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# NONLINEAR ELLIPTIC EQUATIONS WITH MEASURE DATA IN ORLICZ SPACES

# НЕЛІНІЙНІ ЕЛІПТИЧНІ РІВНЯННЯ З ДАНИМИ МІРИ У ПРОСТОРАХ ОРЛІЧА

In this article, we study the existence result of the unilateral problem

$$Au - \mathrm{div}(\Phi(x,u)) + H(x,u,\nabla u) = \mu,$$

where $Au = -\mathrm{div}(a(x,u,\nabla u))$ is a Leray–Lions operator defined on Sobolev–Orlicz space $D(A) \subset W_0^1 L_M(\Omega)$, $\mu \in L^1(\Omega) + W^{-1}E_{\overline{M}}(\Omega)$, where $M$ and $\overline{M}$ are two complementary $N$-functions, the first and the second lower terms $\Phi$ and $H$ satisfies only the growth condition and any sign condition is assumed and $u \geq \zeta$, where $\zeta$ is a measurable function.

Вивчено питання існування для односторонньої задачі

$$Au - \mathrm{div}(\Phi(x,u)) + H(x,u,\nabla u) = \mu,$$

де $Au = -\mathrm{div}(a(x,u,\nabla u))$ — оператор Лере–Ліонса, який визначено у просторі Соболєва–Орліча $D(A) \subset W_0^1 L_M(\Omega)$, $\mu \in L^1(\Omega) + W^{-1}E_{\overline{M}}(\Omega)$, $M$ і $\overline{M}$ — дві додаткові $N$-функції, перший і другий члени $\Phi$ і $H$ задовольняють лише умову зростання та будь-яку умову знака, $u \geq \zeta$, $\zeta$ — вимірна функція.

**1. Introduction.** Let $\Omega$ be a bounded open domain in $I\!R^N$, $N \geq 2$, and consider the following strongly nonlinear Dirichlet problem with an obstacle:

$$\begin{aligned} & u \geq \zeta \quad \text{a.e. in} \quad \Omega, \\ & -\mathrm{div}(a(x,u,\nabla u)) - \mathrm{div}(\Phi(x,u)) + H(x,u,\nabla u) = \mu \quad \text{on} \quad \Omega, \qquad (1) \\ & u = 0 \quad \text{on} \quad \partial\Omega, \end{aligned}$$

where $\mu \in L^1(\Omega) + W^{-1}E_{\overline{M}}(\Omega)$.

Under our assumptions, the problem (1) does not admit, in general, a weak solution since the term $\Phi$ may not belong to $(L^1(\Omega))^N$ and a lack of coercivity for the two terms $\Phi$ and $H$. Thus to overcome this difficulty, we use in this paper the framework of entropy solution, which need less regularity than the usual weak solution. Knowing that the notion of entropy solutions have been developed by P. Bénilan et al. [9] for the study of nonlinear elliptic problems.

In fact, in the classical Sobolev space $W_0^{1,p}(\Omega)$, the paper [10] where $\Phi = 0$, $H$ has polynomial growth and $\mu$ is a measure in $\mathcal{M}_b(\Omega)$, L. Boccardo et al. proved the existence results and in [11] have demonstrated the decomposition theorem of the Radon measure and studied the existence and uniqueness of entropy solution. For more results we refer to [3, 4].

To our knowledge, it is not yet possible to decompose the Radon measure in setting Orlicz space, therefore the authors studied the problem (1) with the second member as the sum of an element from $W^{-1}E_{\overline{M}}(\Omega)$ and of a function from $L^1(\Omega)$. In the first J. P. Gossez and V. Mustonen in [13] proved the existence of entropy solutions for nonlinear problem (1) where $\mu \in W^{-1}E_{\overline{M}}(\Omega)$ and $H$ satisfies sign condition. A. Benkirane and J. Bennouna in [7] proved the existence and uniqueness of the solution of unilateral problem where $\Phi = H = 0$ and $\mu \in L^1(\Omega)$, L. Ahrouch et al. in [5] have proved the existence results where $H = 0$, $\Phi \in \mathcal{C}^0(I\!R^N, I\!R^N)$ and $\mu \in L^1(\Omega) + W^{-1}E_{\overline{M}}(\Omega)$. Recently A. Aberqi et al. in [2] proved the existence and uniqueness results for problem (1) in the parabolic case and $L^1$-sources.

In this article, we are interested in proving the existence of entropy solution for unilateral problem associated to (1) where $\Phi$ depends on $x, u$ and satisfies only the growth condition and $H$ is a nonlinear lower-order term having natural growth with respect to $|\nabla u|$. The second member of (1) as $\mu = f - \operatorname{div}(F)$ with $f \in L^1(\Omega)$ and $F \in (E_{\overline{M}}(\Omega))^N$.

The main difficulties of this problem are in the first the lack of coercivity lower order term $\Phi$ that makes the operator that governs the equation, non coercive. The second lower order term $H$ is controlled by a non-polynomial growth (see (8)) and no sign condition is assumed. Finally, the anisotropic function $M$ defining Orlicz space $W^1L_M(\Omega)$ does not satisfy the $\Delta_2$-condition.

We are not concerned here with the uniqueness of the solution. In fact, the uniqueness problem being a rather delicate one, due to a counter-example of J. Serrin [15]. Note that our result generalizes that of [5, 7, 10], to the case of Orlicz – Sobolev spaces.

This paper is organized as follows. Section 2 contains some preliminaries of Orlicz spaces and a technical lemmas. Section 3 is devoted to the specification of the assumptions on $a, \Phi, K$ and $\mu$. Main results are stated in Section 4, where we give and prove the principal theorem.

**2. Orlicz spaces and technical lemmas.** Let $M : I\!R^+ \to I\!R^+$ be an $N$-function, that is, $M$ is continuous, convex with $M(t) > 0$ for $t > 0$, $M(t)/t \to 0$ as $t \to 0$, and $M(t)/t \to +\infty$ as $t \to +\infty$. Equivalently, $M$ admits the representation $M(t) = \displaystyle\int_0^t a(s)ds$, where $a : I\!R^+ \to I\!R^+$ is nondecreasing, right continuous with $a(0) = 0$, $a(t) > 0$ for $t > 0$, and $a(t) \to +\infty$ as $t \to +\infty$.

