lebesgue measure with $\varrho' \in L^1(\mathbb{R})$ and then $d_1\mu = \varrho'\, dx$ and $\beta_1^\mu = \varrho'/\varrho$.

The situation is similar on $\mathbb{R}^n$: a measure μ on $\mathbb{R}^n$ is Fomin differentiable along all basis vectors e_i if and only if μ has a density ϱ from the Sobolev class $W^{1,1}(\mathbb{R}^n)$, in which case $d_{e_i}\mu$ has density $\partial_{x_i}\varrho$ and $\beta_{e_i}\mu = \partial_{x_i}\varrho/\varrho$.

Thus, Fomin's differentiability corresponds to the topology of setwise convergence. However, it was shown in [5] that it yields an a priori stronger differentiability in the total variation norm: one automatically has

$$\lim_{t\to 0}\left\|\frac{\mu_{th}-\mu}{t} - t d_h\mu\right\| = 0.$$

This definition is quite similar to the definition of the partial derivative $\partial_h f$ for a function f on a locally convex space X and a vector $h \in X$ by the equality

$$\partial_h f(x) = \lim_{t\to 0}\frac{f(x+th)-f(x)}{t},$$

whenever this finite limit exists.

Yet another convenient description of Fomin's differentiability is available for Radon measures on locally convex spaces: such a measure μ is Fomin differentiable along a vector h if and only if there is a Radon measure ν absolutely continuous with respect to μ such that the integration by parts formula

$$\int_X \partial_h f(x)\,\mu(dx) = -\int_X f(x)\,\nu(dx) \tag{2.1}$$

holds for all functions f from the class $\mathcal{FC}(X)$ of all functions on X of the form $f(x) = f_0\big(l_1(x),\ldots,l_n(x)\big)$, where $f_0 \in C_b^\infty(\mathbb{R}^n)$ and $l_1,\ldots,l_n \in X^*$. In this case we have $\nu = d_h\mu$.

This integration by parts formula can be rewritten without using derivatives as follows:

$$\int_X \big[f(x+th)-f(x)\big]\mu(dx) = -\int_0^t\int_X f(x+sh)\,\nu(dx)\,ds \tag{2.2}$$

for all functions f of the form indicated above, hence, also for all bounded Borel functions f, which can be also written as

$$\mu_{th} - \mu = \int_0^t \nu_{sh}\,ds.$$

The equivalence of both identities for smooth cylindrical functions f is verified by differentiation in t. Once identity (2.2) holds for smooth cylindrical functions, it extends to bounded Borel functions, because Radon measures with equal integrals on cylindrical functions must coincide.

Definition 2.2. *A Radon measure μ on a locally convex space X is called Skorohod differentiable along a vector $h \in X$ if there exists a Radon measure ν such that the measures $(\mu_{th} - \mu)/t$ converge weakly to ν as $t \to 0$, that is,*

$$\lim_{t\to 0} \int\limits_X \frac{f(x+th) - f(x)}{t}\, \mu(dx) = -\int\limits_X f(x)\,\nu(dx) \quad \forall\, f \in C_b(X).$$

The measure ν is called the Skorohod derivative of μ along h and denoted by $d_h\mu$.

We shall see below that it is sufficient that the sequence of measures $(\mu_{t_n h} - \mu)/t_n$ be bounded in variation for some sequence $t_n \to 0$ of nonzero numbers. Hence it suffices that this sequence be Cauchy in the weak topology. This was the original definition of Skorohod in the case of Hilbert spaces (see [49], § 21), which was followed by the proof of existence of the weak limit ν. However, there is a general result of A. D. Alexandroff (see [2] or the modern exposition in [11, 16]) that on an arbitrary topological space, every sequence of measures μ_n on the Baire σ-algebra that is Cauchy in the weak topology converges weakly to a measure on the Baire σ-algebra. Hence on complete separable metric spaces (we recall that on such spaces all Baire measures are automatically Radon) the limit measure is always Radon. However, on general spaces the weak limit of a sequence of Radon measures can fail to be Radon. Nevertheless, we shall see below that such a thing cannot occur for sequences arising in connection with Skorohod's differentiation.

Definition 2.3. *A Radon measure μ on a locally convex space X is called continuous along a vector h if $\lim_{t\to 0} \|\mu_{th} - \mu\| = 0$.*

This property was considered in [5] and later studied in [47] and other papers. The continuity along h is equivalent to a formally weaker property that for every Borel set B the function $t \mapsto \mu(B+ + th)$ is continuous. The continuity along h follows from Fomin's or Skorohod's differentiability.

Mixed derivatives $d_{h_1} \ldots d_{h_n}\mu$ and higher order derivatives $d_h^n\mu$ (in the Fomin or Skorohod sense) are defined inductively.

The Fourier transform of a Radon measure μ on a locally convex space X with the topological dual X^* is defined by the equality

$$\widetilde{\mu}(l) = \int\limits_X \exp(il(x))\mu(dx), \quad l \in X^*.$$

Obviously, this function can be also defined for any measure μ defined on the σ-algebra generated by X^*. It is known that Radon measures with equal Fourier transforms coincide (see [11], § 7.13).

The convolution of Radon measures μ and ν on a locally convex space X is defined (see [11, p. 146]) by the formula

$$\mu * \nu(B) = \int\limits_X \mu(B-x)\,\nu(dx).$$

It is known that this is also a Radon measure and $\widetilde{\mu * \nu} = \widetilde{\mu}\widetilde{\nu}$, hence $\mu * \nu = \nu * \mu$.

Good examples of differentiable measures are Gaussian measures. Recall (see [10]) that a Radon probability measure γ on a locally convex space X is called centered Gaussian if every continuous linear functional on X is a centered Gaussian random variable with respect to γ. The Cameron–Martin space H of γ consists of all vectors h with finite norm

$$|h|_H = \sup\bigl\{l(h) : l \in X^*,\ \|l\|_{L^2(\gamma)} \le 1\bigr\}.$$

This subspace coincides with the set of all vectors h such that γ_h is equivalent to γ. It is readily verified that it also coincides with the set of vectors of Fomin's or Skorohod's differentiability.

