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A NOTE ON VARIATIONAL FORMALISM FOR SLOSHING WITH ROTATIONAL FLOWS IN A RIGID TANK WITH AN UNPRESCRIBED MOTION*

ПРО ВАРІАЦІЙНИЙ ФОРМАЛІЗМ ДЛЯ ЗАДАЧІ ПРО ВИХОРОВІ КОЛИВАННЯ РІДИНИ В АБСОЛЮТНО ТВЕРДОМУ БАЦІ ЗА НЕВИЗНАЧЕНОСТІ ЙОГО РУХУ

The Bateman – Luke-type variational formulation of the free-boundary ‘sloshing’ problem is generalized to irrotational flows and unprescribed tank motions, i.e., to the case where both the tank and liquid motions should be found simultaneously for a given set of external forces applied to fixed points of the rigid tank body. We prove that the variational equation, which corresponds to the formulated problem, implies both the dynamic (force and moment) equations of the rigid body and the free-boundary problem, which describes sloshing in terms of the Clebsch potentials.

Варіаційне формулювання типу Бейтмена – Люка для задачі з вільною межею коливання рідини у баці узагальнено для вихорових течій та невизначених рухів бака, тобто для випадку, коли рухи бака та рідини повинні бути одночасно знайдені для фіксованого набору сил, які прикладено до заданих точок твердого тіла. Доведено, що варіаційне рівняння, яке випливає з цього формулювання, приводить як до динамічних (сил та моментів) рівнянь твердого тіла, так і до крайової задачі з вільною межею, яка описує динаміку рідини у баці в термінах потенціалів Клебша.

1. Introduction. Utilising variational approaches to sloshing in a rigid mobile tank is common [1, 2] for irrotational (potential) flows of an ideal incompressible liquid and prescribed tank motions. However, hydrodynamic force and moment affect the rigid tank motions that causes a great interest to variational methods for the coupled liquid-tank problem [3 – 6]. Another challenge is an accounting for the vortical flow component [4, 7, 8]. A Bateman – Luke-type variational formalism for prescribed tank motions and ideal liquid with rotational flows is announced in [9]; it employs the Clebsch potentials [10, 11]. The present paper follows analytical technique from Section 2.9 in [2] for generalising the results to unprescribed tank motions.

2. Variational formulation. ***2.1. Main definitions and preliminary remarks.*** A mobile rigid tank (body) of the shape Q_b is considered partly filled with an inviscid incompressible liquid as shown in Fig. 1. The liquid admits rotational flows. It occupies the time-dependent domain $Q(t)$ confined by the free surface $\Sigma(t)$ and the wetted tank surface $S(t)$.

The rigid body can move with six degrees of freedom which are associated with translational and angular motions of a non-inertial tank-fixed coordinate system $Ox_1x_2x_3$ relatively to an absolute (inertial) coordinate system $O'x'_1x'_2x'_3$. The three translational degrees of freedom are associated with scalar components of the radius-vector $\boldsymbol{r}_O(t) = O'O$ so that $\boldsymbol{r}' = \boldsymbol{r}_O + \boldsymbol{r}$ and $\boldsymbol{r}$ are the radius-vectors in absolute and body-fixed frames, respectively. Three angular degrees of freedom could be introduced via the Euler angles but, following Chapter 2 in [1], we consider instead the instant angular velocity $\boldsymbol{\omega}(t)$ of the rigid body and, if needed in variational equations, virtual angular displacement $\delta\boldsymbol{\theta}$. The total virtual displacement of the rigid body reads then as