The $N$-function $\overline{M}$ conjugate to $M$ is defined by $\overline{M}(t) = \displaystyle\int_0^t \overline{a}(s)ds$, where $\overline{a} : I\!R^+ \to I\!R^+$ is given by $\overline{a}(t) = \sup\{s : a(s) \leq t\}$. (See [1] for more details.) We will extend these $N$-functions into even functions on all $I\!R$.

***Example* 1.** For $M(t) = \dfrac{|t|^p}{p}$, $\overline{M}(t) = \dfrac{|t|^q}{q}$, where $\dfrac{1}{p} + \dfrac{1}{q} = 1$ and $p, q \in (1; +\infty)$. For $M(t) = \exp(t) - 1 - |t|$, $\overline{M}(t) = (1 + |t|)\ln(1 + |t|) - |t|$.

Let $P$ and $Q$ be two $N$-functions. $P \ll Q$ means that $P$ grows essentially less rapidly than $Q$, that is, for each $\varepsilon > 0$, $\lim_{t\to+\infty} \dfrac{P(t)}{Q(\varepsilon t)} = 0$.

**Proposition 1.** *$P \ll M$ if and only if, for all $\varepsilon > 0$, there exists a constant $c_\varepsilon$ such that*

$$P(t) \leq M(\varepsilon t) + c_\varepsilon \quad \textit{for all} \quad t \geq 0. \tag{2}$$

***Proof.*** Let $\varepsilon > 0$, then, by the definition of $P \ll M$, there exists $t_\varepsilon > 0$ such that, for all $t > t_\varepsilon$, $P(t) \leq M(\varepsilon t)$. On the other hand, for $t \in [0, t_\varepsilon]$, by the continuity of $P$, there exists a constant $C_\varepsilon$ such that $P(t) \leq C_\varepsilon$, where $C_\varepsilon = \sup_{t\in[0,t_\varepsilon]} P(t)$. So, from the above we have (2).

The Orlicz class $K_M(\Omega)$ (resp., the Orlicz space $L_M(\Omega)$) is defined as the set of (equivalence classes) real valued measurable functions $u$ on $\Omega$ such that

$$\int_\Omega M(u(x))dx < +\infty \left(\text{resp.,} \quad \int_\Omega M\left(\frac{u(x)}{\lambda}\right)dx < +\infty \quad \text{for some} \quad \lambda > 0\right).$$

The set $L_M(\Omega)$ is Banach space under the norm

$$\|u\|_{M,\Omega} = \inf\left\{\lambda > 0\colon \int_\Omega M\left(\frac{u(x)}{\lambda}\right)dx \le 1\right\}$$

and $K_M(\Omega)$ is a convex subset of $L_M(\Omega)$ . The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega)$. The dual $E_M(\Omega)$ can be identified with $L_{\overline{M}}(\Omega)$ by means of the pairing $\int_\Omega uvdx$, and the dual norm of $L_{\overline{M}}(\Omega)$ is equivalent to $\|u\|_{\overline{M},\Omega}$.

We now turn to the Orlicz–Sobolev space, $W^1L_M(\Omega)$ (resp., $W^1E_M(\Omega)$) is the space of all functions $u$ such that $u$ and its distributional derivatives up to order 1 lie in $L_M(\Omega)$ $\big($resp., $E_M(\Omega)\big)$. It is a Banach space under the norm

$$\|u\|_{1,M} = \|u\|_{M,\Omega} + \sum_{1\le i\le N}\left\|\frac{\partial u}{\partial x_i}\right\|_{M,\Omega}.$$

Thus, $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspaces of product of $(N+1)$ copies of $L_M(\Omega)$ . Denoting this product by $\Pi L_M$, we will use the weak topologies $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ and $\sigma(\Pi L_M, \Pi L_{\overline{M}})$. The space $W_0^1E_M(\Omega)$ is defined as the (norm) closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^1E_M(\Omega)$ and the space $W_0^1L_M(\Omega)$ as the $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ closure of $\mathcal{D}(\Omega)$ in $W^1L_M(\Omega)$.

Let $W^{-1}L_{\overline{M}}(\Omega)$ (resp., $W^{-1}E_{\overline{M}}(\Omega)$) denote the space of distributions on $\Omega$ which can be written as sums of derivatives of order $\le 1$ of functions in $L_{\overline{M}}(\Omega)$ (resp., $E_{\overline{M}}(\Omega)$). It is a Banach space under the usual quotient norm (see [1]). We recall the following lemmas.

**Lemma 1** [14]. *For all* $u \in W_0^1L_M(\Omega)$ *with* $\operatorname{meas}(\Omega) < +\infty$ *one has*

$$\int_\Omega M\left(\frac{|u|}{\delta}\right)dx \le \int_\Omega M(|\nabla u|)dx, \tag{3}$$

*where* $\delta = \operatorname{diam}(\Omega)$ *is the diameter of* $\Omega$.

**Lemma 2** [8]. *Let* $\Omega$ *be an open subset of* $I\!R^N$ *with finite measure. Let* $M, P$ *and* $Q$ *be* $N$-*functions such that* $Q \ll P$, *and let* $f\colon \Omega\times I\!R \to I\!R$ *be a Carathéodory function such that*

$$|f(x,s)| \le c(x) + k_1P^{-1}M(k_2|s|) \quad a.e.\ x\in\Omega, \quad for\ all \quad s\in I\!R,$$

*where* $k_1, k_2$ *are real constants and* $c(x) \in E_Q(\Omega)$.

*Then the Nemytskii operator defined by* $N_f(u)(x) = f(x,u(x))$ *is strongly continuous from* $P\left(E_M(\Omega), \dfrac{1}{k_2}\right) = \left\{u \in L_M(\Omega)\colon d(u, E_M(\Omega)) < \dfrac{1}{k_2}\right\}$ *into* $E_Q(\Omega)$.

**Lemma 3** [14]. *Let* $u_n$ *and* $u$ *belong to* $L_M(\Omega)$. *If* $u_n \to u$ *with respect to the modular convergence, then* $u_n \to u$ *for* $\sigma(\Pi L_M, \Pi L_{\overline{M}})$.