The fact that here the set of vectors of differentiability is much smaller than the whole space (and has measure zero) is not occasional: this is always the case for any nonzero measure on an infinite-dimensional space.

An immediate corollary of Skorohod's definition is the integration by parts formula

$$\int\limits_X \partial_h f(x)\,\mu(dx) = -\int\limits_X f(x)\,d_h\mu(dx)$$

for every function $f \in C_b(X)$ such that $\partial_h f(x)$ exists and is uniformly bounded. However, we shall see below that this formula remains valid even for Borel functions f integrable with respect to $d_h\mu$ such that $\partial_h f$ exists and is μ-integrable.

3. Basic properties of Skorohod differentiable measures. Here we give a brief overview of basic properties of Skorohod differentiable measures, some results are given with proofs, others with references.

First of all, it is readily verified that if μ is Skorohod differentiable along vectors h and k, then it is also differentiable along $sh + tk$ and

$$d_{sh+tk}\mu = sd_h\mu + td_k\mu.$$

Next, for any Radon measure σ on the same space X, we obtain

$$d_h(\mu * \sigma) = d_h\mu * \sigma.$$

If σ is a Radon measure on a locally convex space Y, then

$$d_{(h,0)}(\mu \otimes \sigma) = d_h\mu \otimes \sigma.$$

Finally, if $T\colon X \to Y$ is a continuous affine mapping, then for the image $\mu \circ T^{-1}$ of μ under T we have

$$d_{Th}(\mu \circ T^{-1}) = (d_h\mu) \circ T^{-1}.$$

Let us proceed to some less obvious properties.

Theorem 3.1. *Let μ be a Radon measure on a locally convex space X Skorohod differentiable along a vector h. Then* (2.2) *holds with ν equal to its Skorohod derivative $d_h\nu$ for all bounded Borel functions f. Conversely, if there is a Radon measure ν such that* (2.2) *holds for all smooth cylindrical functions f, then μ is Skorohod differentiable along h and ν is its Skorohod derivative.*

Proof. As explained above, (2.2) for smooth cylindrical functions is equivalent to the same identity for all bounded Borel functions. If μ is Skorohod differentiable, then we have (2.1) for smooth cylindrical functions f. Then (2.2) also holds, which is verified by differentiation in t.

Conversely, if we have (2.2) for smooth cylindrical functions, then it holds for $f \in C_b(X)$, but for such functions the right-hand side of this equality is differentiable in t and the corresponding derivative at zero is the integral of $-f$ against ν, which shows that μ is Skorohod differentiable and ν is its Skorohod derivative.

Theorem 3.1 is proved.

We shall see below that it suffices to have (2.2) with some Baire measure ν that is not a priori Radon, but then automatically such a measure extends to a Radon measure (of course, this may be of interest only for those who deal with spaces on which not all Baire measures are Radon).

***Corollary* 3.1.** *Suppose that a Radon measure μ is Skorohod differentiable along a vector h. It is Fomin differentiable along h if and only if $d_h\mu \ll \mu$, which is also equivalent to the continuity of $d_h\mu$ along h.*

Proof. If μ is Fomin differentiable along h, then $d_h\mu \ll \mu$ (see [5], [12] or [25]). Conversely, if $d_h\mu$ is absolutely continuous with respect to μ, then $d_h\mu$ is continuous along h (see [12]), but then the right-hand side of (2.2) is differentiable in t, because the function under the integral sign is continuous in s.

It follows that if μ is twice Skorohod differentiable along h, then it is Fomin differentiable. Therefore, if μ is Skorohod differentiable along h, then $\mu * \mu$ is Fomin differentiable along h.

***Corollary* 3.2.** *A Radon measure μ on a locally convex space X is Skorohod differentiable along a vector h is and only if there is a Radon measure ν such that*

$$\widetilde{\nu}(l) = il(h)\widetilde{\mu}(l) \quad \forall\, l \in X^*.$$

In this case, $\nu = d_h\mu$.

Proof. Both implications follow easily from (2.2). If $d_h\mu$ exists, then the integration by parts formula gives the desired equality, since $\partial_h \exp(il) = il(h)\exp(il)$. Conversely, if ν satisfies the stated equality, then (2.2) holds for $f = \exp(il)$, which is easily verified by differentiation in t. Then it follows that

$$\mu_{th} - \mu = \int_0^t \nu_{sh}\, ds,$$

which gives our claim.

The following simple observation seems to be new, it is of interest for the following reason: in general locally convex spaces, where the Prokhorov theorem fails, a weakly convergent sequence of Radon measures with a Radon limit need not be uniformly tight, but in the specific situation of measures of the form $(\mu_{th} - \mu)/t$ the situation is the same as in Polish spaces.

Theorem 3.2. *Suppose that a Radon measure μ on a locally convex space X is Skorohod differentiable along a vector h. Then the family of measures $(\mu_{th} - \mu)/t$, where $0 < |t| \le 1$, is uniformly tight.*

Proof. Let $\varepsilon > 0$. There is a compact set $K \subset X$ such that $|d_h\mu|(X\backslash K) < \varepsilon$. The set $S = \{k + sh\colon k \in K, |s| \le 2\}$ is also compact as the image of the compact set $K \times [-2, 2]$ under the continuous mapping $(k, s) \mapsto k + sh$. Let us show that $|\mu_{th} - \mu|(X\backslash S) \le \varepsilon|t|$ if $|t| \le 1$. Let $t > 0$. It follows from (2.2) that for $\nu = d_h\mu$ we have

$$|\mu_{th} - \mu|(X\backslash S) \le \int_0^t |\nu_{sh}|(X\backslash S)\, ds \le t \sup_{s\in[0,1]} |\nu_{sh}|(X\backslash S).$$

It remains to observe that

$$|\nu_{sh}|(X\backslash S) = |\nu|_{sh}(X\backslash S) \le |\nu|(X\backslash K) \le \varepsilon,$$

because $|\sigma_h| = |\sigma|_h$ for any measure σ, which can be readily seen from the definition of $|\sigma|$, and $(X\backslash S) - sh \subset X\backslash K$. The latter is verified as follows: if $y - sh \in K$ with $|s| \le 1$, then $y \in K + sh \subset S$. The case $t < 0$ is similar.

