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CHARACTERIZATION OF SOME FINITE SIMPLE GROUPS BY THE SET OF ORDERS OF VANISHING ELEMENTS AND ORDER

ХАРАКТЕРИЗАЦІЯ ДЕЯКИХ СКІНЧЕННИХ ПРОСТИХ ГРУП МНОЖИНОЮ ПОРЯДКІВ ЗНИКАЮЧИХ ЕЛЕМЕНТІВ ТА ПОРЯДКУ

Let G be a finite group. We say that an element g of G is a vanishing element if there exists an irreducible complex character χ of G such that $\chi(g) = 0$. Ghasemabadi, Iranmanesh, Mavadatpour (2015), present the following conjecture: Let G be a finite group and M a finite non-Abelian simple group such that $Vo(G) = Vo(M)$ and $|G| = |M|$. Then $G \cong M$. We answer in affirmative this conjecture for $M = {}^2D_{r+1}(2)$, where $r = 2^n - 1 \geq 3$ and either $2^r + 1$ or $2^{r+1} + 1$ is a prime number and $M = {}^2D_r(3)$, where $r = 2^n + 1 \geq 5$ and either $(3^{r-1} + 1)/2$ or $(3^r + 1)/4$ is prime.

Нехай G — скінченна група. Елемент $g \in G$ є зникаючим елементом, якщо існує незвідний комплексний характер $\chi \in G$ такий, що $\chi(g) = 0$. Гасемабаді, Іранманеш та Мавадатпур (2015) запропонували гіпотезу: якщо G — скінченна група, а M — скінченна неабелева проста група, для яких $Vo(G) = Vo(M)$ і $|G| = |M|$, то $G \cong M$. Ми доводимо цю гіпотезу для $M = {}^2D_{r+1}(2)$, де $r = 2^n - 1 \geq 3$, якщо або $2^r + 1$, або $2^{r+1} + 1$ є простим числом, і для $M = {}^2D_r(3)$, де $r = 2^n + 1 \geq 5$, якщо або $(3^{r-1} + 1)/2$, або $(3^r + 1)/4$ є простим.

1. Introduction. Let G be a finite group. It is well-known that the set of values $cd(G) = \{\chi(1) : \chi \in Irr(G)\}$ has a strong influence on the group structure of G, where $Irr(G)$ denotes the set of irreducible complex characters of G. We say that an element g of G is a vanishing element if there exists an irreducible complex character χ of G such that $\chi(g) = 0$. Denote $Van(G)$ the set $\{g \in G : \chi(g) = 0 \text{ for some } \chi \in Irr(G)\}$, $Vo(G)$ the set $\{o(g) : g \in Van(G)\}$ consisting of the orders of the elements in $Van(G)$.

In [16], it is shown that if G is a finite group such that $Vo(G) = Vo(A_5)$, then $G \cong A_5$. In [17] it is proved that if G is a finite group such that $Vo(G) = Vo(Sz(2^{2m+1}))$, where $m \geq 1$, then $G \cong Sz(2^{2m+1})$. But not all finite simple groups are characterizable by the set of orders of their vanishing elements. For example, $Vo(PSL(3,5)) = Vo(Aut(PSL(3,5)))$, but $PSL(3,5) \ncong Aut(PSL(3,5))$. The following conjecture is one of the important problem:

Conjecture. Let G be a finite group and M a finite non-Abelian simple group such that $Vo(G) = Vo(M)$ and $|G| = |M|$. Then $G \cong M$. The above conjecture was proved for simple groups $PSL(2,q)$, where $q \in \{5,7,8,9,17\}$, $PSL(3,4)$, A_7, $Sz(8)$ and $Sz(32)$. Then in [9], it is proved that sporadic simple groups, alternating groups, projective special linear groups $PSL(2,p)$, where p is an odd prime, and finite simple K_n-groups where $n \in \{3,4\}$, satisfying this conjecture. Now, we prove this conjecture for some finite simple groups as follows:

Theorem A. *If G is a finite group such that $Vo(G) = Vo({}^2D_{r+1}(2))$ and $|G| = |{}^2D_{r+1}(2)|$, where $r = 2^n - 1 \geq 3$ and either $2^r + 1$ or $2^{r+1} + 1$ is prime, then $G \cong {}^2D_{r+1}(2)$.*

Theorem B. *If G is a finite group such that $Vo(G) = Vo({}^2D_r(3))$ and $|G| = |{}^2D_r(3)|$, where $r = 2^n + 1 \geq 5$ and either $(3^{r-1} + 1)/2$ or $(3^r + 1)/4$ is prime, then $G \cong {}^2D_r(3)$.*

Let X be a finite set of positive integers. The prime graph $\Pi(X)$ is a graph whose vertices are the prime divisors of elements of X divisable by pq. For a finite group G, we denote by $\omega(G)$ the set of element orders of G, and by $\pi(G)$ the set of prime divisors of $|G|$. The graph $\Pi(\omega(G))$ is denoted by $GK(G)$ and is called the Gruenberg – Kegel graph of G. We denote by $t(G)$ the number of connected components of $GK(G)$ and by $\pi_i(G), i = 1, 2, \ldots, t(G)$, the vertex set of the ith connected components of $GK(G)$. If $2 \in \pi(G)$, we always assume that $2 \in \pi_1(G)$. The prime graph $\Pi(Vo(G))$ is denoted by $\Gamma(G)$ and is called the vanishing prime graph of G. Obviously the vanishing prime graph of G is a subgraph of Gruenberg – Kegel graph of G.