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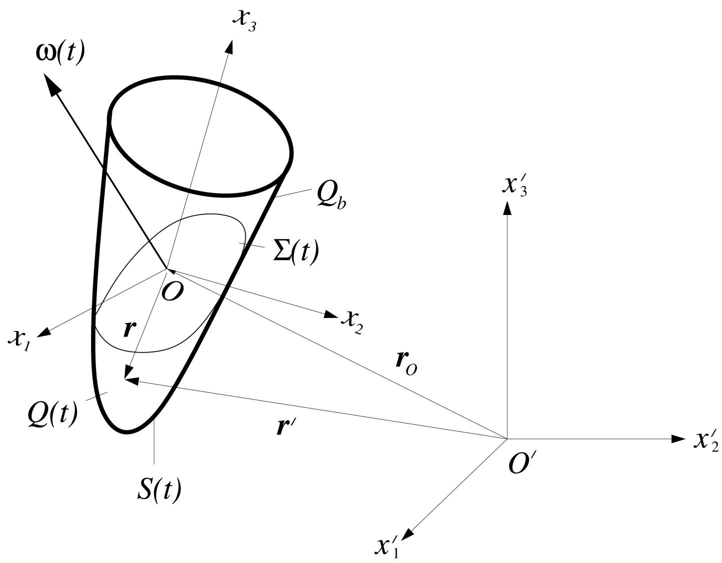

Fig. 1. Sketch of a moving rigid tank. Nomenclature.

$$\delta \boldsymbol{r}' = \delta \boldsymbol{r}_O + \delta \boldsymbol{\theta} \times \boldsymbol{r},$$

where $\delta \boldsymbol{r}_O$ is the translational virtual displacements. The Euler formula introduces the absolute velocity

$$\boldsymbol{v}_b = \boldsymbol{v}_O + \boldsymbol{\omega} \times \boldsymbol{r}$$

of a fixed point of the rigid body.

If $\rho_b(x_1, x_2, x_3)$ and $\rho_l = \mathrm{const}$ are the rigid-body and liquid density, then

$$M_b = \int\limits_{Q_b} \rho_b dQ = \text{const} \quad \text{and} \quad M_l = \rho_l \int\limits_{Q(t)} dQ = \rho_l V_l = \text{const} \tag{1}$$

are the body and liquid mass, respectively. The body-liquid mechanical system is affected by the gravity forces whose gravity potential takes the form

$$U(x_1, x_2, x_3, t) = -\boldsymbol{g} \cdot \boldsymbol{r}' = \boldsymbol{g} \cdot (\boldsymbol{r}_O + \boldsymbol{r}). \tag{2}$$

The free surface $\Sigma(t)$ is defined in the tank-fixed coordinate system, implicitly, $Z(x_1, x_2, x_3, t) = 0$ so that its outer normal is $\boldsymbol{n} = -\nabla Z / |\nabla Z|$. The liquid-mass conservation condition in (1) implies *geometric constraint* on Z.

Following [9], the *absolute velocity* field $\boldsymbol{v}(x_1, x_2, x_3, t) = (v_1, v_2, v_3)$ in $Q(t)$ can be described by the three Clebsch potentials [10, 11] $\varphi(x_1, x_2, x_3, t)$, $m(x_1, x_2, x_3, t)$, and $\phi(x_1, x_2, x_3, t)$ so that

$$\boldsymbol{v} = \nabla \varphi + m \nabla \phi. \tag{3}$$

***Remark* 2.1.** The three Clebsch potentials in (3) do not provide a unique representation of the velocity field (substitution $m := C\,m$, $\phi := \phi / C$, where C is a non-zero constant, confirms that). Alike in [9], the potentials can be assumed being three independent functions in the forthcoming analysis. Irrotational flows corresponds to either $m = 0$ or $\phi = \mathrm{const}$.

***Remark* 2.2.** As discussed in Chapter 2 of [1], modelling the liquid sloshing deals with absolute (in the inertial coordinate system $O'x'_1x'_2x'_3$) velocities and other vector and scalar values, which are defined in the body-fixed coordinate system $Ox_1x_2x_3$. The absolute vector $\boldsymbol{a} = (a_1, a_2, a_3)$ (in the $Ox_1x_2x_3$ coordinates) admits, therefore, the time-differentiation rule

$$\dot{\boldsymbol{a}} = \overset{*}{\boldsymbol{a}} + \boldsymbol{\omega} \times \boldsymbol{a}, \quad \overset{*}{\boldsymbol{a}} = (\dot{a}_1, \dot{a}_2, \dot{a}_3).$$