Theorem 3.2 is proved.

There is a useful characterization of differentiability in terms of conditional measures. We mention here only the formulation with Skorohod's differentiability, but a similar result holds for Fomin's differentiability (see [12], Chapter 3).

We recall (see [11], Chapter 10 or [12], Chapter 1) that for every Radon measure μ on a locally convex space X and every finite-dimensional subspace L in X one can find a closed subspace Y such that $X = L \oplus Y$ is a direct topological sum and there are Radon measures μ^y on L (called conditional measures), $y \in Y$, for which

$$\mu = \int\limits_Y \mu^y \, |\mu|_Y(dy),$$

where $|\mu|_Y$ is the projection of $|\mu|$ on Y under the natural projection operator. This representation is understood in the following way: for every bounded Borel measurable function f on $L \oplus Y$, the function

$$y \mapsto \int\limits_L f(u,y)\, \mu^y(du)$$

is Borel measurable on Y and

$$\int\limits_X f(x)\, \mu(dx) = \int\limits_Y \int\limits_L f(u,y)\, \mu^y(du)\, |\mu|_Y(dy).$$

In addition, one has

$$\|\mu\| = \int\limits_Y \|\mu^y\| \, |\mu|_Y(dy),$$

and conditional measures are unique up to a redefinition for y from a set of $|\mu|_Y$-measure zero.

Theorem 3.3. *Let $e_1, \ldots, e_n$ be a basis in L. The measure μ is Skorohod differentiable along all vectors in L precisely when the Skorohod derivatives $d_{e_i}\mu^y$, $i = 1, \ldots, n$, exist for $|\mu|_Y$-almost all y and*

$$\int\limits_Y \|d_{e_i}\mu^y\| \, |\mu|_Y(dy) < \infty.$$

In this case,

$$d_{e_i}\mu = \int\limits_Y d_{e_i}\mu^y \, |\mu|_Y(dy).$$

This criterion implies the following result that is far from being obvious from the definition.

***Corollary* 3.3.** *Let μ be a Radon measure on a locally convex space X Skorohod differentiable along a vector h. Then its positive part μ^+, its negative part μ^- and the total variation $|\mu|$ are also Skorohod differentiable along h. In addition*,

$$\|(d_h|\mu|)\| \le \|d_h\mu\|, \qquad \|d_h\mu^+\| \le \|d_h\mu\|, \qquad \|d_h\mu^-\| \le \|d_h\mu\|.$$

By using conditional measures and the one-dimensional integration by parts formula, we obtain the following sharper version of the integration by parts formula.

***Corollary* 3.4.** *Let μ be a Radon measure on a locally convex space X Skorohod differentiable along a vector h and let f be a Borel function integrable with respect to $|d_h\mu|$ such that $\partial_h f$ exists everywhere and is $|\mu|$-integrable. Then*

$$\int_X \partial_h f(x)\mu(dx) = -\int_X f(x)\, d_h\mu(dx).$$

Note that this formula is not immediate on the real line where it gives the equality

$$\int f'(x)\varrho(x)\, dx = -\int f(x)\, d\varrho(x)$$

for an everywhere differentiable function f and an integrable function ϱ of bounded variation such that $f'\varrho$ is integrable with respect to Lebesgue measure and f is integrable with respect to $d\varrho$. Justification is easily reduced to bounded functions f. Now, if ϱ is continuously differentiable, then this is a classical result (although, not obvious, because it is rather involved to show that $f\varrho$ is absolutely continuous on bounded intervals). Finally, the general case follows by using convolutions.

***Example* 3.1.** Suppose that a Radon probability measure μ on a locally convex space X is Skorohod differentiable along a vector h and B is a Borel set such that the sections $B \cap (\mathbb{R}h + x)$ are convex or empty. Then the measure $I_B \cdot \mu$ is Skorohod differentiable along h and $\|d_h(I_B \cdot \mu)\| \leq$ $\leq 2\|d_h\mu\|$.

Indeed, if $X = \mathbb{R}$, μ is Fomin differentiable, ϱ is the absolutely continuous density of μ and B is a bounded interval (a, b), then the generalized derivative of $I_B\varrho$ is $\varrho(a)\delta_a - \varrho(b)\delta_b + I_B\varrho'$, so the variation of this measure does not exceed

$$2\sup_x \big|\varrho(x)\big| + \|\varrho'\|_{L^1} \leq 3\|\varrho'\|_{L^1}.$$

The same is true if (a, b) is a ray. The case of Skorohod's differentiability is similar, moreover, it can be deduced from the considered case by using approximations with convolutions. The general case follows from this estimate for differentiable conditional measures μ^y. Note that the same reasoning applies if the indicated sections are unions of k intervals with some fixed k.

4. Setwise characterizations of Skorohod's differentiability. In this section, we discuss various properties of functions $t \mapsto \mu(A + th)$. We know that the continuity of such functions (for a given vector h) is the continuity of μ along h, and their differentiability is Fomin's differentiability. Another possible property is analyticity (see [12] and [25]). The place of Skorohod's differentiability in these terms is this.

Theorem 4.1. *A Radon measure μ on a locally convex space X is Skorohod differentiable along a vector h precisely if, for every Borel set A, the function $t \mapsto \mu(A + th)$ is Lipschitz on* $[0, 1]$. *In this case, such functions have a common Lipschitz constant.*

Proof. There are several different proofs of this fact (of course, one implication is easy and follows from (2.2) for indicator functions of Borel sets). Probably, the shortest one is based on Theorem 3.3 about conditional measures, but here we include a more naive justification in the spirit of the original proof in [8] where this fact was first proved. However, the proof below is slightly shorter.