Throughout this paper, we denote by $\pi(n)$ the set of prime divisors of integer n. All further notation can be found in [4], for instance.

2. Preliminaries. A 2-Frobenius group is a group G which has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, where K and G/H are Frobenius groups with kernels H and K/H, respectively. Also, we know that 2-Frobenius groups are solvable.

Definition 2.1 [18]. *Let a and n be integers greater than 1. Then a Zsigmondy prime of $a^n - 1$ is a prime l such that $l \mid (a^n - 1)$ but $l \nmid (a^i - 1)$ for $1 \leq i < n$.*

Lemma 2.1 [18]. *Let a and n be integers greater than 1. Then there exists a Zsigmondy prime of $a^n - 1$, unless $(a, n) = (2, 6)$ or $n = 2$ and $a = 2^s - 1$ for some natural number s.*

***Remark* 2.1.** If l is a Zsigmondy prime of $a^n - 1$, then Fermat's little theorem shows that $n \mid l-1$. Put

$$Z_n(a) = \{l : l \text{ is a Zsigmondy prime of } a^n - 1\}.$$

If $r \in Z_n(a)$ and $r \mid a^m - 1$, then we can see at once that $n \mid m$.

Lemma 2.2 [3]. *Let G be a Frobenius group of even order with kernel K and complement H. Then $t(G) = 2$, the prime graph components of G are $\pi(H)$ and $\pi(K)$ and the following assertions hold*:

(1) *K is nilpotent*;

(2) $|K| \equiv 1 \pmod{|H|}$.

Lemma 2.3 [3]. *Let G be a 2-Frobenius group. Then:*

(a) *$t(G) = 2$, $\pi_1 = \pi(G/K) \cup \pi(H)$ and $\pi_2 = \pi(K/H)$;*

(b) *G/K and K/H are cyclic, $|G/K| \mid (|K/H| - 1)$ and $G/K \leq \mathrm{Aut}(K/H)$.*

Lemma 2.4 [15]. *If G is a finite group such that $t(G) \geq 2$, then G has one of the following structures:*

(a) *G is a Frobenius group or 2-Frobenius group*;

(b) *G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $\pi(H) \cup \pi(G/K) \subseteq \pi_1$ and K/H is a non-Abelian simple group. In particular, H is nilpotent, $G/K \lesssim \mathrm{Out}(K/H)$ and the odd order components of G are the odd order components of K/H.*

Lemma 2.5 [7, 8]. (i) *If G is a finite non-Abelian simple group except A_7, then $GK(G) = \Gamma(G)$.*

(ii) *If G is a solvable group, then $\Gamma(G)$ has at most 2 connected components.*

Lemma 2.6 [7]. *Let G be a finite nonsolvable group. If $\Gamma(G)$ is disconnected. Then G has a unique non-Abelian composition factor S, and $t(S)$ is greater than or equal to the number of connected components of $\Gamma(G)$, unless G is isomorphic to A_7.*

Lemma 2.7 [7]. *Let G be a group and K a nilpotent normal subgroup of G. If $K \bigcap Van(G) \neq \neq 0$, then there exists $g \in K \bigcap Van(G)$ whose order is divisable by every prime in $\pi(K)$.*

The following lemma is an easy consequence of [12] (Corollary 22.26).

Lemma 2.8. *If $\chi \in Irr(G)$ vanishes on a p-element for some prime p, then $p \mid \chi(1)$.*

Let p be a prime number. A character $\chi \in Irr(G)$ is said to be of p defect zero, if $p \nmid |G|/\chi(1)$. Also, if $\chi \in Irr(G)$ is of p defect zero, then for every element $g \in G$ such that $p \mid o(g)$, we have $\chi(g) = 0$ [11] (Theorem 8.17).

Lemma 2.9 [6]. *The equation* $p^m - q^n = 1$, *where* p *and* q *are primes and* $m, n > 1$ *has only solution, namely,* $3^2 - 2^3 = 1$.

Lemma 2.10 [6]. *With the exceptions of the relations* $(239)^2 - 2(13)^4 = -1$ *and* $3^5 - 2(11)^2 = 1$ *every solution of the equation*

$$p^m - 2q^n = \pm 1, \quad p, q \ prime, \quad m, n > 1,$$

has exponents $m = n = 2$; *i.e., it comes from a unit* $p - q.2^{1/2}$ *of the quadratic field* $\mathbb{Q}(2^{1/2})$ *for which the coefficients* p *and* q *are primes.*