Furthermore, as remarked in [1, p. 47], the spatial derivatives in the inertial (∂'_i) and non-inertial (∂_i) coordinate systems remain the same, but the time-derivatives (∂'_t and ∂_t, respectively) possess the rule

$$\partial'_i = \partial_i, \quad \partial'_t = \partial_t - \boldsymbol{v}_M \cdot \nabla, \quad d'_t = \partial'_t + \boldsymbol{v} \cdot \nabla = \partial_t + (\boldsymbol{v} - \boldsymbol{v}_M) \cdot \nabla. \tag{4}$$

The coupled body-liquid problem implies finding the rigid tank motions (defined by $\boldsymbol{v}_O(t)$ and $\boldsymbol{\omega}(t)$), the free-surface (determined by $Z(x_1, x_2, x_3, t)$), and the absolute velocity field (the Clebsch potentials $\varphi(x_1, x_2, x_3, t)$, $m(x_1, x_2, x_3, t)$, and $\phi(x_1, x_2, x_3, t)$) as functions of the *prescribed external forces* $\boldsymbol{P}_k(t)$, $k = 1, \ldots, N$, applied to the body-fixed points M_k.

***Remark* 2.3.** For irrotational liquid flows [2], Z, $\boldsymbol{v}_O(t)$, and $\boldsymbol{\omega}(t)$ fully determine the liquid velocity field in $Q(t)$. But this is not true for rotational flows.

Based on the differentiation rules (4) and definitions in [12, p. 164], we introduce the Bateman–Luke-type Lagrangian

$$BL(\varphi, m, \phi, Z, \boldsymbol{v}_O, \boldsymbol{\omega}, \boldsymbol{r}'_{lC}) = \int\limits_{Q(t)} P\, dQ = -\rho_l \int\limits_{Q(t)} \left[\partial'_t\varphi + m\, \partial'_t\phi + \frac{1}{2}|\boldsymbol{v}|^2 + U\right] dQ =$$

$$= -\rho_l \int\limits_{Q(t)} \left[\partial_t\varphi + m\, \partial_t\phi - (\boldsymbol{v}_O + \boldsymbol{\omega} \times \boldsymbol{r})\boldsymbol{v} + \frac{1}{2}|\boldsymbol{v}|^2 + U\right] dQ \tag{5}$$

as a functional with respect to the independent Clebsch potentials, the translational and instant angular velocities $\boldsymbol{v}_O$ and $\boldsymbol{\omega}$ and the gravity potential U by (2):

$$-\rho_l \int\limits_{Q(t)} U dQ = M_l \boldsymbol{g} \cdot \boldsymbol{r}'_{lC} = M_l \boldsymbol{g} \cdot (\boldsymbol{r}_O + \boldsymbol{r}_{lC}),$$

where $\boldsymbol{r}'_{lC}$ and $\boldsymbol{r}_{lC}$ are the liquid-mass centre in absolute and body-fixed coordinate systems, respectively;

$$\boldsymbol{r}_{lC}(t) = \int\limits_{Q(t)} \boldsymbol{r} dQ / V_l. \tag{6}$$

The rigid-body motions are subject of the classical Lagrangian

$$L(\boldsymbol{v}_O, \boldsymbol{\omega}, \boldsymbol{r}'_{bC}) = T_b - \Pi_b,$$

where T_b and Π_b are the kinetic and potential energy of the rigid body,

$$T_b = \frac{1}{2}\int\limits_{Q_b} \rho_b \boldsymbol{v}^2 dQ = \frac{1}{2} M_b \boldsymbol{v}_O^2 + M_b(\boldsymbol{v}_O \times \boldsymbol{\omega}) \cdot \boldsymbol{r}_{bC} + \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{J}^O \cdot \boldsymbol{\omega},$$

$$\Pi_b = -\int\limits_{Q_b} \rho_b \boldsymbol{g} \cdot (\boldsymbol{r}_O + \boldsymbol{r}) dQ = -M_b \boldsymbol{g} \cdot (\boldsymbol{r}_O + \boldsymbol{r}_{bC}) = -M_b \boldsymbol{g} \cdot \boldsymbol{r}'_{bC},$$

in which M_b is the body mass by (1),

$$\boldsymbol{r}_{bC} = \frac{1}{M_b}\int\limits_{Q_b} \rho_b \boldsymbol{r} dQ$$

is the rigid-body mass centre in the body-fixed coordinate system $Ox_1x_2x_3$, and $\mathbf{J}^O = \{J^O_{ij}\}$ is the tensor of inertia at the origin O whose scalar components are computed by the formula

$$J^O_{ij} = (2\delta_{ij} - 1)\int\limits_{Q_b} \rho_b \, x_i x_j dQ,$$

where δ_{ij} is the Kronecker delta.