If μ has compact support K, the proof is almost immediate. From the Nikodym theorem we conclude that the measures $(\mu_{th} - \mu)/t$ with $t \in (0, 1]$ are uniformly bounded in variation. These measures are concentrated on the compact set

$$S = K + I, \quad \text{where} \quad I = \{sh\colon |s| \le 1\}.$$

Hence by the Alaoglu – Banach – Bourbaki theorem about weak-$*$ compactness of balls in duals to Banach spaces (combined with the fact that the dual to $C(S)$ can be identified with the space of Radon measures on S) there is a Radon measure ν on S that is a limit point of the sequence of measures $n(\mu_{h/n} - \mu)$ in the weak topology. It follows that $\widetilde{\nu}(l) = \exp(il(h))\widetilde{\mu}(l)$ for all $l \in X^*$, hence ν is the Skorohod derivative of μ along h. By construction, the derivative is concentrated on S, but the same reasoning shows that it is concentrated on every compact set $K + \varepsilon I$, hence on the their intersection K.

We now turn to the general case. We have $\|\mu_{th} - \mu\| \le Ct$ for some C and all $|t| \le 1$. Take increasing compact sets K_n with $|\mu|(X\backslash K_n) \to 0$. Next, construct inductively larger increasing compact sets S_n such that $S_{n+1} = K_n + nI$, where $I = \{sh\colon |s| \le 1\}$. It is clear that $S = \bigcup_{n=1}^{\infty} S_n$ is invariant under translations by all vectors th and $|\mu|(X\backslash S) = 0$. Let us set

$$f_{n+1}(x) = \int\limits_{-1/2}^{1/2} I_{S_{n+1}}(x + sh)\, ds.$$

Then $0 \le f_n \le 1$, $f_n(x) = 1$ if $x \in S_n$, $f_n(x) = 0$ if $x \notin S_{n+2}$, and

$$\big|f(x + th) - f(x)\big| \le 2|t| \quad \text{for all} \quad t.$$

The measures $\mu_n = f_n \cdot \mu$ given by densities f_n with respect to μ converge to the original measure μ in variation and $\mu_{n+1} = \mu_n$ on S_n. In addition,

$$\big\|(\mu_n)_{th} - \mu_n\big\| \le \big(C + 2\|\mu\|\big)|t| = M|t|,$$

which is seen from the equality

$$(f_n \cdot \mu)_{th} - f.\mu = (f_n \cdot \mu)_{th} - f_n \cdot \mu_{th} + f_n \cdot \mu_{th} - f_n \cdot \mu$$

and the estimates

$$\big\|(f_n \cdot \mu)_{th} - f_n \cdot \mu_{th}\big\| \le 2|t|\, \|\mu\|$$

and $\|\mu_{th} - \mu\| \le C$, $|f_n| \le 1$. Since the measure μ_n with compact support in S_{n+2} is Skorohod differentiable along h, its Skorohod derivative is concentrated on S_{n+2} and $\|d_h\mu_n\| \le M$.

We also observe that $d_h\mu_{n+2}$ and $d_h\mu_{n+1}$ coincide on S_n. Indeed, let ψ be a bounded Borel function vanishing outside $S_n + I/2$ such that the functions $t \mapsto \psi(x + th)$ are continuous for all $x \in X$. Then, for each $t \in [0, 1/2]$, we have

$$\int\limits_X \big[\psi(x + th) - \psi(x)\big]\, \mu_{n+1}(dx) = \int\limits_0^t \int\limits_X \psi(x + sh)\, d_h\mu_{n+1}(dx)\, ds,$$

and the same is true for μ_{n+2}, but the left-hand sides for μ_{n+1} and μ_{n+2} coincide. By the continuity of ψ along h the right-hand sides are differentiable in t, hence their derivatives coincide, which means that ψ has equal integrals against $d_h\mu_{n+1}$ and $d_h\mu_{n+2}$. So, in order to prove the coincidence of both measures on S_n it suffices to verify the following fact: if σ is a Radon measure such that the integral against σ for every function ψ with the stated properties is zero, then $|\sigma|(S_n) = 0$. If not, there is a compact set $K \subset S_n$ with $\sigma(K) > 0$ or $\sigma(K) < 0$. We can assume that $\sigma(K) > 0$. There is a closed hyperplane Y such that X is the direct topological sum of Y and $\mathbb{R}h$. By using this decomposition, we can construct a sequence of Borel functions ψ_n with $n > 1$ such that $0 \le \psi_n \le 1$, $\psi_n(x) = 1$ if $x \in K$, $\psi_n(x) = 0$ if $x \notin S_n + I/2$, the functions $t \mapsto \psi_n(x + th)$ are continuous for every $x \in X$, and $\psi_n(x) \to I_K(x)$ for all x. To this end, we set

$$\theta(y,t) = \inf\bigl\{s \ge 0 \colon y + th + sh \in K \text{or} y + th - sh \in K\bigr\}, \qquad y \in Y, \quad t \in \mathbb{R},$$

$$\psi_n = 1 - \min(n\theta, 1).$$

When y is fixed, the function $\theta(y,t)$ of t behaves as follows: it vanishes on some compact set and is of the form $c - t$ or $c + t$ on the intervals of the complement of this compact set (more precisely, $c - t$ on the left halves of such bounded intervals and $c + t$ on the right halves). Note that if $\psi_n(y,t) > 0$, then $\theta(y,t) < 1/n$, so there is $s \in [0, 1/n)$ such that either $y + th + sh \in K$ or $y + th - sh \in K$, which means that $y + th \in K + I/2 \subset S_n + I/2$. Thus, $\psi_n = 0$ outside $S_n + I/2$. If $y + th \in K$, then $\theta(y,t) = 0$. If $\theta(y,t) = \alpha > 0$, then $n\theta(y,t) > 1$ for all n large enough, so that $\psi_n(y,t) = 0$. By the dominated convergence theorem the integral of ψ_n against σ is positive for all n large enough, which is a contradiction.

Finally, we observe that $|d_h\mu_n|(X\backslash S) = 0$. Indeed, otherwise there is a compact set $K \subset X\backslash S$ with $d_h\mu_n(K) > 0$ or $d_h\mu_n(K) > 0$. There is an absolutely convex neighborhood of zero V such that $(K + V) \cap (S_{n+2} + V) = \varnothing$. There is also a bounded continuous function f with support in $K + V$ such that its integral against $d_n\mu_n$ is nonzero. However, the integrals of $f(x + th)$ and $f(x)$ against μ_n are zero for sufficiently small t, because $f(x) = f(x + th) = 0$ for all $x \in S_{n+2}$, so by differentiating (2.2) we arrive at a contradiction.