3. Proofs of the main results. ***Proof of Theorem* A.** By the assumption $Vo(G) = Vo(^2D_{r+1}(2))$, it is obvious that $\Gamma(G) = \Gamma(^2D_{r+1}(2))$. By Lemma 2.6, we know that $\Gamma(^2D_{r+1}(2)) = GK(^2D_{r+1}(2))$ has 3 connected components including an isolated vertex p, where $p \in \{2^r + 1, 2^{r+1} + 1\}$. Also, note that

$$|G| = 2^{r(r+1)}(2^r - 1)(2^r + 1)(2^{r+1} + 1)\prod_{i=1}^{r-1}(2^{2i} - 1).$$

Since $p \in Vo(^2D_{r+1}(2))$ and $Vo(G) = Vo(^2D_{r+1}(2))$, so $p \in Vo(G)$. Thus there exist an element $g \in G$ and irreducible character $\chi \in Irr(G)$ such that $o(g) = p$ and $\chi(g) = 0$. So $p \mid \chi(1)$ and since $|G|_p = p$, we conclude that $p \nmid |G|/\chi(1)$. Therefore, χ is a p-defect zero, and, hence, for every element $h \in G$ such that $p \mid o(h)$, we have $\chi(h) = 0$. So, by the fact p is an isolated vertex in $\Gamma(G)$, we conclude that p is an isolated vertex in $GK(G)$. Hence, $t(G) \geq 2$.

Since $\Gamma(G)$ has three connected components, Lemma 2.6 implies that G is not a solvable group and consequently G is not a 2-Frobenius group. We also claim that G is not a Frobenius group. Suppose that G is a Frobenius group with kernel K and complement H. So $|G| = |H||K|$ and $|H| \mid |K| - 1$. Lemma 2.2 implies that $GK(G)$ has two connected components $\pi(H)$ and $\pi(K)$, and since $|H| < |K|$, it follows that $|H| = p$ and $|K| = |G|/p$. In both cases $p = 2^r + 1$ and $p = 2^{r+1}+1$, one can get a contradiction by the fact that $|H| \mid |K|-1$. Therefore G is not a Frobenius group. So, by Lemma 2.4, G has a normal $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $\pi(H) \cup \pi(G/K) \subseteq \pi_1$ and K/H is a non-Abelian simple group and $G/K \leq Aut(K/H)$. By Lemma 2.6, we have $t(K/H) \geq 3$. In both cases $p = 2^r + 1$ and $p = 2^{r+1} + 1$, we use the classification of finite non-Abelian simple groups with more than two Gruenberg–Kegel graph connected components to prove that K/H is isomorphic to $^2D_{r+1}(2)$.

Case 1. First suppose that $p = 2^r + 1$.

Step 1. K/H is not an sporadic simple group.

Suppose that K/H is an sporadic simple group. Then $p = 2^r + 1 \in \{5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71\}$. If $K/H \cong Fi_{22}$, then $p = 2^r + 1 = 17, 23$ or 29. The only possibility is $r = 4$, but $r = 2^n - 1 \geq 3$, which is impossible. For other sporadic simple groups one get a contradiction similarly.

Step 2. K/H is not an alternating group.

Let $K/H \cong A_{p'}$, where p' and $p'-2$ are primes. If $p'-2 = p = 2^r+1$, then $p' = 2^r+3$ is a prime number, which is impossible. Let $p' = p = 2^r + 1$ and $p' > 7$. Since $p' - 7 = 2(2^{r-1} - 3) \mid |K/H|$, we have $2^{r-1} - 3 \mid |G|$, which is impossible. If $p' = 7$, then $p' = 2^r + 1$, which is impossible. For $p' = 5$, we have $2^r + 1 = 5$ and hence $r = 2$, but $r = 2^n - 1 \geq 3$, which is a contradiction.

Step 3. K/H is not a simple group of lie type, except ${}^2D_{r+1}(2)$.

If K/H is isomorphic to ${}^2A_5(2),\ E_7(2),\ E_7(3),\ A_2(4)$ or ${}^2E_6(2)$, then we easily get a contradiction similar to sporadic simple groups.

a) Let $K/H \cong A_1(q')$, where $q' = 2^m > 2$. Therefore $q' - 1 = p$ or $q' + 1 = p$. If $q' - 1 = p = 2^r + 1$, then $2^m - 2^r = 2$. Since $m \geq 2$ and $r \geq 3$, we get a contradiction. So $q' + 1 = p = 2^r + 1$ and, hence, $m = r$ and $|K/H| = q'(q' - 1)(q' + 1) = 2^r(2^r - 1)(2^r + 1)$. On the other hand, $G/K \leq Out(K/H)$, which implies that $|G/K| \mid r$. Therefore, $2^{r+1}(2^{r+1}+1)\prod_{i=1}^{r-1}(2^{2i}-1) \mid |H|$. By considering $\Gamma(G)$ we conclude that there exist $g \in G$ and $\chi \in Irr(G)$ such that $\pi(o(g)) \subseteq \pi(2^{r+1} + 1)$ and $\chi(g) = 0$. Since $\pi(o(g)) \subseteq \pi(2^{r+1} + 1)$, $(2^{r+1} + 1, 2^r + 1) = 1$ and $H \trianglelefteq G$, we conclude that $g \in H$. Therefore, H is a nilpotent normal subgroup of G such that $H \bigcap Van(G) \neq \phi$. Now, Lemma 2.7 implies that there exist a vanishing element whose order is divisible by all prime divisors of $|H|$. So all prime divisors of $|H|$ are adjacent in $\Gamma(G)$, which is a contradiction by Table 9 of [14].

b) Let $K/H \cong A_1(q')$, where $3 < q' \equiv \varepsilon(\mathrm{mod}\ 4)$ for $\varepsilon = \pm 1$. Hence $q' = 2^r + 1 = p$ or $(q' + \varepsilon)/2 = 2^r + 1 = p$. First let $(q' + \varepsilon)/2 = 2^r + 1$. If $\varepsilon = 1$, then $q' - 2^{r+1} = 1$, which is a contradiction with Lemma 2.9.