**Lemma 4** [14]. *Let $\Omega$ has the segment property. Then, for each $v \in W_0^1 L_M(\Omega)$, there exists a sequence $v_n \in \mathcal{D}(\Omega)$ such that $v_n$ converges to $v$ for the modular convergence in $v \in W_0^1 L_M(\Omega)$. Furthermore, if $v \in W_0^1 L_M(\Omega) \cap L^\infty(\Omega)$, then*

$$\|v_n\|_{L^\infty(\Omega)} \le (N+1)\|v\|_{L^\infty(\Omega)}.$$

**3. Formulation of the problem.** Let $\Omega$ be an open bounded subset of $I\!R^N$, $N \ge 2$, and let $M$ and $P$ be two $N$-functions such that $P \ll M$.

$a\colon \Omega \times I\!R \times I\!R^N \to I\!R^N$ is Carathéodory function such that, for a.e. $x \in \Omega$ and for all $s \in I\!R$, $\xi$, $\xi^* \in I\!R^N$, $\xi \ne \xi^*$,

$$|a(x,s,\xi)| \le \beta(a_0(x) + \overline{M}^{-1} P(k_1|s|)) + \overline{M}^{-1} M(k_2|\xi|)) \tag{4}$$

with $\beta, k_1, k_2 > 0$ and $a_0(\cdot) \in E_{\overline{M}}(\Omega)$,

$$(a(x,s,\xi) - a(x,s,\xi^*))(\xi - \xi^*) > 0, \tag{5}$$

$$a(x,s,\xi).\xi \ge \alpha M(|\xi|) + M(|s|). \tag{6}$$

$\Phi : \Omega \times I\!R \to I\!R^N$ is a Carathéodory function such that

$$|\Phi(x,s)| \le c(x)\overline{M}^{-1} M(\alpha_0|s|), \tag{7}$$

where $c(\cdot) \in L^\infty(\Omega)$ such that $\|c(\cdot)\|_{L^\infty(\Omega)} < \dfrac{\alpha}{2}$ and $0 < \alpha_0 < \min\left(1, \dfrac{1}{\alpha}\right)$.

$H : \Omega \times I\!R \times I\!R^N \to I\!R$ is a Carathéodory function such that

$$|H(x,s,\xi)| \le h(x) + \rho(s)M(|\xi|), \tag{8}$$

$\rho\colon I\!R \to I\!R^+$ is a continuous positive function which belongs $L^1(I\!R) \cap L^\infty(I\!R)$ and $h$ belongs to $L^1(\Omega)$.

Let $\mu \in L^1(\Omega) + W^{-1}E_{\overline{M}}(\Omega)$ such that

$$\mu = f - \operatorname{div}(F) \quad \text{with} \quad f \in L^1(\Omega) \quad \text{and} \quad F \in (E_{\overline{M}}(\Omega))^N. \tag{9}$$

Given a negative measurable obstacle function $\zeta : \Omega \to I\!R$,

$$K_\zeta = \{u \in W_0^1 L_M(\Omega) : u \ge \zeta \ \text{ a.e. in } \ \Omega\}, \tag{10}$$

and we suppose that $K_\zeta \cap L^\infty(\Omega) \ne \varnothing$.

Throughout the paper, $T_k$ denotes the truncation at height $k \ge 0$:

$$T_k(r) = \max(-k; \min(k, r)).$$

**Definition 1.** *A measurable function $u$, defined on $\Omega$, is said an entropy solution of problem* (1), *if it satisfies the following conditions*:

$$u \in D(A) \cap W_0^1 L_M(\Omega), \quad u \ge \zeta,$$

$$\int_{\Omega} a(x,u,\nabla u)\nabla T_k(u-v)dx + \int_{\Omega} \Phi(x,u)\nabla T_k(u-v)dx +$$

$$+ \int_{\Omega} H(x,u,\nabla u)\nabla T_k(u-v)dx \leq \int_{\Omega} fT_k(u-v)dx + \int_{\Omega} F\nabla T_k(u-v)dx$$

$$\forall v \in K_\zeta \cap L^\infty(\Omega) \quad \forall k > 0. \tag{11}$$

**4. Main results.**

**Theorem 1.** *Assume that* (4)–(10) *hold true. Then there exists at least one solution of the unilateral problem* (1) *in the sense of the Definition* 1.

***Remark*** **1.** 1. The condition (6) can be replaced by the weaker one

$$a(x,s,\xi)\xi \geq \alpha M(|\xi|) + M(|s|) - b(x),$$

where $b(x)$ is in $L^1$-function.

2. The results obtained in Theorem 1, remain true if we replace (7) by the general growth condition

$$|\Phi(x,s)| \leq c(x)\overline{P}^{-1}P(|s|), \tag{12}$$

where $c(\cdot) \in E_P(\Omega)$ and $P \ll M$.

3. For any $s \in I\!R$ and $\alpha' > 0$, we have

$$\exp\left(-\frac{\|\rho\|_{L^1(I\!R)}}{\alpha'}\right) \leq \exp\left(\pm G(s)\right) \leq \exp\left(\frac{\|\rho\|_{L^1(I\!R)}}{\alpha'}\right), \tag{13}$$

where $G(s) = \displaystyle\int_0^s \frac{\rho(r)}{\alpha'}dr$.

***Remark*** **2.** 1. For the sake of the simplification, the explicit dependence on $x$ of the functions $a, \Phi$ and $H$ will be omited so that $a(x,u,\nabla u) = a(u,\nabla u)$, $\Phi(x,u) = \Phi(u)$ and $H(x,u,\nabla u) = H(u,\nabla u)$.

2. We will denote by $C_i$ with $i = 1, 2, \ldots$ any constant which depends on the various quantities of the problem but not on $n$.

***Proof of Theorem*** **1.** *Step* 1: Approximate problem. For each $n > 0$, we define the approximations

$$a_n(x,s,\xi) = a(T_n(s),\xi), \quad \Phi_n(x,s) = \Phi(T_n(s)), \quad H_n(x,s,\xi) = \frac{H(s,\xi)}{1 + \frac{1}{n}|H(s,\xi)|} \quad \text{a.e.} \quad x \in \Omega,$$

for all $s \in I\!R$ and $\xi \in I\!R^N$.

Let $(f_n)_n$ be a sequence of a smooth functions such that $f_n \to f$ strongly in $L^1(\Omega)$. Let us now consider the approximate problem

$$u_n \in K_\zeta \cap D(A),$$

$$\int_\Omega a_n(u_n, \nabla u_n)\nabla(u_n - v)dx + \int_\Omega \Phi_n(u_n)\nabla(u_n - v)dx +$$

$$+\int_\Omega H_n(u_n, \nabla u_n)(u_n - v)dx \leq \int_\Omega f_n(u_n - v)dx + \int_\Omega F\nabla(u_n - v)dx \quad \forall v \in K_\zeta. \tag{14}$$

Since $H_n$ is bounded for any fixed $n > 0$, there exists at least one solution $u_n \in W_0^1 L_M(\Omega)$ of (14) (see [13]).