Based on BL and L, one can introduce the actions

$$W_l(\varphi, m, \phi, Z, \boldsymbol{v}_O, \boldsymbol{\omega}, \boldsymbol{r}'_{lC}) = \int\limits_{t_1}^{t_2} \left[BL - p_0 \int\limits_{Q(t)} dQ \right] dt, \tag{7a}$$

$$W_b(\boldsymbol{v}_O, \boldsymbol{\omega}, \boldsymbol{r}'_{bC}) = \int\limits_{t_1}^{t_2} L \, dt = \int\limits_{t_1}^{t_2} (T_b - \Pi_b) \, dt \tag{7b}$$

for any fixed instant times $t_1 < t_2$. The Lagrange multiplier p_0 is a consequence of the liquid-mass conservation constraint (1). The multiplier implies the ullage (atmospheric) pressure.

The *variational principle* sounds as

$$\delta W_l + \delta W_b + \delta' A = 0, \tag{8}$$

where $\delta' A$ is the elementary work of external forces $\boldsymbol{F}_b^{(i)}$ applied to points $\boldsymbol{r}_i$, $i = 1, \ldots, N$, of the rigid body and the variations are made by all independent generalised coordinates of the mechanical system. According to Remark 2.1 and the Bateman – Luke-type variational formalism, the Clebsch potentials φ, m, ϕ and the free-surface shape by Z can be adopted as generalised coordinates for the liquid motions. Because the elementary work reads, by definition, as

$$\delta' A = \underbrace{\sum_{i=1}^{N} \boldsymbol{F}_b^{(i)}}_{\boldsymbol{F}_b} \cdot \delta \boldsymbol{r}_O + \underbrace{\sum_{i=1}^{N} \boldsymbol{r}_i \times \boldsymbol{F}_b^{(i)}}_{\boldsymbol{M}_O^b} \cdot \delta \boldsymbol{\theta}, \tag{9}$$

where $\boldsymbol{F}_b$ is the resulting (principal) external force, and $\boldsymbol{M}_O^b$ is the resulting (principal) external moment.

Because the actions W_l and W_b formally depend on $\boldsymbol{v}_O, \boldsymbol{\omega}$ and $\boldsymbol{r}'_{lC}$, the following formulas from [2] are useful to compute their variations by $\boldsymbol{r}_O$ and $\boldsymbol{\theta}$:

$$\delta\boldsymbol{\omega} = (\overset{*}{\delta\boldsymbol{\theta}}) + \boldsymbol{\omega}\times\delta\boldsymbol{\theta}, \quad \delta\boldsymbol{v}_O = (\overset{*}{\delta\boldsymbol{r}_O}) + \boldsymbol{\omega}\times\delta\boldsymbol{r}_O + \boldsymbol{v}_O\times\delta\boldsymbol{\theta}, \quad \delta\boldsymbol{r}' = \delta\boldsymbol{r}_O + \delta\boldsymbol{\theta}\times\boldsymbol{r}, \tag{10}$$

where the $*$-time derivative is defined in Remark 2.2.

2.2. Hydrodynamic equations. Only W_l depends on the Clebsch potentials φ, m, ϕ and the free-surface shape by Z, therefore, the variational principle (8) reduces to $\delta W_l = 0$ for the hydrodynamic part. Henceforth, we assume that the Clebsch potentials are smooth functions in $Q(t)$, which admit, for any instant time t, an analytical continuation through the smooth (provided by admissible Z) free surface $\Sigma(t)$.