If $\varepsilon = -1$, then $q' \equiv -1(\mathrm{mod}\ 4)$. Since $4 \mid (q' + 1)$, we can conclude that $q' = u^\alpha$, where u is odd prime. Thus $p \in Z_\alpha(u)$ and hence by Remark 2.1, $\alpha \mid p - 1 = 2^r$. Therefore, $\alpha = 2^t$, which implies that $q = u^\alpha \equiv 1(\mathrm{mod}\ 4)$, which is a contradiction. Now let $q' = 2^r + 1 = p$. So $q' - 2^r = 1$ and, by Lemma 2.9, $q' = 9$, which implies that $r = 3$. Therefore, $|G| = 2^{12} \times 3^4 \times 5 \times 7 \times 17$, $|K/H| = 2^3 \times 3^2 \times 5$ and $|G/K| \mid 2$. Hence, $|H| = 2^9 \times 3^2 \times 7 \times 17$. Now, similar to the above case, we can conclude that all prime divisors of order of H are adjacent in $\Gamma(G)$, which is impossible.

c) Let $K/H \cong E_8(q')$. Then $p = 2^r + 1$ is an element of the set

$$\{q'^8 \pm q'^7 \mp q'^5 - q'^4 \mp q'^3 \pm q' + 1, q'^8 - q'^6 + q'^4 - q'^2 + 1, q'^8 - q'^4 + 1\}.$$

So, $p = 2^r + 1 < (q'^8 + q'^7 + q'^6 + q'^5 + q'^4 + q'^3 + q'^2 + q' + 1)(q' - 1) = q'^9 - 1 < q'^9 + 1$, which implies that $2^r < q'^9$ and, hence, $|K/H| > |G|$, which is impossible.

d) Let $K/H \cong Sz(q')$, where $q' = 2^{2m+1} > 2$. If $2^{2m+1} - 1 = p = 2^r + 1$, then $2^{2m+1} - 2^r = 2$, which is impossible. If $2^{2m+1} \pm 2^{m+1} + 1 = 2^r + 1$, then $2^{m+1}(2^m \pm 1) = 2^r$, which is impossible.

e) Let $K/H \cong {}^2F_4(q')$, where $q' = 2^{2m+1} > 2$. Then $2^{2(2m+1)} \pm 2^{3m+2} + 2^{2m+1} \pm 2^{m+1} + 1 = 2^r + 1$, which implies that $2^{m+1}(2^{3m+1} \pm 2^{2m+1} + 2^m \pm 1) = 2^r$, which is a contradiction.

f) Let $K/H \cong {}^2G_2(q')$ for $q' = 3^{2m+1} > 3$. Therefore $3^{2m+1} \pm 3^{m+1} + 1 = 2^r + 1$, and, consequently, $3^{m+1}(3^m \pm 1) = 2^r$, which is impossible. If $K/H \cong G_2(q')$, where $q' \equiv 0(\mathrm{mod}\ 3)$ and $K/H \cong {}^2B_2(q')$, one can get a contradiction similarly.

g) Let K/H be isomorphic to ${}^2D'_p(3)$, where $p' = 2^m + 1$. Then either $(3^{p'} + 1)/4 = 2^r + 1$ or $(3^{p'-1} + 1)/2 = 2^r + 1$. Now, if $(3^{p'} + 1)/4 = 2^r + 1$, then $3^{p'} - 3 = 2^{r+2}$, which is impossible. If $(3^{p'-1} + 1)/2 = 2^r + 1$, then $3^{p'-1} - 2^{r+1} = 1$, which is impossible by Lemma 2.9.

h) Therefore $K/H \cong {}^2D_{r'+1}(2)$, where $r' = 2^m - 1 \geq 3$. Obviously $m \leq n$. Since $p \in \pi(K/H)$, it follows that $p = 2^r + 1$ is a divisor of

$$2^{r'(r'+1)}(2^{r'} - 1)(2^{r'} + 1)(2^{r'+1} + 1)\prod_{i=1}^{r'-1}(2^{2i} - 1).$$

Note that p is a primitive prime divisors of $2^r + 1$. Now, if $m < n$, then $p \nmid |G|$, a contradiction. Therefore $m = n$ and, hence, $r' = r$. Thus, $K/H \cong {}^2D_{r+1}(2)$.

Case 2. Now suppose that $p = 2^{r+1} + 1$.

Step 1. K/H is not an sporadic simple group.