Now, let show that $u_n$ converges to a function $u$, where $u$ is the solution of the unilateral problem (1).

*Step* 2: A priori estimates.

**Lemma 5.** *Let choose* $\{u_n\}_n$ *be a solution of the approximate problem* (14). *Then, for all* $k > 0$, *there exist two positive constants* $C_1$ *and* $C_2$ *such that*

$$\int_\Omega M(|\nabla T_k(u_n)|)dx \leq kC_1 + C_2. \tag{15}$$

***Proof.*** Let $v_0 \in K_\zeta \cap L^\infty(\Omega) \cap W_0^1 E_M(\Omega)$ and fix $k > 0$. Let $u_n - \exp(G(u_n))T_k(u_n - v_0)^+$ as a test function in problem (14), where $G(s) = \int_0^s \frac{\rho(r)}{\alpha'}dr$, and $\alpha' > 0$ is a parameter to be specified later. We get

$$\int_\Omega a_n(u_n, \nabla u_n)\nabla\left(\exp(G(u_n))T_k(u_n - v_0)^+\right)dx +$$

$$+\int_\Omega \Phi_n(u_n)\nabla\left(\exp(G(u_n))T_k(u_n - v_0)^+\right)dx +$$

$$+\int_\Omega H_n(u_n, \nabla u_n)\exp(G(u_n))T_k(u_n - v_0)^+dx \leq$$

$$\leq k\exp\left(\frac{\|\rho\|_{L^1}}{\alpha'}\right)\|f_n\|_{L^1(\Omega)} + \int_\Omega F\nabla\left(\exp(G(u_n))T_k(u_n - v_0)^+\right)dx. \tag{16}$$

In the second term of the left-hand side of (16) we use (7), Lemma 1 and Young inequality to get

$$\int_\Omega \Phi_n(u_n)\nabla(\exp(G(u_n))T_k(u_n - v_0)^+)dx \leq$$

$$\leq \frac{\|c(\cdot)\|_{L^\infty(\Omega)}}{\alpha'}\Bigg[\alpha_0\int_\Omega M(u_n)\rho(u_n)\exp(G(u_n))T_k(u_n - v_0)^+dx +$$

$$+\int_\Omega M(\nabla u_n)\rho(u_n)\exp(G(u_n))T_k(u_n - v_0)^+dx\Bigg] +$$

$$+2\alpha_0\|c(\cdot)\|_{L^\infty(\Omega)}\int\limits_{\{0\le u_n-v_0\le k\}} M(u_n)\exp(G(u_n))dx+$$

$$+\|c(\cdot)\|_{L^\infty(\Omega)}\int\limits_{\{0\le u_n-v_0\le k\}} M(|\nabla T_k(u_n-v_0)^+|)\exp(G(u_n))dx+c_1.$$

For the third term of the left-hand side of (16), we have

$$\int\limits_\Omega H_n(u_n,\nabla u_n)\exp(G(u_n))T_k(u_n-v_0)^+dx\le k\exp\left(\frac{\|\rho\|_{L^1}}{\alpha'}\right)\int\limits_\Omega |h(x)|dx+$$

$$+\int\limits_\Omega \rho(u_n)\exp(G(u_n))M(|\nabla u_n|)T_k(u_n-v_0)^+dx.$$

For the second term of the right-hand side of (16), we obtain

$$\int\limits_\Omega F\nabla\left(\exp(G(u_n))T_k(u_n-v_0)^+\right)dx\le\frac{k}{\alpha'}\exp\left(\frac{\|\rho\|_{L^1}}{\alpha'}\right)\|\rho\|_{L^\infty}\int\limits_\Omega \overline{M}\left(\frac{|F|}{\varepsilon_1}\right)dx+$$

$$+\frac{\varepsilon_1}{\alpha'}\int\limits_\Omega \rho(u_n)\exp(G(u_n))M(|\nabla u_n|)T_k(u_n)^+dx+$$

$$+2\exp\left(\frac{\|\rho\|_{L^1}}{\alpha'}\right)\int\limits_\Omega \overline{M}\left(\frac{|F|}{\varepsilon_1}\right)dx+\varepsilon_1\int\limits_{\{0\le u_n-v_0\le k\}}\exp(G(u_n))M(|\nabla u_n|)dx+c_2.$$

Finally, by using the above and (6) in (16), we get

$$\frac{1-\alpha_0\|c(\cdot)\|_{L^\infty(\Omega)}}{\alpha'}\int\limits_\Omega M(|u_n|)\rho(u_n)\exp(G(u_n))T_k(u_n-v_0)^+dx+$$

$$+\left[\frac{\alpha}{\alpha'}-\frac{\|c(\cdot)\|_{L^\infty(\Omega)}}{\alpha'}-1-\frac{\varepsilon_1}{\alpha'}\right]\int\limits_\Omega \rho(u_n)\exp(G(u_n))M(|\nabla u_n|)T_k(u_n-v_0)^+dx+$$

$$+\int\limits_\Omega a(u_n,\nabla u_n)\exp(G(u_n))\nabla(T_k(u_n-v_0)^+)dx\le$$

$$\le 2\alpha_0\|c(\cdot)\|_{L^\infty(\Omega)}\int\limits_{\{0\le u_n-v_0\le k\}}\exp(G(u_n))M(|\nabla u_n|)dx+$$

$$+(\|c(\cdot)\|_{L^\infty(\Omega)}+\varepsilon_1)\int\limits_{\{0\le u_n-v_0\le k\}}\exp(G(u_n))M(|\nabla u_n|)dx+c_3k+c_4,$$

where $c_4=c_1+c_2$.