After these preparations we obtain a bounded measure ν on the set S such that on every fixed set S_n this measure coincides with $d_h\mu_k$ for all $k \ge n + 2$. We extend ν by zero outside S. It follows that (2.2) holds for every bounded Borel function f vanishing outside some S_n. Since the measures μ_{th}, μ, ν are concentrated on S, we conclude that (2.2) holds for all bounded Borel functions, so that ν is the Skorohod derivative of μ along h.

Theorem 4.1 is proved.

***Corollary* 4.1.** *Let $\{\mu_\alpha\}$ be a net of Radon measures on a locally convex space X Skorohod differentiable along vectors h_α such that these vectors converge weakly to a vector h and there is a Radon measure μ with $\widetilde{\mu}(l) = \lim\limits_\alpha \widetilde{\mu_\alpha}(l)$ for every $l \in X^*$. Suppose also that*

$$\sup_\alpha \|d_{h_\alpha}\mu_\alpha\| \le C < \infty.$$

Then μ is Skorohod differentiable along h and $\|d_h\mu\| \le C$.

Proof. Suppose first that $X = \mathbb{R}^n$. Then for every function f that is a linear combination of functions of the form $\exp(i(v,x))$ we have

$$\int\limits_{\mathbb{R}^n} \bigl[f(x - th) - f(x)\bigr]\, \mu(dx) = \lim_\alpha \int\limits_{\mathbb{R}^n} \bigl[f(x - th) - f(x)\bigr]\, \mu_\alpha(dx) =$$

$$= \int\limits_0^t \int\limits_{\mathbb{R}^n} f(x - sh)\, d_{h_\alpha}\mu_\alpha(dx)\, ds,$$

which does not exceed $C|t|\sup_x |f(x)|$ in absolute value. Hence this bound extends to bounded continuous functions and further to bounded Borel functions. Therefore, in the general case we obtain the same estimate for functions of the form $f(l_1,\dots,l_n)$, where f is a bounded Borel function and $l_i \in X^*$. It follows that $\|\mu_{th}-\mu\| \le C|t|$.

***Corollary* 4.2.** *Let μ be a Radon measure on a locally convex space X and $h\in X$. Suppose that there is a measure ν on the σ-algebra generated by X^* such that identity* (2.2) *holds for all smooth cylindrical functions. Then ν uniquely extends to a Radon measure that is the Skorohod derivative of μ along h.*

Proof. It follows from the assumption that $\|\mu_{th}-\mu\| \le |t|\|\nu\|$. By the theorem μ is Skorohod differentiable along h and its Skorohod derivative $d_h\mu$ is a Radon measure. Now (2.2) yields the equality $\widetilde{\nu} = \widetilde{d_h\mu}$, hence $\nu = d_h\mu$ on the the σ-algebra generated by X^*. A Radon extension is unique, because Radon measures with equal Fourier transforms coincide.

It also follows in the same manner that in Corollary 3.2 it suffices to have the measure ν only on the σ-algebra generated by X^*.

***Corollary* 4.3.** *If μ is a Radon measure on a locally convex space X such that for some $h\in X$ there is a sequence of nonzero numbers $t_n\to 0$ with the property that, for every bounded continuous function f on X, the sequence of integrals of f with respect to the measures $(\mu_{t_nh}-\mu)/t_n$ is bounded, then μ is Skorohod differentiable along h.*

Proof. By the Banach–Steinhaus theorem the norms of $(\mu_{t_nh}-\mu)/t_n$ in $C_b(X)^*$ are uniformly bounded by some number C, but these are exactly the variations of $(\mu_{t_nh}-\mu)/t_n$. Let us take the standard Gaussian measure γ on the real line and denote by γ^ε its image under the function $t\mapsto \varepsilon t$. The image of γ^ε under the mapping $t\mapsto th$ will be denoted by σ^ε. Let $\mu^\varepsilon = \mu*\sigma^\varepsilon$. Then

$$\widetilde{\mu^\varepsilon}(l) = \widetilde{\mu}(l)\widetilde{\sigma^\varepsilon}(l) = \widetilde{\mu}(l)\widetilde{\sigma^1}(\varepsilon l) \to \widetilde{\mu}(l)$$

pointwise as $\varepsilon\to 0$. Obviously, the measures μ^ε are Fomin differentiable along h. Let us show that $\|d_h\mu^\varepsilon\| \le C$. For any function $f\in C_b(X)$, we have

$$t_n^{-1}\int\limits_X \big[f(x-t_nh)-f(x)\big]\,\mu^\varepsilon(dx) =$$

$$= t_n^{-1}\int\limits_{\mathbb{R}}\int\limits_X \big[f(x-t_nh+\varepsilon sh)-f(x+\varepsilon sh)\big]\,\mu(dx)\,\gamma(ds) =$$

$$= \int\limits_{\mathbb{R}}\int\limits_X f(x-\varepsilon sh)t_n^{-1}\big[\mu_{t_nh}-\mu\big](dx)\,\gamma(ds),$$

which is estimated in absolute value by $C\sup_x |f(x)|$. Hence the integral of f against $d_h\mu^\varepsilon$ does not exceed $C\sup_x|f(x)|$, which implies the announced bound.

We now show that the integration by parts formula characterizing Skorohod's differentiability is equivalent to an inequality. This is a complete analog of the classical characterization of functions in $BV(\mathbb{R})$ by the estimate

$$\int \varphi'(x)f(x)\,dx \le C\sup_x|\varphi(x)|$$

for all smooth compactly supported functions φ.