Suppose that K/H is an sporadic simple group. Then $p = 2^{r+1} + 1 \in \{5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71\}$. If $K/H \cong Fi_{23}$, then $p = 2^{r+1}+1 = 17, 23$ or 29. The only possibility is $r = 3$. But $|Fi_{23}| \nmid |{}^2D_4(2)|$, a contradiction. For other sporadic simple groups, one get a contradiction similarly.

Step 2. K/H is not an alternating group.

Let $K/H \cong A_{p'}$, where p' and $p' - 2$ are primes. If $p' = 2^{r+1} + 1$, then $p' - 2 = 2^{r+1} - 1$ is a prime number, which is a contradiction. If $p' - 2 = 2^{r+1} + 1$, then $p' = 2^{r+1} + 3$ is a divisor of

$$|G| = 2^{r(r+1)}(2^r - 1)(2^r + 1)(2^{r+1} + 1)\prod_{i=1}^{r-1}(2^{2i} - 1),$$

which is impossible.

Step 3. K/H is not a simple group of lie type, except ${}^2D_{r+1}(2)$.

If K/H is isomorphic to ${}^2A_5(2)$, $E_7(2)$, $E_7(3)$, $A_2(4)$ or ${}^2E_6(2)$, then we easily get a contradiction similar to sporadic simple groups.

a) Let $K/H \cong A_1(q')$, where $q' = 2^m > 2$. Therefore $q' - 1 = p$ or $q' + 1 = p$. If $q' - 1 = p = 2^{r+1}+1$, then $2^m - 2^{r+1} = 2$, which is impossible. If $q'+1 = p = 2^{r+1}+1$, then $m = r+1$ and $|K/H| = 2^{r+1}(2^{r+1} - 1)(2^{r+1} + 1)$. On the other hand, $G/K \leq Out(K/H)$, which implies that $|G/K| \mid r + 1$. Therefore $2(2^r - 1)(2^r + 1)\prod_{i=1}^{r-1}(2^{2i} - 1) \mid |H|$. By considering $\Gamma(G)$ we conclude that there exist $g \in G$ and $\chi \in Irr(G)$ such that $\pi(o(g)) \subseteq \pi(2^r+1)$ and $\chi(g) = 0$. Since $\pi(o(g)) \subseteq \pi(2^r + 1)$, $(2^{r+1} + 1, 2^r + 1) = 1$ and $H \trianglelefteq G$, we conclude that $g \in H$. Therefore, H is a nilpotent normal subgroup of G such that $H \bigcap Van(G) \neq \phi$. Now, Lemma 2.7 implies that there exist a vanishing element whose order is divisible by all prime divisors of $|H|$. So all prime divisors of $|H|$ are adjacent in $\Gamma(G)$, which is a contradiction by Table 9 of [14].

b) Let $K/H \cong A_1(q')$, where $3 < q' \equiv \varepsilon (\mathrm{mod}\ 4)$ for $\varepsilon = \pm 1$. Hence $q' = 2^{r+1} + 1 = p$ or $(q' + \varepsilon)/2 = 2^{r+1} + 1 = p$. First let $(q' + \varepsilon)/2 = 2^{r+1} + 1$. If $\varepsilon = 1$, then $q' - 2^{r+2} = 1$, which is a contradiction with Lemma 2.9.

If $\varepsilon = -1$, then $q' \equiv -1 (\mathrm{mod}\ 4)$. Since $4 \mid (q' + 1)$, we can conclude that $q' = u^\alpha$, where u is odd prime. Thus $p \in Z_\alpha(u)$ and hence by Remark 2.1, $\alpha \mid p - 1 = 2^{r+1}$. Therefore, $\alpha = 2^t$, which implies that $q = u^\alpha \equiv 1 (\mathrm{mod}\ 4)$, which is a contradiction.

Now let $q' = 2^{r+1} + 1 = p$. So $q' - 2^{r+1} = 1$ and, by Lemma 2.9, $q' = 9$, which implies that $r = 2$. Since $r = 2^n - 1 \geq 3$, we get a contradiction.

c) Let $K/H \cong E_8(q')$. Then $p = 2^{r+1} + 1$ is an element of the set

$$\{q'^8 \pm q'^7 \mp q'^5 - q'^4 \mp q'^3 \pm q' + 1, q'^8 - q'^6 + q'^4 - q'^2 + 1, q'^8 - q'^4 + 1\}.$$

So $p = 2^{r+1} + 1 < (q'^8 + q'^7 + q'^6 + q'^5 + q'^4 + q'^3 + q'^2 + q' + 1)(q' - 1) = q'^9 - 1 < q'^9 + 1$, which implies that $2^{r+1} < q'^9$ and hence $|K/H| > |G|$, which is impossible.

d) Let $K/H \cong Sz(q')$, where $q' = 2^{2m+1} > 2$. If $2^{2m+1} - 1 = p = 2^{r+1} + 1$, then $2^{2m+1} - 2^{r+1} = 2$, which is impossible. If $2^{2m+1} \pm 2^{m+1} + 1 = 2^{r+1} + 1$, then $2^{m+1}(2^m \pm 1) = 2^{r+1}$, which is impossible.