If we choose $\alpha'$ and $\varepsilon_1$ such that $\alpha' = \dfrac{\alpha}{2}$, $\varepsilon_1 < \dfrac{\alpha}{2} - \|c(\cdot)\|_{L^\infty(\Omega)}$, by using that $T_k(u_n - v_0)^+ = u_n - v_0$ for $x \in \{x \in \Omega : 0 \le u_n - v_0 \le k\}$, we obtain

$$\int\limits_{\{0\le u_n - v_0 \le k\}} a_n(u_n, \nabla u_n)\exp(G(u_n))\nabla u_n dx \le$$

$$\le \frac{\alpha_0(\varepsilon_1 + \|c(\cdot)\|_{L^\infty(\Omega)})}{\alpha}\left[\int\limits_{\{0\le u_n - v_0 \le k\}} \exp(G(u_n))M(|u_n|)dx\right] +$$

$$+ \left[(\varepsilon_1 + \|c(\cdot)\|_{L^\infty(\Omega)})\int\limits_{\{0\le u_n - v_0 \le k\}} \exp(G(u_n))M(|\nabla u_n|)dx\right] +$$

$$+ \int\limits_{\{0\le u_n - v_0 \le k\}} a_n(u_n, \nabla u_n)\exp(G(u_n))\nabla v_0 dx + c_3 k + c_4.$$

Since $\alpha_0\alpha < 1$, by using (6), we get

$$\left[1 - \frac{\varepsilon_1 + \|c(\cdot)\|_{L^\infty(\Omega)}}{\alpha}\right]\int\limits_{\{0\le u_n - v_0 \le k\}} a_n(u_n, \nabla u_n)\exp(G(u_n))\nabla u_n dx \le$$

$$\le \int\limits_{\{0\le u_n - v_0 \le k\}} a_n(u_n, \nabla u_n)\exp(G(u_n))\nabla v_0 dx + c_3 k + c_4.$$

By using (5), we get, for any $\alpha_1 > 0$ ($\alpha_1$ is a parameter to be specified later),

$$\alpha_1 \int\limits_{\{0\le u_n - v_0 \le k\}} a_n(u_n, \nabla u_n)\exp(G(u_n))\frac{\nabla v_0}{\alpha_1} dx \le$$

$$\le \alpha_1 \int\limits_{\{0\le u_n - v_0 \le k\}} a_n(u_n, \nabla u_n)\exp(G(u_n))\nabla u_n dx -$$

$$- \int\limits_{\{0\le u_n - v_0 \le k\}} a_n(u_n, \nabla u_n)\exp(G(u_n))\nabla\left(u_n - \frac{\nabla v_0}{\alpha_1}\right) dx.$$

Then

$$c' \int\limits_{\{0\le u_n - v_0 \le k\}} a_n(u_n, \nabla u_n)\exp(G(u_n))\nabla u_n dx \le$$

$$\le \alpha_1 \int\limits_{\{0\le u_n - v_0 \le k\}} \left|a_n\left(u_n, \frac{\nabla v_0}{\alpha_1}\right)\right|\exp(G(u_n))|\nabla u_n| dx +$$

$$+\alpha_1 \int\limits_{\{0\le u_n - v_0 \le k\}} \left| a_n\left(u_n, \frac{\nabla v_0}{\alpha_1}\right)\right| \exp(G(u_n)) \left|\frac{\nabla v_0}{\alpha_1}\right| dx. \tag{17}$$

Taking $c' = \left[1 - \dfrac{\varepsilon_1 + \|c(\cdot)\|_{L^\infty(\Omega)}}{\alpha} - \alpha_1\right]$ such that $\alpha_1 < 1 - \dfrac{\varepsilon_1 + \|c(\cdot)\|_{L^\infty(\Omega)}}{\alpha}$, using (2), (4) and Young inequality, we have

$$\left| a_n\left(u_n, \frac{\nabla v_0}{\alpha_1}\right)\right| \exp(G(u_n)) |\nabla u_n| \le$$

$$\le \beta \exp\left(\frac{\|\rho\|_{L^1(I\!R)}}{\alpha'}\right) \left[\overline{M}(a(x)) + M\left(\frac{|u_n|}{\delta}\right) + C' + M\left(k_2 \frac{|\nabla v_0|}{\alpha_1}\right) + 3M(|\nabla u_n|)\right] \tag{18}$$

and

$$\left| a_n\left(u_n, \frac{\nabla v_0}{\alpha_1}\right)\right| \exp(G(u_n)) \left|\frac{\nabla v_0}{\alpha_1}\right| \le$$

$$\le \beta \exp\left(\frac{\|\rho\|_{L^1(I\!R)}}{\alpha'}\right) \left[\overline{M}(a(x)) + M\left(\frac{|u_n|}{\delta}\right) + C' + M\left(k_2 \frac{|\nabla v_0|}{\alpha_1}\right) + 3M\left(\frac{|\nabla v_0|}{\alpha_1}\right)\right]. \tag{19}$$

Since $\dfrac{\nabla v_0}{\alpha_1} \in (E_M(\Omega))^N$, one pass to the integral in (18), (19), by using Lemma 1, (13) and (17), we get

$$c' \int\limits_{\{0\le u_n - v_0 \le k\}} a_n(u_n, \nabla u_n) \exp(G(u_n)) \nabla u_n dx \le$$

$$\le 4\alpha_1 \beta \exp\left(\frac{\|\rho\|_{L^1(\Omega)}}{\alpha'}\right) \int\limits_{\{0\le u_n - v_0 \le k\}} M(|\nabla u_n|) dx + c_5 k + c_6.$$

Taking also $\alpha_1$ such that $4\alpha_1\beta \exp\left(\dfrac{\|\rho\|_{L^1(\Omega)}}{\alpha'}\right) < \dfrac{c'}{2} \exp\left(-\dfrac{\|\rho\|_{L^1(\Omega)}}{\alpha'}\right)$ and using (6), we obtain

$$\int\limits_{\{0\le u_n - v_0 \le k\}} M(|\nabla u_n|) dx \le c_7 k + c_8. \tag{20}$$

Similarly, taking $u_n + \exp(-G(u_n)) T_k(u_n - v_0)^-$ as a test function in problem (14), we get

$$c' \int\limits_{\{-k\le u_n - v_0 \le 0\}} a_n(u_n, \nabla u_n) \exp(-G(u_n)) \nabla u_n dx \le$$

$$\le \frac{c'}{2} \int\limits_{\{-k\le u_n - v_0 \le 0\}} \exp(-G(u_n)) M(|\nabla u_n|) dx + c_9 k + c_{10} \tag{21}$$

and

$$\int\limits_{\{-k\le u_n-v_0\le 0\}} \exp(-G(u_n))M(|\nabla u_n|)dx \le c_{11}k + c_{12}.$$

Combining now (20) and (21), we deduce (15).