Lemma 2.1. *Under the assumption on the smoothness of the Clebsch potentials and the free surface $\Sigma(t)$, the zero first variation condition*

$$\delta_\varphi W_l = 0 \quad \textit{subject to} \quad \delta\varphi|_{t=t_1,t_2} = 0$$

is equivalent to the continuity equation

$$\nabla\cdot(\boldsymbol{v}-\boldsymbol{v}_b) \equiv \nabla\cdot\boldsymbol{v} = 0 \quad \textit{in} \quad Q(t) \tag{11}$$

and the kinematic boundary conditions

$$(\boldsymbol{v}-\boldsymbol{v}_b)\cdot\boldsymbol{n} = 0 \quad \textit{on} \quad S(t), \quad (\boldsymbol{v}-\boldsymbol{v}_b)\cdot\boldsymbol{n} = -\frac{\partial_t Z}{|\nabla Z|} \quad \textit{on} \quad \Sigma(t) \tag{12}$$

implying the normal velocity is defined by that of the rigid wall and the liquid particles are kept on the free surface $\Sigma(t)$, respectively.

Proof. Derivation of $\delta_\varphi W_l$ is similar (but not the same) to that for potential flows (see [1, p. 58, 59]). Consequently, employing the Reynolds transport and divergence theorems, and $\delta\varphi|_{t=t_1,t_2} = 0$ yields the derivation line

$$\delta_\varphi W_l = -\rho\int_{t_1}^{t_2}\int_{Q(t)} \left(\partial_t(\delta\varphi) + (\boldsymbol{v}-\boldsymbol{v}_b)\cdot\nabla(\delta\varphi)\right) dQdt =$$

$$= -\rho\int_{t_1}^{t_2}\left(\left[\frac{d}{dt}\int_{Q(t)}\delta\varphi\, dQ + \int_{\Sigma(t)}\frac{\partial_t Z}{|\nabla Z|}\delta\varphi\, dS\right] + \right.$$

$$\left. + \left[\int_{S(t)+\Sigma(t)}(\boldsymbol{v}-\boldsymbol{v}_b)\cdot\boldsymbol{n}\,\delta\varphi\, dS - \int_{Q(t)}\nabla\cdot(\boldsymbol{v}-\boldsymbol{v}_b)\,\delta\varphi\, dQ\right]\right) dt =$$

$$= -\rho\int_{t_1}^{t_2}\left(\int_{\Sigma(t)}\left[(\boldsymbol{v}-\boldsymbol{v}_b)\cdot\boldsymbol{n} + \frac{\partial_t Z}{|\nabla Z|}\right]\delta\varphi\, dS + \right.$$

$$+\int\limits_{S(t)} [(\boldsymbol{v}-\boldsymbol{v}_b)\cdot\boldsymbol{n}]\,\delta\varphi\,dS - \int\limits_{Q(t)} [\nabla\cdot(\boldsymbol{v}-\boldsymbol{v}_b)]\,\delta\varphi\,dQ\Bigg) = 0,$$

which deduces (11) and (12).

Lemma 2.2. *Under the assumption on the smoothness of the Clebsch potentials and the free surface* $\Sigma(t)$, *the zero first variation condition*

$$\delta_m W_l = 0$$

is equivalent to the governing equation

$$d'\phi \equiv \partial_t'\phi + \boldsymbol{v}\cdot\nabla\phi \equiv \partial_t\phi + (\boldsymbol{v}-\boldsymbol{v}_b)\cdot\nabla\phi = 0 \quad in \quad Q(t), \tag{13}$$

which indicates that the vortex lines contain the same fluid particles.

Proof. The variation by m derives the variational equality

$$\delta_m W_l = -\rho\int\limits_{t_1}^{t_2}\int\limits_{Q(t)} [\partial_t\phi + (\boldsymbol{v}-\boldsymbol{v}_b)\cdot\nabla\phi]\,\delta m\,dQdt = 0,$$

which proves the lemma.