e) Let $K/H \cong {}^2F_4(q')$, where $q' = 2^{2m+1} > 2$. Then $2^{2(2m+1)} \pm 2^{3m+2} + 2^{2m+1} \pm 2^{m+1} + 1 = 2^{r+1} + 1$, which implies that $2^{m+1}(2^{3m+1} \pm 2^{2m+1} + 2^m \pm 1) = 2^{r+1}$, which is a contradiction.

f) Let $K/H \cong {}^2G_2(q')$ for $q' = 3^{2m+1} > 3$. Therefore $3^{2m+1} \pm 3^{m+1} + 1 = 2^{r+1} + 1$, and consequently $3^{m+1}(3^m \pm 1) = 2^{r+1}$, which is impossible. If $K/H \cong G_2(q')$, where $q' \equiv 0 (\mathrm{mod}\ 3)$ and $K/H \cong {}^2B_2(q')$, one can get a contradiction similarly.

g) Let K/H be isomorphic to ${}^2D'_p(3)$, where $p' = 2^m + 1$. Then either $(3^{p'} + 1)/4 = 2^{r+1} + 1$ or $(3^{p'-1} + 1)/2 = 2^{r+1} + 1$. Now, if $(3^{p'} + 1)/4 = 2^{r+1} + 1$, then $3^{p'} - 3 = 2^{r+3}$, which is impossible. If $(3^{p'-1} + 1)/2 = 2^{r+1} + 1$, then $3^{p'-1} - 2^{r+2} = 1$, which is impossible by Lemma 2.9.

h) Therefore $K/H \cong {}^2D_{r'+1}(2)$, where $r' = 2^m - 1 \geq 3$. Obviously $m \leq n$. Since $p \in \in \pi(K/H)$, it follows that $p = 2^{r+1} + 1$ is a divisor of

$$2^{r'(r'+1)}(2^{r'} - 1)(2^{r'} + 1)(2^{r'+1} + 1) \prod_{i=1}^{r'-1} (2^{2i} - 1).$$

Note that p is a primitive prime divisors of $2^{r+1} + 1$. Now, if $m < n$, then $p \nmid |G|$, a contradiction. Therefore $m = n$ and hence $r' = r$. Thus $K/H \cong {}^2D_{r+1}(2)$. So in both cases $K/H \cong {}^2D_{r+1}(2)$ and since $|G| = |{}^2D_{r+1}(2)|$, it is obvious that $H = 1$ and $G = K$, hence, $G \cong {}^2D_{r+1}(2)$.

Theorem A is proved.

***Proof of Theorem* B.** By the assumption $Vo(G) = Vo({}^2D_r(3))$, it is obvious that $\Gamma(G) = = \Gamma({}^2D_r(3))$. By Lemma 2.6, we know that $\Gamma({}^2D_r(3)) = GK({}^2D_r(3))$ has 3 connected components including an isolated vertex p, where $p \in \{(3^{r-1} + 1)/2, (3^r + 1)/4\}$. Also, note that

$$|G| = 3^{r(r-1)}(3^r + 1) \prod_{i=1}^{r-1} (3^{2i} - 1).$$

Since $p \in Vo({}^2D_r(3))$ and $Vo(G) = Vo({}^2D_r(3))$, so $p \in Vo(G)$. Thus there exist an element $g \in G$ and irreducible character $\chi \in Irr(G)$ such that $o(g) = p$ and $\chi(g) = 0$. So $p \mid \chi(1)$ and since $|G|_p = p$, we conclude that $p \nmid |G|/\chi(1)$. Therefore χ is a p-defect zero, and hence for every element $h \in G$ such that $p \mid o(h)$, we have $\chi(h) = 0$. So, by the fact p is an isolated vertex in $\Gamma(G)$, we conclude that p is an isolated vertex in $GK(G)$. Hence, $t(G) \geq 2$.

Since $\Gamma(G)$ has three connected components, Lemma 2.6 implies that G is not a solvable group and consequently G is not a 2-Frobenius group. We also claim that G is not a Frobenius group. Suppose that G is a Frobenius group with kernel K and complement H. So $|G| = |H||K|$ and $|H| \mid |K| - 1$. Lemma 2.2 implies that $GK(G)$ has two connected components $\pi(H)$ and $\pi(K)$, and since $|H| < |K|$, it follows that $|H| = p$ and $|K| = |G|/p$. In both cases $p = (3^{r-1} + 1)/2$ and $p = (3^r + 1)/4$, one can get a contradiction by the fact that $|H| \mid |K| - 1$. Therefore G is not a Frobenius group. So, by Lemma 2.4, G has a normal $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $\pi(H) \cup \pi(G/K) \subseteq \subseteq \pi_1$ and K/H is a non-Abelian simple group and $G/K \leq Aut(K/H)$. By Lemma 2.6, we have $t(K/H) \geq 3$. In both cases $p = (3^{r-1} + 1)/2$ and $p = (3^r + 1)/4$, we use the classification of finite nonabelian simple groups with more than two Gruenberg – Kegel graph connected components to prove that K/H is isomorphic to ${}^2D_r(3)$.

Case 1. First suppose that $p = (3^{r-1} + 1)/2$.