Since $\{x\in\Omega; |u_n|\le k\}\subset\{x\in\Omega; |u_n-v_0|\le k+|v_0|_\infty\}$, we obtain

$$\int\limits_\Omega M(|\nabla T_k(u_n)|)dx = \int\limits_{\{|u_n|\le k\}} M(|\nabla T_k(u_n)|)dx \le$$

$$\le \int\limits_{\{|u_n-v_0|\le k+\|v_0\|_\infty\}} M(|\nabla T_k(u_n)|)dx \le kC_1 + C_2.$$

We conclude that $\{T_k(u_n)\}_n$ is bounded in $W_0^1L_M(\Omega)$ independently of $n$ and for any $k>0$, so there exists a subsequence still denoted by $u_n$ such that

$$T_k(u_n) \rightharpoonup \xi_k \quad \text{weakly in} \quad W_0^1L_M(\Omega). \tag{22}$$

On the other hand, by using Lemma 1, we have

$$M\left(\frac{k}{\delta}\right)\text{meas}\{|u_n|>k\} \le \int\limits_{\{|u_n|>k\}} M\left(\frac{|T_k(u_n)|}{\delta}\right)dx \le$$

$$\le \int\limits_\Omega M(|\nabla T_k(u_n)|)dx \le kC_1 + C_2.$$

Then

$$\text{meas}\,\{|u_n|>k\} \le \frac{kC_3+C_4}{M\left(\dfrac{k}{\delta}\right)} \quad \text{for all} \quad n \quad \text{and} \quad k.$$

Thus, we get

$$\lim_{k\to\infty} \text{meas}\,\{|u_n|>k\} = 0.$$

*Step* 3: Now we turn to prove the almost every convergence of $\{u_n\}_n$ and the convergence of $a_n(T_k(u_n),\nabla T_k(u_n))$.

**Proposition 2.** *Let $u_n$ be a solution of the approximate problem* (14), *then*

$$u_n \to u \quad \textit{a.e. in} \quad \Omega, \tag{23}$$

$$a_n(T_k(u_n),\nabla T_k(u_n)) \rightharpoonup \varpi_k \quad \textit{in} \quad (L_{\overline{M}}(\Omega))^N \quad \textit{for} \quad \sigma(\Pi L_{\overline{M}},\Pi E_M), \tag{24}$$

*for some $\varpi_k\in(L_{\overline{M}}(\Omega))^N$.*

***Proof. Proof of* (23).** Let $\eta>0$, $\varepsilon>0$ and $k>0$, then

$$\text{meas}\{|u_n-u_m|>\eta\} \le \text{meas}\{|u_n|>k\} + \text{meas}\{|u_m|>k\} +$$

$$+\text{meas}\{|T_k(u_n)-T_k(u_m)|>\eta\},$$

by (22), we can assume that $(T_k(u_n))_n$ is a Cauchy sequence in measure in $\Omega$. Thus, there exists $k(\varepsilon) > 0$ such that meas$\{|T_k(u_n) - T_k(u_m)| > \eta\} < \varepsilon$ for all $n, m > n_0$. Hence, $\{u_n\}_n$ is a Cauchy sequence in measure in $\Omega$ and then converges almost everywhere to some measurable function $u$.

***Proof of* (24).** We shall prove that $\{a(T_k(u_n), \nabla T_k(u_n))\}_n$ is bounded in $\left(L_{\overline{M}}(\Omega)\right)^N$ for all $k > 0$.

Let $w \in (E_M(\Omega))^N$ be arbitrary. By (5) we have

$$(a(u_n, \nabla u_n) - a(u_n, w))(\nabla u_n - w) > 0.$$

Then

$$\int\limits_{\{|u_n| \le k\}} a(u_n, \nabla u_n) w dx \le \int\limits_{\{|u_n| \le k\}} a(u_n, \nabla u_n) \nabla u_n dx + \int\limits_{\{|u_n| \le k\}} a(u_n, w)(w - \nabla u_n) dx.$$

By using (4), the convexity of $\overline{M}$ and the definition of $T_k$, we get

$$\int\limits_{\{|u_n| \le k\}} \overline{M} \left( \frac{a\left(u_n, \dfrac{w}{k_2}\right)}{3\beta} \right) dx \le \frac{\beta}{3\nu} \int\limits_{\Omega} \left[ \overline{M}(a_0(x)) + P(k_1 |T_k(u_n)|) + M(|w|) \right] dx \le$$

$$\le \frac{\beta}{3\nu} \int\limits_{\Omega} \left[ \overline{M}(a_0(x)) + P(k_1 k) dx + M(|w|) \right] dx \quad \text{for} \quad \nu > \beta.$$

Thus, $\left\{ a\left( T_k(u_n), \dfrac{w}{k_2} \right) \right\}_n$ is bounded in $(L_{\overline{M}}(\Omega))^N$. By (15), (21) and by Banach – Steinhaus theorem, the sequence $\{a(T_k(u_n), \nabla T_k(u_n))\}_n$ remains bounded in $(L_{\overline{M}}(\Omega))^N$ and we conclude (24).

*Step* 4: Almost everywhere convergence of the gradients. To have that the gradient converges almost everywhere, we need to prove this proposition.

**Proposition 3.** *Let* $\{u_n\}_n$ *be a solution of the approximate problem* (14), *then*

$$\lim_{m \to \infty} \limsup_{n \to \infty} \int\limits_{\{m \le |u_n| \le m+1\}} a(u_n, \nabla u_n) \nabla u_n dx = 0 \tag{25}$$

*and, for a subsequence as* $n \to \infty$,

$$\nabla u_n \to \nabla u \quad \text{a.e. in} \quad \Omega.$$

***Proof.*** Choosing, in the equation (14), the test function $Z_m(u_n) = u_n + \exp(-G(u_n))\psi_m(u_n)^-$, where $\psi_m(u_n) = T_1(u_n - T_m(u_n))$, we get

$$\int\limits_{\Omega} a_n(u_n, \nabla u_n) \nabla(\exp(-G(u_n))\psi_m(u_n)^-) dx +$$

$$+ \int\limits_{\Omega} \Phi_n(u_n) \nabla(\exp(-G(u_n))\psi_m(u_n)^-) dx +$$

$$+\int_\Omega H_n(u_n,\nabla u_n)\exp(-G(u_n))\psi_m(u_n)^- dx \geq$$

$$\geq \int_\Omega f_n \exp(-G(u_n))\psi_m(u_n)^-\, dx + \int_\Omega F\nabla(\exp(-G(u_n))\psi_m(u_n)^-)dx. \qquad (26)$$