Lemma 2.3. *Under the assumption on the smoothness of the Clebsch potentials and the free surface* $\Sigma(t)$, *the zero variation condition*

$$\delta_\phi W_l = 0 \quad subject\ to \quad \delta\phi|_{t_1,t_2} = 0 \tag{14}$$

and the kinematic problem (11), (12) *is equivalent to*

$$d'm \equiv \partial_t' m + \boldsymbol{v}\cdot\nabla m \equiv \partial_t m + (\boldsymbol{v}-\boldsymbol{v}_b)\cdot\nabla m = 0 \quad in \quad Q(t). \tag{15}$$

Proof. The variation by ϕ yields the variational equation

$$\delta_\phi W_l = -\rho\int\limits_{t_1}^{t_2}\int\limits_{Q(t)} m\left(\partial_t(\delta\phi) + (\boldsymbol{v}-\boldsymbol{v}_b)\cdot\nabla(\delta\phi)\right)dQ\,dt =$$

$$= -\rho\int\limits_{t_1}^{t_2}\left(\left[\frac{d}{dt}\int\limits_{Q(t)} m\,\delta\phi\,dQ - \int\limits_{Q(t)}\partial_t m\,\delta\phi\,dQ + \int\limits_{\Sigma(t)}\frac{\partial_t Z}{|\nabla Z|}m\,\delta\phi dS\right] + \right.$$

$$+\left[\int\limits_{S(t)+\Sigma(t)} m\,(\boldsymbol{v}-\boldsymbol{v}_b)\cdot\boldsymbol{n}\,\delta\phi\,dS\ - \right.$$

$$\left.\left. - \int\limits_{Q(t)}\delta\phi\,\left(m\,\nabla\cdot(\boldsymbol{v}-\boldsymbol{v}_b) + (\boldsymbol{v}-\boldsymbol{v}_b)\cdot\nabla m\right)\,dQ\right]\right)\,dt =$$

$$= \rho \int_{t_1}^{t_2} \int_{Q(t)} \delta\phi \, [\partial_t m + (\boldsymbol{v} - \boldsymbol{v}_b) \cdot \nabla m] \, dQ \, dt = 0, \tag{16}$$

whose derivation adopted the Reynolds transport and divergence theorems, condition (14) at $t = t_1$ and t_2, and the kinematic conditions (11), (12). The variational equality (16) proves the lemma.

***Remark* 2.4.** Function P in (5) is, generally speaking, not the pressure. One can show that it turns into the pressure, $p = P + f(t)$ ($f(t)$ is an arbitrary function) when (13) and (15) are satisfied because the Euler equation

$$d'\boldsymbol{v} = -\frac{1}{\rho} \left(\nabla P + \nabla U\right) \quad \text{in} \quad Q(t) \tag{17}$$

is then formally fulfilled. This follows from the left-hand side of (17), i.e.,

$$d'(\nabla\varphi + m\nabla\phi) = \left[\nabla(\partial_t'\varphi) + m\nabla(\partial_t'\phi) + \partial_t' m \nabla\phi\right] + \underbrace{\boldsymbol{v}\cdot\nabla(\nabla\varphi + m\nabla\phi)}_{\boldsymbol{v}\cdot\nabla\nabla\varphi + m\boldsymbol{v}\cdot\nabla\nabla\phi + \nabla\phi(\nabla m\cdot\boldsymbol{v})} =$$

$$= \boxed{\nabla(\partial_t'\varphi) + m\nabla(\partial_t'\phi) + \boldsymbol{v}\cdot\nabla\nabla\varphi + m\boldsymbol{v}\cdot\nabla\nabla\phi + \nabla\phi(\nabla m\cdot\boldsymbol{v})} + \nabla\phi\,[d'm]$$

and the right-hand side (after annihilliating the U-term)

$$\nabla\left(\partial_t'\varphi + m\partial_t'\phi + \frac{1}{2}|\boldsymbol{v}|^2\right) = [\nabla(\partial_t'\varphi) + m\nabla(\partial_t'\phi) + \partial_t'\phi\nabla m] +$$

$$+\boldsymbol{v}\cdot\nabla\nabla\varphi + m\boldsymbol{v}\cdot\nabla\nabla\phi + \nabla m(\nabla\phi\cdot\boldsymbol{v}) =$$

$$= \boxed{\nabla(\partial_t'\varphi) + m\nabla(\partial_t'\phi) + \boldsymbol{v}\cdot\nabla\nabla\varphi + m\boldsymbol{v}\cdot\nabla\nabla\phi + \nabla\phi(\nabla m\cdot\boldsymbol{v})} + \nabla m\,[d'\phi],$$

in which the framed terms are identical but the residual terms vanish as (13) and (15) hold true.