Step 1. K/H is not an sporadic simple group.

Suppose that K/H is an sporadic simple group. Then $p = (3^{r-1}+1)/2 \in \{5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71\}$. If $K/H \cong F_1$, then $p = (3^{r-1} + 1)/2 = 41$. The only possibility is $r = 5$. But $|F_1| \nmid |{}^2D_5(3)|$, which is impossible. For other sporadic simple groups one get a contradiction.

Step 2. K/H is not an alternating group.

Let $K/H \cong A_{p'}$, where p' and $p'-2$ are primes. If $p'-2 = p = (3^{r-1}+1)/2$, then $p' = (3^{r-1}+5)/2$ is a prime number, which is impossible. Let $p' = p = (3^{r-1}+1)/2$, then $p'-2 = (3^{r-1}-3)/2$ is a prime number, which is a contradiction.

Step 3. K/H is not a simple group of lie type, except ${}^2D_r(3)$.

If K/H is isomorphic to ${}^2A_5(2)$, $E_7(2)$, $E_7(3)$, $A_2(4)$ or ${}^2E_6(2)$, then we easily get a contradiction similar to sporadic simple groups.

a) Let $K/H \cong A_1(q')$, where $q' = 2^m > 2$. therefore $q'-1 = p$ or $q'+1 = p$. If $q'-1 = p = (3^{r-1}+1)/2$, then $2q' = 3^{r-1}+3$ and hence $2^{m+1} = 3(3^{r-2}+1)$, which is impossible. If $q'+1 = p = (3^{r-1}+1)/2$, then $3^{r-1}-2^{m+1} = 1$ and, by Lemma 2.10, $r-1 = 2$. Since $r = 2^n+1 \geq 5$, we get a contradiction.

b) Let $K/H \cong A_1(q')$, where $3 < q' \equiv \varepsilon (\mathrm{mod}\ 4)$ for $\varepsilon = \pm 1$. Hence $q' = (3^{r-1}+1)/2 = p$ or $(q'+\varepsilon)/2 = (3^{r-1}+1)/2 = p$. First let $(q'+\varepsilon)/2 = (3^{r-1}+1)/2$. If $\varepsilon = 1$, then $q' = 3^{r-1}$ and $|K/H| = 3^{r-1}(3^{r-1}-1)(3^{r-1}+1)/2$. On the other hand, $G/K \leq Out(K/H)$, which implies that $|G/K| \mid r-1$. Therefore $3^r(3^r+1)/4 \mid |H|$. By considering $\Gamma(G)$ we conclude that there exist $g \in G$ and $\chi \in Irr(G)$ such that $\pi(o(g)) \subseteq \pi((3^r+1)/4)$ and $\chi(g) = 0$. Since $\pi(o(g)) \subseteq \pi((3^r+1)/4)$, $((3^r+1)/4, (3^{r-1}+1)/2) = 1$ and $H \trianglelefteq G$, we conclude that $g \in H$. Therefore H is a nilpotent normal subgroup of G such that $H \bigcap Van(G) \neq \phi$. Now, Lemma 2.7 implies that there exist a vanishing element whose order is divisible by all prime divisors of $|H|$. So all prime divisors of $|H|$ are adjacent in $\Gamma(G)$, which is a contradiction by Table 9 of [14].

If $\varepsilon = -1$, then $q' = 3^{r-1}+2$ and $|K/H| = (3^{r-1}+1)(3^{r-1}+2)(3^{r-1}+3)$. Since $(3^{r-1}+2) \nmid |G|$, we get a contradiction.

If $q' = (3^{r-1}+1)/2 = p$, then $|K/H| = 3/8\big((3^{r-1}-1)(3^{r-1}+1)(3^{r-2}+1)\big)$. On the other hand, $G/K \leq Out(K/H)$, which implies that $|G/K| \mid 2$. Now, similar to the above for $\varepsilon = +1$, we can get a contradiction.

c) Let $K/H \cong E_8(q')$. Then $(3^{r-1}+1)/2$ is an element of the set

$$\{q'^8 \pm q'^7 \mp q'^5 - q'^4 \mp q'^3 \pm q' + 1, q'^8 - q'^6 + q'^4 - q'^2 + 1, q'^8 - q'^4 + 1\}.$$

So $p = (3^{r-1}+1)/2 < (q'^8 + q'^7 + q'^6 + q'^5 + q'^4 + q'^3 + q'^2 + q' + 1)(q'-1) = q'^9 - 1 < q'^9 + 1$, which implies that $3^{r-1} < q'^{10}$ and hence $|K/H| > |G|$, which is impossible.

d) Let $K/H \cong Sz(q')$, where $q' = 2^{2m+1} > 2$. If $2^{2m+1} - 1 = p = (3^{r-1}+1)/2$, then $2^{2m+2} = 3^r + 3$, which is impossible.

e) Let $K/H \cong {}^2F_4(q')$, where $q' = 2^{2m+1} > 2$. Then $2^{2(2m+1)} \pm 2^{3m+2} + 2^{2m+1} \pm 2^{m+1} + 1 = (3^{r-1}+1)/2$, which implies that $2^{m+1}(2^{3m+1} \pm 2^{2m+1} + 2^m \pm 1) = 3^{r-1}$, which is a contradiction.