In the second term of the left-hand side of (26), we use (3), (7) and Young inequality to get

$$\int_\Omega \Phi_n(u_n)\nabla(\exp(-G(u_n))\psi_m(u_n)^-)dx \leq$$

$$\leq \frac{\|c(\cdot)\|_{L^\infty(\Omega)}}{\alpha'}\alpha_0 \int_\Omega M(|u_n|)\rho(u_n)\exp(-G(u_n))\psi_m(u_n)^- dx +$$

$$+\frac{\|c(\cdot)\|_{L^\infty(\Omega)}}{\alpha'}\int_\Omega M(|\nabla u_n|)\exp(-G(u_n))\psi_m(u_n)^- dx +$$

$$+\alpha_0\|c(\cdot)\|_{L^\infty(\Omega)}\int_{\{-(m+1)\leq u_n\leq -m\}} M(|u_n|)\exp(-G(u_n))dx +$$

$$+\|c(\cdot)\|_{L^\infty(\Omega)}\int_\Omega M(|\nabla\psi_m(u_n)^-|)\exp(-G(u_n))dx.$$

For the second term of the right-hand side of (26), we have

$$\int_\Omega F\nabla(\exp(-G(u_n))\psi_m(u_n)^-)dx \leq \frac{\|\rho\|_{L^\infty}}{\alpha'}\exp\left(\frac{\|\rho\|_{L^1}}{\alpha'}\right)\int_\Omega \overline{M}\left(\frac{|F|}{\varepsilon_1}\right)\psi_m(u_n)^- dx +$$

$$+\frac{\varepsilon_1}{\alpha'}\int_\Omega \rho(u_n)\exp(-G(u_n))M(|u_n|)\psi_m(u_n)^- dx +$$

$$+\exp\left(\frac{\|\rho\|_{L^1}}{\alpha'}\right)\int_{\{-(m+1)\leq u_n\leq -m\}} M\left(\frac{|F|}{\varepsilon_1}\right)dx +$$

$$+\varepsilon_1\int_\Omega \exp(-G(u_n))M(|\nabla\psi_m(u_n)^-|)dx.$$

By using the same argument is step 2, we obtain

$$\int_\Omega a_n(u_n,\nabla u_n)\nabla(\psi_m(u_n)^-)dx \leq$$

$$\leq \exp\left(\frac{\|\rho\|_{L^1}}{\alpha'}\right) C\left(\int\limits_{\Omega} f_n Z_m(u_n)dx + \int\limits_{\Omega} h(x)Z_m(u_n)dx\right) +$$

$$+C\frac{\|\rho\|_{L^1}}{\alpha'}\exp\left(\frac{\|\rho\|_{L^1}}{\alpha'}\right)\left[\int\limits_{\Omega} M\left(\frac{|F|}{\varepsilon_1}\right)Z_m(u_n)dx + \int\limits_{\Omega}\overline{M}\left(\frac{|F|}{\varepsilon_1}\right)dx\right],$$

where $\dfrac{1}{C} = \left[1 - \dfrac{\|c(x)\|_{L^\infty(\Omega)} + \varepsilon_1}{\alpha}\right]\exp\left(-\dfrac{\|\rho\|_{L^1}}{\alpha'}\right)$.

Passing to limit as $n \to +\infty$, since the pointwise convergence of $u_n$ and strongly convergence in $L^1(\Omega)$ of $f_n$, we get

$$\lim_{n\to+\infty}\int\limits_{\Omega} a_n(u_n, \nabla u_n)\nabla(\psi_m(u_n)^-)dx \leq$$

$$\leq \exp\left(\frac{\|\rho\|_{L^1}}{\alpha'}\right) C\left[\int\limits_{\Omega} fZ_m(u)\,dx + \int\limits_{\Omega} h(x)Z_m(u)dx +\right.$$

$$\left.+\frac{\|\rho\|_{L^\infty}}{\alpha'}\int\limits_{\Omega}\overline{M}\left(\frac{|F|}{\varepsilon_1}\right)Z_m(u)dx + \int\limits_{\{-(m+1)\leq u\leq -m\}}\overline{M}\left(\frac{|F|}{\varepsilon_1}\right)dx\right].$$

By using Lebesgue's theorem and passing to limit as $m \to +\infty$, in the all term of the right-hand side, we get

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\int\limits_{\{-(m+1)\leq u_n\leq -m\}} a_n(u_n, \nabla u_n)\nabla u_n dx = 0.$$

In the same way, we take $Z_m(u_n) = u_n - \exp(G(u_n))\psi_m(u_n)^+$ and choosing in approximation equation (14), the test function $Z_m(u_n)$, we also obtain

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\int\limits_{\{m\leq u_n\leq m+1\}} a_n(u_n, \nabla_n)\nabla u_n = 0.$$

On the above we get (25).

For the almost everywhere convergence of the gradient (see Appendix).

*Step* 5: Compactness of the nonlinearities. We shall prove that $H_n(u_n, \nabla u_n) \to H(u, \nabla u)$ strongly in $L^1(\Omega)$.

Let $\chi$ be the characteristic function and consider $g_0(u_n) = \displaystyle\int_0^{u_n}\rho(s)\chi_{\{s>h\}}ds$. Choosing $u_n +$ $+\exp(G(u_n))g_0(u_n)$ in the equation (14), we get, after using the same technique in step 2,

$$\int\limits_{\{u_n>h\}} a_n(u_n, \nabla u_n)\rho(u_n)\chi_{\{u_n>h\}}\nabla u_n dx \leq$$

$$\leq C\left(\int\limits_{h}^{+\infty}\rho(s)dx\right)\left[\|f\|_{L^1(\Omega)}+\|h(x)\|_{L^1(\Omega)}+\frac{\|\rho\|_{L^\infty(I\!R)}}{\alpha'}\int\limits_{\Omega}\overline{M}\left(\frac{F}{\varepsilon_1}\right)dx\right]+$$

$$+\exp\left(\frac{\|\rho\|_{L^1(I\!R)}}{\alpha'}\right)\|\rho\|_{L^1(I\!R)}\int\limits_{\{u_n>h\}}\overline{M}\left(\frac{F}{\varepsilon_1}\right)dx,$$

where $\dfrac{1}{C}=\left[1-\dfrac{\|c(x)\|_{L^\infty(\Omega)}+\varepsilon_1}{\alpha}\right]\exp\left(-\dfrac{2\|\rho\|_{L^1}}{\alpha'}\right)$.