Theorem 2.1. *Under the assumption on the smoothness of the Clebsch potentials and the free surface* $\Sigma(t)$, *the variational equation*

$$\delta_\varphi W_l + \delta_m W_l + \delta_\phi W_l + \delta_Z W_l = 0$$

subject to

$$\delta\varphi|_{t_1,t_2} = \delta\phi|_{t_1,t_2} = 0$$

is equivalent to the free-surface sloshing problem for prescribed tank motions; it includes (11) – (13) *and* (15) *as well as the dynamic boundary condition*

$$p - p_0 = -\rho\left(\partial_t\varphi + m\,\partial_t\phi - \boldsymbol{v}_b\cdot\boldsymbol{v} + \frac{1}{2}|\boldsymbol{v}|^2 + U\right) - p_0 = 0 \quad \textit{on} \quad \Sigma(t) \tag{18}$$

implying the pressure is equal to the ullage pressure p_0 *on the free surface. The mass conservation condition* (1) *should also be added.*

Proof. The assertion follows from Lemmas 2.1, 2.2 and 2.3, Remark 2.4, and the variational equality

$$\delta_Z W_l = -\int_{t_1}^{t_2}\int_{\Sigma(t)} (p - p_0)\,\frac{\delta Z}{|\nabla Z|}\,dS\,dt = 0$$

which derives the dynamic boundary condition (18).

2.3. The rigid-body dynamic equations. The both actions W_l and W_b in (8) are functions of the rigid-body motions.

Theorem 2.2. *The variational equation*

$$\delta_{\boldsymbol{r}_O,\boldsymbol{\theta}} W_b + \delta_{\boldsymbol{r}_O,\boldsymbol{\theta}} W_l + \delta' A = 0 \quad \textit{subject to} \quad \delta\boldsymbol{r}_O = \delta\boldsymbol{\theta} = 0 \quad \textit{at} \quad t = t_1, t_2$$

by the independent generalised coordinates $\boldsymbol{r}_O, \boldsymbol{\theta}$, where W_l and W_b are defined by (7) *and the virtual work is determined by* (9) *for the prescribed resulting (principal) external force $\boldsymbol{F}_b$ and moment $\boldsymbol{M}_O^b$ leads to the dynamic equations*

$$M_b\left[\overset{*}{\boldsymbol{v}}_O + \boldsymbol{\omega}\times\boldsymbol{v}_O + \dot{\boldsymbol{\omega}}\times\boldsymbol{r}_C + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}_C) - \boldsymbol{g}\right] = \boldsymbol{F}_l + \boldsymbol{F}_b, \tag{19a}$$

$$\boldsymbol{J}^O\cdot\dot{\boldsymbol{\omega}} + \boldsymbol{\omega}\times(\boldsymbol{J}^O\cdot\boldsymbol{\omega}) + M_b\boldsymbol{r}_C\times\left[\overset{*}{\boldsymbol{v}}_O + \boldsymbol{\omega}\times\boldsymbol{v}_O - \boldsymbol{g}\right] = \boldsymbol{M}_O^l + \boldsymbol{M}_O^b, \tag{19b}$$

where the hydrodynamic force, $\boldsymbol{F}_l$, and moment with respect to O, $\boldsymbol{M}_O^l$, are computed by the formulas

$$\boldsymbol{F}_l = -\dot{\boldsymbol{M}} + M_l\boldsymbol{g}, \quad \boldsymbol{M}_O^l = -\dot{\boldsymbol{G}}_O - \boldsymbol{v}_O\times\boldsymbol{M} + M_l\boldsymbol{r}_{lC}\times\boldsymbol{g}, \tag{20}$$

in which

$$\boldsymbol{M} = \rho_l\int_{Q(t)}\boldsymbol{v}\,dQ \quad \textit{and} \quad \boldsymbol{G}_O = \int_{Q(t)}\boldsymbol{r}\times\boldsymbol{v}\,dQ$$

are the liquid momentum and angular momentum, respectively, and $\boldsymbol{r}_{lC}$ is the liquid mass centre by (6).