f) Let $K/H \cong {}^2G_2(q')$ for $q' = 3^{2m+1} > 3$. Therefore $3^{2m+1} \pm 3^{m+1} + 1 = (3^{r-1}+1)/2$, and consequently $3^{m+1}(3^m \pm 1) = (3^{r-1}+1)/2$, which is impossible. If and $K/H \cong {}^2B_2(q')$, similarly we get a contradiction.

g) Let K/H be isomorphic to ${}^2D_{p'+1}(2)$, where $p' = 2^n - 1$, $n \geq 2$. Therefore $2^{p'} + 1 = (3^{r-1}+1)/2$ or $2^{p'+1} + 1 = (3^{r-1}+1)/2$. If $2^{p'} + 1 = (3^{r-1}+1)/2$, then $3^{r-1} - 2^{p'+1} = 1$ and, by Lemma 2.10, $r-1 = 2$. Since $r = 2^n + 1 \geq 5$, we get a contradiction. For $2^{p'+1} + 1 = (3^{r-1}+1)/2$, similar to the above we get a contradiction.

h) Therefore $K/H \cong {}^2D_{r'}(3)$, where $r' = 2^m + 1 \geq 5$. Obviously $m \leq n$. Since $p \in \pi(K/H)$, it follows that $p = (3^{r-1}+1)/2$ is a divisor of

$$3^{r'(r'-1)}(3^{r'}+1)\prod_{i=1}^{r'-1}(3^{2i}-1).$$

Note that p is a primitive prime divisors of $(3^{r-1}+1)/2$. Now, if $m<n$, then $p\nmid|G|$, a contradiction. Therefore $m=n$ and hence $r'=r$. Thus, $K/H\cong {}^2D_r(3)$.

Case 2. If $p=(3^r+1)/4$, then similar to case 1, we can conclude that $K/H\cong {}^2D_r(3)$ and by the fact that $|G|=|{}^2D_r(3)|$, we have $H=1$, $G=K$ and $G\cong {}^2D_r(3)$ as required.

Theorem B is proved.

References

1. S. Asgary, N. Ahanjideh, *Characterization of $PSL(3,q)$ by nse*, Math. Rep., **19 (60)**, № 4, 425 – 438 (2017).
2. S. Asgary, N. Ahanjideh, nse *Characterization of some finite simple groups*, Sci. Ann. Comput. Sci., **3 (2)**, 797 – 806 (2016).
3. G. Chen, *Further reflections on Thompson's conjecture*, J. Algebra, **218**, 276 – 285 (1999).
4. J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, R. A. Wilson, *Atlas of finite groups*, Clarendon Press, New York (1985).
5. G. Y. Chen, *On Frobenius and 2-Frobenius group*, J. Southwest China Normal Univ., **20**, № 5, 485 – 487 (1995).
6. P. Crescenzo, *A Diophantine equation which arises in the theory of finite groups*, Adv. Math., **17**, 25 – 29 (1975).
7. S. Dolfi, E. Pacific, L. Sanus, P. Spiga, *On the vanishing prime graph of finite groups*, J. London Math. Soc., Ser. 2, **82**, 167 – 183 (2010).
8. S. Dolfi, E. Pacific, L. Sanus, P. Spiga, *On the vanishing prime graph of solvable groups*, J. Group Theory, **13**, 189 – 206 (2010).
9. M. F. Ghasemabadi, A. Iranmanesh, M. Ahanjideh, *A new characterization of some families of finite simple groups*, Rend. Semin. Mat. Univ. Padova, **137**, 57 – 74 (2017).
10. M. F. Ghasemabadi, A. Iranmanesh, F. Mavadatpur, *A new characterization of some finite simple groups*, Sib. Math. J., **56**, 78 – 82 (2015).
11. I. M. Isaacs, *Character theory of finite groups*, Pure and Appl. Math., vol. 69, Acad. Press, New York (1976).
12. G. James, M. Liebeck, *Representations and characters of groups*, Cambridge Math. Textbooks (1993).
13. H. Shi, G. Y. Chen, *Relation between $B_p(3)$ and $C_p(3)$ with their order components where p is an odd prime*, J. Appl. Math. Inform., **27**, 653 – 659 (2009).
14. A. V. Vasil'ev, E. P. Vdovin, *An adjacency ceriterion for the prime graph of a finite simple group*, Algebra and Logic, **44**, 381 – 406 (2005).
15. J. S. Williams, *Prime graph components of finite groups*, J. Algebra, **69**, 487 – 513 (1981).
16. J. Zhang, Z. Li, C. Shao, *Finite groups whose irreducible characters vanish only on elements of prime power order*, Int. Electron. J. Algebra, **9**, 114 – 123 (2011).
17. J. Zhang, C. Shao, Z. Shen, *A new characterization of Suzuki,s simple groups*, J. Algebra and Appl., **16**, 1 – 6 (2017).
18. K. Zsigmondy, *Zur Theorie der Potenzreste*, Monatsh. Math. Phys., **3**, 265 – 284 (1892).

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