***Corollary* 4.4.** *A Radon measure μ on a locally convex space X is Skorohod differentiable along a vector $h \in X$ precisely when there is a number $C \geq 0$ such that*

$$\int\limits_X \partial_h\varphi(x)\,\mu(dx) \leq C \sup_x |\varphi(x)| \quad \forall\, \varphi \in \mathcal{FC}(X).$$

Proof. We have

$$\int\limits_X \big[\varphi(x-th)-\varphi(x)\big]\,\mu(dx) = \int\limits_X \int\limits_0^t \partial_h\varphi(x-sh)\,ds\,\mu(dx) =$$

$$= \int\limits_0^t \int\limits_X \partial_h\varphi(x-sh)\,\mu(dx)\,ds,$$

which is estimated by $Ct\sup_x|\varphi(x)|$ for every $\varphi \in \mathcal{FC}(X)$ and $t \geq 0$. Therefore, we have

$$\|\mu_{th}-\mu\| \leq C|t|.$$

***Corollary* 4.5.** *A Radon measure μ on a locally convex space X is infinitely differentiable along a vector $h \in X$ (in Fomin's or Skorohod's sense) precisely when there are numbers $C_n \geq 0$ such that*

$$\int\limits_X \partial_h^n\varphi(x)\,\mu(dx) \leq C_n \sup_x |\varphi(x)| \quad \forall\, \varphi \in \mathcal{FC}(X).$$

Proof. We conclude that $d_h\mu$ exists and continue inductively using the integration by parts formula. Fomin's differentiability follows, since a measure twice Skorohod differentiable is Fomin differentiable.

Once the situation with the Lipschitz property is clarified, it is natural to ask about absolute continuity. Similarly to the Lipschitz property, one can think of the absolute continuity of the mapping $t \mapsto \mu_{th}$ with values in the Banach space of measures and the absolute continuity of scalar functions $t \mapsto \mu(A+th)$. Moreover, such absolutely continuity can be required on an interval or on the whole real line. These subtle and interesting questions had remained open for quite a long time until A. Shaposhnikov published his short note [48] with complete proofs of the following results. We reproduce here the proof of only one of these results, since it is quite illuminating.

Theorem 4.2. *A Radon measure μ on a locally convex space X is Skorohod differentiable along a vector h if and only if the mapping $t \mapsto \mu_{th}$ with values in the Banach space of Borel measures equipped with the total variation norm is absolutely continuous on the interval $[0,1]$.*

Proof. By the definition of absolute continuity, for every $\varepsilon > 0$ there is $\delta > 0$ such that, for any disjoint intervals $[s_1,t_1],\ldots,[s_k,t_k]$ in $[0,1]$ of total length

$$\sum_{i=1}^k |t_i - s_i| < \delta,$$

we have

$$\sum_{i=1}^k \|\mu_{t_ih} - \mu_{s_ih}\| \leq \varepsilon.$$

We show that for every Borel set A in X the function $t \mapsto \mu(A - th)$ is Lipschitz on $[0,1]$. To this end, we employ the following simple, but useful observation due to G. M. Fichtenholz (see [33] or [27], Exercise 4.5.18): if f is a function on the interval $[0,1]$ such that, for every $\varepsilon > 0$, there is $\delta > 0$ with the property that $\sum_{i=1}^{n} \big|f(b_i) - f(a_i)\big| < \varepsilon$ for every finite collection of intervals $[a_i, b_i]$ in $[0,1]$ (possibly, overlapping) with $\sum_{i=1}^{n} |b_i - a_i| < \delta$, then f is Lipschitz.

Given $\varepsilon > 0$, we take the corresponding $\delta \in (0,1)$. Then, for any finite collection of intervals $[a_1, b_1], \ldots, [a_k, b_k]$ (possibly, overlapping) in $[0,1]$ with

$$\sum_{i=1}^{k} |a_i - b_i| < \delta < 1,$$

we can pick numbers $c_1, c_2, \ldots, c_k$ such that the intervals

$$\big[a_1 + c_1, b_1 + c_1\big],\ \big[a_2 + c_2, b_2 + c_2\big], \ldots,\ \big[a_k + c_k, b_k + c_k\big]$$

also belong to $[0,1]$, but are disjoint (here it is important that $\delta < 1$). Then

$$\sum_{i=1}^{k} \big|\mu(A - a_i h) - \mu(A - b_i h)\big| \le \sum_{i=1}^{k} \|\mu_{a_i h - c_i h} - \mu_{b_i h - c_i h}\| \le \varepsilon.$$

Therefore, our function $t \mapsto \mu(A - th)$ is Lipschitz on $[0,1]$.

Theorem 4.2 is proved.

Theorem 4.3. *Let μ be a Radon measure on a locally convex space X and $h \in X$. The following conditions are equivalent*:

(i) *for each Borel set $A \subset X$, the function $t \mapsto \mu(A + th)$ is absolutely continuous on* $[0,1]$;

(ii) *for each open set $U \subset X$, the function $t \mapsto \mu(U + th)$ is absolutely continuous on* $[0,1]$;

(iii) *for each bounded Borel function f on X, the function*

$$t \mapsto \int\limits_X f(x)\, \mu_{th}(dx) = \int\limits_X f(x + th)\, \mu(dx)$$

is absolutely continuous on $[0,1]$.

***Remark* 4.1.** If the measure μ is compactly supported, then the properties specified above are also equivalent to the absolute continuity of the functions $t \mapsto \mu(K + th)$ for all compact sets K.

The reader may wonder why the conditions in (i)–(iii) are not symmetric with respect to t unlike the definition of differentiability, in which t of any sign is allowed. However, a closer look reveals that no symmetry is lost, because for A we can also take $A - h$, so that $A - th = (A - h) + (1 - t)h$ with nonnegative $1 - t$ for $t \in [0,1]$.

It should be noted that the equivalences mentioned in the previous theorem do not include the assertion that μ is Skorohod differentiable. As shown in [48] by an explicit (but highly nontrivial example), they do not imply differentiability.