Proof. We consider

$$\delta_{\boldsymbol{v}_O,\boldsymbol{\omega},\boldsymbol{r}'_{bC}} W_b + \delta_{\boldsymbol{v}_O,\boldsymbol{\omega},\boldsymbol{r}'_{lC}} W_l =$$

$$= \int_{t_1}^{t_2}\Bigg[M_b\left[\boldsymbol{v}_O\cdot(\delta\boldsymbol{v}_O) + \boldsymbol{r}_C\cdot\{(\delta\boldsymbol{v}_O)\times\boldsymbol{\omega} + \boldsymbol{v}_O\times(\delta\boldsymbol{\omega})\} + \boldsymbol{g}\cdot(\delta\boldsymbol{r}'_{bC})\right] +$$

$$+\boldsymbol{\omega}\cdot\boldsymbol{J}^O\cdot(\delta\boldsymbol{\omega}) + \rho_l\int_{Q(t)}(\delta\boldsymbol{v}_O + \delta\boldsymbol{\omega}\times\boldsymbol{r})\cdot\boldsymbol{v}\,dQ + M_l\boldsymbol{g}\cdot\delta\boldsymbol{r}'_{lC}\Bigg]\,dt,$$

substitute (10) to deal with the virtual displacements $\delta\boldsymbol{r}_O$, $\delta\boldsymbol{\theta}$ and their $*$-time derivatives, integrate by part to exclude the $*$-derivatives, and use the vector algebra to get $\ldots\delta\boldsymbol{r}_O$, $\ldots\delta\boldsymbol{\theta}$ in all expressions. The result is

$$\delta_{\boldsymbol{r}_O,\boldsymbol{\theta}} W_b + \delta_{\boldsymbol{r}_O,\boldsymbol{\theta}} W_l + \delta' A = \int_{t_1}^{t_2}\Bigg(\Bigg\{-M_b\left[\overset{*}{\boldsymbol{v}}_O + \boldsymbol{\omega}\times\boldsymbol{v}_O + \dot{\boldsymbol{\omega}}\times\boldsymbol{r}_C + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}_C) - \boldsymbol{g}\right] -$$

$$-\dot{\boldsymbol{M}} + M_l\boldsymbol{g} + \boldsymbol{F}_b\Big\} \cdot (\delta \boldsymbol{r}_O) - \Big\{\boldsymbol{J}^O \cdot \dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (\boldsymbol{J}^O \cdot \boldsymbol{\omega}) + M_b \boldsymbol{r}_C \times \Big[\overset{*}{\boldsymbol{v}}_O + \boldsymbol{\omega} \times \boldsymbol{v}_O - \boldsymbol{g}\Big] +$$

$$+\dot{\boldsymbol{G}}_O + \boldsymbol{v}_O \times \boldsymbol{M} - M_l \boldsymbol{r}_{lC} \times \boldsymbol{g} - \boldsymbol{M}_O^b\Big\} \cdot (\delta\boldsymbol{\theta})\Big) dt = 0,$$

which is equivalent to (19) because (20) was proven in [1] (see Eqs. (2.38) and (7.26)) for arbitrary inviscid rotational flows.

Remark **2.5.** Chapter 7 in [1] proves the Lukovsky formula for the resulting (principal) hydrodynamic force

$$\boldsymbol{F}_l = M_l\boldsymbol{g} - M_l\left[\overset{*}{\boldsymbol{v}}_O + \boldsymbol{\omega} \times \boldsymbol{v}_O + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_{lC}) + \dot{\boldsymbol{\omega}} \times \boldsymbol{r}_{lC} + 2\boldsymbol{\omega} \times \overset{*}{\boldsymbol{r}}_{lC} + \overset{**}{\boldsymbol{r}}_{lC}\right]$$

for rotational liquid flows of an ideal liquid. Unfortunately, the Lukovsky formula (see Eq. (7.32) in [1]) for the resulting hydrodynamic moment holds only true for irrotational flows.

3. Conclusions. Utilising the Bateman – Luke variational formalism for the contained ideal incompressible liquid with rotational flows makes it possible to derive the full set of governing equations (11), (13), (15), boundary conditions (12), (18) for the liquid sloshing dynamics as well as the dynamic equations (19) for the carrying rigid body whose motions are not prescribed but affected by a set of external forces applied to the body. The generalised Bateman – Luke-type formulation can be a base for the nonlinear multimodal method.

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