Theorem 4.4. *There is a Borel probability measure μ on the real line such that the functions $t \mapsto \mu(A - t)$ are absolutely continuous on the interval $[0,1]$ for all Borel sets A, but not all these functions are Lipschitz*, *so that μ is not Skorohod differentiable.*

The corresponding example constructed in [48] is as follows (we omit the details of verification that are unexpectedly involved and can be found in [48]). For every $m \in \mathbb{N}$ we set $\varrho_m(x) = \sin^2 mx$ and

$$\varrho(x) := (m^{5/3}2^m)^{-1}\varrho_m(x), \quad x \in \big[2\pi a_m, 2\pi(a_m + 2^m)\big],$$

where a_m are natural numbers picked such that the distances between the indicated intervals are greater than 100π, i.e., $a_{m+1} - a_m > 2^m + 50$. At all remaining points we set $\varrho(x) := 0$.

Thus, the absolute continuity of the functions $t \mapsto \mu(A - th)$ on all intervals is weaker than the Skorohod differentiability. It turns out that the situation is different if we impose the absolute continuity on the whole real line in place of every compact interval.

Theorem 4.5. *Let μ be a Radon measure on a locally convex space X and $h \in X$. Suppose that, for every Borel set $A \subset X$, the function $t \mapsto \mu(A - th)$ is absolutely continuous on the entire real line in the sense that, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for every finite set of disjoint intervals $[s_1, t_1]$, $[s_2, t_2], \ldots, [s_k, t_k]$ with total length $\sum_{i=1}^k |t_k - s_k| < \delta$, the inequality $\sum_{i=1}^k \big|\mu(A - t_i h) - \mu(A - s_i h)\big| < \varepsilon$ holds. Then the measure μ is Skorohod differentiable along h.*

The proof of this result also employs the Fichtenholz observation.

5. The classes BV and SBV. In the last two decades, there has been a growing interest in analogs of classes BV for measures on infinite-dimensional spaces, introduced in [36, 37] and further studied by many authors, see [3, 4, 13, 22–24, 28–31, 38, 39, 43, 45, 46], and related constructions of surface measures appearing in connection with such classes. About surface measures associated with differentiable measures, see [1, 12, 15, 21, 52]; note also that Skorohod suggested a construction of surface measures in [49].

The characterization of Skorohod differentiable measures on $\mathbb{R}^n$ as measures with densities from the class $BV(\mathbb{R}^n)$ of functions of bounded variation leads naturally to the idea of defining classes BV on infinite-dimensional spaces via Skorohod differentiable measures. The idea is this: for a fixed nice reference measure μ (say, a Gaussian measure), a function f is of bounded variation if the measure $f \cdot \mu$ is Skorohod differentiable. Actually, even two classes of functions appear depending on whether the corresponding derivatives are vector measures of bounded variation or bounded semivariation. Let us recall that a measure Λ with values in a separable Hilbert space H defined on $\mathcal{B}(X)$ is said to be of bounded semivariation if every scalar measure $\Lambda(k) = (k, \Lambda)$ is countably additive. It is known that in this case

$$V(\Lambda) = \sup_{|k|_H \le 1} \|\Lambda(k)\| < \infty.$$

However, this does not mean that Λ is of bounded variation defined by

$$Var(\Lambda) = \sup \sum_{i=1}^n |\Lambda(B_i)|_H,$$

where sup is taken over all finite partitions of X into disjoint Borel sets B_i.

We first consider the case of Gaussian reference measures. Suppose that γ is a centered Radon–Gaussian measure on a locally convex space X and H is its Cameron–Martin space. Every vector $h \in H$ generates a measurable linear functional $\widehat{h}$ that belongs to the closure of X^* in $L^2(\gamma)$ and is defined by

$$l(h) = \int_X l(x)\widehat{h}(x)\,\gamma(dx), \quad l \in X^*.$$

For example, if γ is the countable power of the standard Gaussian measure on $\mathbb{R}$, then $\widehat{h}(x) = \sum_{n=1}^{\infty} h_n x_n$, $h = (h_n) \in l^2$. The series above converges in $L^2(\gamma)$ and also almost everywhere.

Definition 5.1. *The class $BV(\gamma)$ consists of all functions $f \in L^1(\gamma)$ such that $f\widehat{h} \in L^1(\gamma)$ for all $h \in H$ and there is an H-valued measure Λf of bounded variation satisfying the identity*

$$\int_X \varphi(x)\,(\Lambda f, h)_H(dx) = -\int \bigl[f(x)\partial_h\varphi(x) - \widehat{h}(x)f(x)\varphi(x)\bigr]\gamma(dx) \tag{5.1}$$

for all $\varphi \in \mathcal{FC}(X)$ and all $h \in H$.

It is easy to show that the Sobolev class $W^{1,1}(\gamma)$ is contained in $BV(\gamma)$ and $\Lambda f = D_H f \cdot \gamma$.

Definition 5.2. *The class $SBV(\gamma)$ consists of all functions $f \in L^1(\gamma)$ such that $f\widehat{h} \in L^1(\gamma)$ for all $h \in H$ and there is an H-valued measure Λf of bounded semivariation satisfying* (5.1).

In the finite-dimensional case the classes $BV(\gamma)$ and $SBV(\gamma)$ coincide as sets, but their norms are different. Even for smooth functions f, where Λf is given by a vector density $D_H f$ with respect to γ, the BV-norm may be much larger, since it involves the integral of $|D_H f|_H$, while the SBV-norm deals with the integrals of $|\partial_h f|$ with $|h| \le 1$.

The space $BV(\gamma)$ is Banach with the norm

$$\|f\|_{BV} = \|f\|_1 + \mathrm{Var}(\Lambda f).$$

The space $SBV(\gamma)$ is Banach with the norm

$$\|f\|_{SBV} = \|f\|_1 + V(\Lambda f).$$

We now consider any differentiable reference measures. Let H be a separable Hilbert space continuously embedded into a locally convex space X.

Let us observe that for any measure σ on X Skorohod differentiable along all vectors in H we obtain an H-valued measure $D\sigma$ defined by the equality

$$\bigl(D\sigma(B), h\bigr)_H = d_h\sigma(B).$$

As noted above, this measure is automatically of bounded semivariation. If H is infinite-dimensional and σ is not zero, then the measure $D\sigma$ has unbounded variation (see [12], Proposition 7.3.2). However, if $D\sigma$ is regarded as an X-valued measure and X is a Banach space, then under broad assumptions this X-valued measure has bounded variation (e.g., if the embedding $H \to X$ is absolutely summing, see [11], Chapter 7). If μ is a centered Gaussian measure and H is its Cameron–Martin space, then $D\mu$ as an X-valued measure has vector density $-x$ with respect to μ.

We now fix a Radon probability measure μ on a locally convex space X Fomin differentiable along all vectors from a separable Hilbert space H continuously embedded into X. Set

$$M_H(\mu) = \bigl\{f \in L^1(\mu):\ f\beta_h^\mu \in L^1(\mu)\ \ \forall h \in H\bigr\}.$$

Definition 5.3. *Let*

$$SV(\mu) = \bigl\{f \in L^1(\mu):\ \textit{the Skorohod derivative } d_h(f\cdot\mu) \textit{ exists for all } h \in H\bigr\},$$

$$SBV(\mu) = SV(\mu) \cap M_H(\mu).$$

In other words, the class $SBV(\mu)$ consists of all functions $f \in L^1(\mu)$ for which

$$\sup_{|h|\le 1} |f\beta_h|_{L^1(\mu)} < \infty$$

and there exists an H-valued measure Λf of bounded semivariation such that the Skorohod derivative $d_h(f\cdot\mu)$ exists and equals $(\Lambda f, h)_H + f\beta_h\mu$ for each $h \in H$.

It follows from Example 3.1 that if V is a Borel convex set, then its indicator function I_V belongs to $SBV(\mu)$.

Definition 5.4. *Let $BV(\mu)$ be the class of all functions $f \in SBV(\mu)$ such that the H-valued measure Λf has bounded variation.*

Theorem 5.1. *The set $M_H(\mu)$ is a Banach space with the norm*

$$\|f\|_M := \|f\|_{L^1(\mu)} + \sup_{|h|_H\le 1} \|f\beta_h^\mu\|_{L^1(\mu)}.$$

Theorem 5.2. (i) *The set $SV(\mu)$ is a Banach space with the norm*

$$\|f\|_{SV} := \|f\|_{L^1(\mu)} + \sup_{|h|_H\le 1} \|d_h(f\cdot\mu)\|,$$

and for every function $f \in SV(\mu)$ there is an H-valued measure $D(f\cdot\mu)$ of bounded semivariation such that $d_h(f\cdot\mu) = \big(D(f\cdot\mu), h\big)_H$ for all $h \in H$.

(ii) *The set $SBV(\mu)$ is a Banach space with the norm*

$$\|f\|_{SBV} := \|f\|_M + \|f\|_{SV},$$

and for every function $f \in SBV(\mu)$ there is an H-valued measure Λf of bounded semivariation such that $d_h(f\cdot\mu) = (\Lambda f, h)_H + f\cdot d_h\mu$ for all $h \in H$.

Theorem 5.3. *The set $BV(\mu)$ is a Banach space with the norm*

$$\|f\|_{BV} := \|f\|_{SBV} + \|\Lambda f\|.$$

See [22 – 24] for the proofs.

I would like to close this discussion by mentioning a number of challenging open problems connected with Skorohod's differentiability of measures. Probably, the most intriguing one concerns the so-called logarithmically concave measures (which are called convex measures in my book [11], although this term is also used in the literature for some broader class of measures). Let us recall that a Radon probability measure μ on a locally convex space X is called logarithmically concave (log-concave) if for all compact sets A and B one has

$$\mu(tA + (1-t)B) \ge \mu^t(A)\mu^{1-t}(B) \quad \forall\, t \in [0,1].$$

This is equivalent to the following: for every continuous linear operator $T\colon X \to \mathbb{R}^n$ the induced measure $\mu\circ T^{-1}$ is either given by a density of the form $\exp(-V)$ with a convex function V or is concentrated on a proper affine subspace and is given there by such a density. It was shown by [41] that every absolutely continuous log-concave measure on $\mathbb{R}^n$ is Skorohod differentiable along all vectors, i.e., has a density of class BV. Therefore, if a log-concave measure on a finite-dimensional space is not a Dirac measure, it must have nonzero vectors of differentiability. It is a long-standing open problem whether the same is true in infinite dimensions. However, the following nice inequality was established by [41]: if a log-concave measure μ on a locally convex space is Skorohod differentiable along a vector $h,$ then

$$\|\mu_h - \mu\| \geq 2 - \exp(-\|d_h\mu\|/2).$$

It follows from this universal bound that $d_h\mu$ exists provided that μ_{th} is not mutually singular with μ for some $t \neq 0$. Obviously, the converse is also true. On differentiability of log-concave measures, see also [42]. We recall that log-concave measures on Hilbert spaces are weak limits of sequences of uniform distributions on finite-dimensional convex compact sets (see [16], Exercise 2.7.52). So for the study of differentiability of log-concave measures on infinite-dimensional spaces it would be useful to look for possible uniforms bounds for the Skorohod derivatives of uniform distributions on finite-dimensional convex bodies.

Another interesting open question concerns the subspace of Skorohod differentiability of general measures. It is known (see [6, 7, 9] and [12], Chapter 5) that for any nonzero Radon measure μ on a locally convex space X the set $D_C(\mu)$ of all vectors in X along which μ is Skorohod differentiable is a linear space that can be equipped with the norm $\|h\|_{D_C} = \|d_h\mu\|$ with respect to which it is a Banach space whose closed unit ball is compact in X. The subspace $D(\mu)$ of vectors of Fomin differentiability is a closed linear subspace in $D_C(\mu)$. If X is a Fréchet space (say, a separable Hilbert space), then $D(\mu)$ is separable with this norm $\big($and isometric to a closed subspace in $L^1(\mu)\big)$. However, it is not known whether $D_C(\mu)$ is always separable for measures on Hilbert spaces.

Skorohod differentiable measures are involved in the study of distributions of functionals on infinite-dimensional spaces and fractional Besov-type classes on such spaces, see [14, 17 – 20, 40]. An interesting problem is connected with extensions of Sobolev and BV classes from infinite-dimensional domains. Skorohod differentiable measures fit naturally this framework, and one can expect that further investigation of their properties will be fruitful and inspiring.

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