Since $\rho\in L^1(I\!R)$ and by (6), we get

$$\lim_{n\to\infty}\sup_{n\in I\!N}\int\limits_{\{u_n>h\}}\rho(u_n)M(\nabla u_n)dx=0.$$

Similarly, let $g_0(u_n)=\displaystyle\int_{u_n}^{0}\rho(s)\chi_{\{s<-h\}}dx$ and choosing in (14) the test function $u_n+$ $+\exp(-G(u_n))g_0(u_n)$, we have also

$$\lim_{n\to\infty}\sup_{n\in I\!N}\int\limits_{\{u_n<-h\}}\rho(u_n)M(\nabla u_n)dx=0.$$

We conclude that

$$\lim_{n\to\infty}\sup_{n\in I\!N}\int\limits_{\{|u_n|>h\}}\rho(u_n)M(\nabla u_n)dx=0.$$

Let $D\subset\Omega$, then

$$\int\limits_{D}\rho(u_n)M(\nabla u_n)dx\leq\max_{\{|u_n|\leq h\}}(\rho(x))\int\limits_{D\cap\{|u_n|\leq h\}}M(\nabla u_n)dx+$$

$$+\int\limits_{D\cap\{|u_n|>h\}}\rho(u_n)M(\nabla u_n)dx.$$

Consequently, $\rho(u_n)M(\nabla u_n)$ is equiintegrable. Then $\rho(u_n)M(\nabla u_n)$ converges to $\rho(u)M(\nabla u)$ strongly in $L^1(I\!R)$. Hence, by (8), we get our result.

*Step* 6: We show that $u$ satisfies (11). Let $v\in K_\zeta\cap L^\infty(\Omega)$, then by Lemma 4 there exists $v_j\in\mathcal{D}(\Omega)$ such that $v_j\to v$ in for the modular convergence in $W_0^1L_M(\Omega)$, with $\|v_j\|_{L^\infty(\Omega)}\leq$ $\leq(N+1)\|v\|_{L^\infty(\Omega)}$ and we can take $v_j\in K_\zeta$.

By choosing, in the approximate equation (14), the test function $T_k(u_n-v_j)$, we get

$$\int\limits_{\Omega}a_n(u_n,\nabla u_n)\nabla T_k(u_n-v_j)dx+\int\limits_{\Omega}\Phi_n(u_n)\nabla T_k(u_n-v_j)dx+$$

$$+\int\limits_{\Omega}H_n(u_n,\nabla u_n)T_k(u_n-v_j)dx=\int\limits_{\Omega}f_nT_k(u_n-v_j)dx-\int\limits_{\Omega}F\nabla T_k(u_n-v_j)dx. \qquad (27)$$

We pass to the limit in (27), as $n\to+\infty$ and $j\to+\infty$:

We can follow same way as in [7] to prove that

$$\liminf_{j\to\infty}\liminf_{n\to\infty}\int\limits_{\Omega} a(u_n,\nabla u_n)\nabla T_k(u_n-v_j)dx \geq \int\limits_{\Omega} a(u,\nabla u)\nabla T_k(u-v)dx.$$

For $n \geq \mathbf{K}$, where $\mathbf{K} = k + (N+1)\|v\|_{L^\infty(\Omega)}$, we have

$$\Phi_n(u_n)\nabla T_k(u_n-v_j) = \Phi(T_{\mathbf{K}}(u_n))\nabla T_k(u_n-v_j).$$

The pointwise convergence of $u_n$ to $u$ as $n \to +\infty$ and (12), gives $\Phi(T_{\mathbf{K}}(u_n))\nabla T_k(u_n-v_j) \rightharpoonup$ $\rightharpoonup \Phi(T_{\mathbf{K}}(u))\nabla T_k(u-v_j)$ weakly for $\sigma(\Pi L_M, \Pi L_{\overline{M}})$. In a similar way, we obtain

$$\lim_{j\to\infty}\int\limits_{\Omega}\Phi(T_{\mathbf{K}}(u))\nabla T_k(u-v_j)dx = \int\limits_{\Omega}\Phi(T_{\mathbf{K}}(u))\nabla T_k(u-v)dx =$$

$$= \int\limits_{\Omega}\Phi(u)\nabla T_k(u-v)dx.$$

Limit of $H_n(u_n,\nabla u_n)T_k(u_n-v_j)$: Since $H_n(u_n,\nabla u_n)$ converges strongly to $H(u,\nabla u)$ in $L^1(\Omega)$ and the pointwise convergence of $u_n$ to $u$ as $n \to +\infty$, it is possible to prove that $H_n(u_n,\nabla u_n)T_k(u_n-v_j)$ converges to $H(u,\nabla u)T_k(u-v_j)$ in $L^1(\Omega)$ and

$$\lim_{j\to\infty}\int\limits_{\Omega} H(u,\nabla u)T_k(u-v_j)dx = \int\limits_{\Omega} H(u,\nabla u)T_k(u-v)dx.$$

Since $f_n$ converges strongly to $f$ in $L^1(\Omega)$ and $T_k(u_n-v_j) \to T_k(u-v_j)$ *-weakly in $L^\infty(\Omega)$, we have $\int_\Omega f_n T_k(u_n-v_j)dx \to \int_\Omega f T_k(u-v_j)dx$ as $n\to\infty$ and also $\int_\Omega f T_k(u-v_j)dx \to$ $\to \int_\Omega f T_k(u-v)dx$ as $j\to\infty$, then it easy to get

$$\lim_{j\to\infty}\lim_{n\to\infty}\int\limits_{\Omega} F\nabla T_k(u_n-v_j)dx = \int\limits_{\Omega} F\nabla T_k(u-v)dx.$$

Theorem 1 is proved.

***Example* 2.** As an examples of equations to which the present result entropy solutions can be applied, we give

1) for $M(t) = \dfrac{1}{p}|u|^p$, $a(x,u,\nabla u) = |\nabla u|^{p-2}\nabla u$, $\Phi(x,u) = c(x)|\alpha_0 u|^{\frac{p}{q}}$, $c(\cdot) \in L^\infty(\Omega)$ and $F \in (E_{\overline{M}}(\Omega))^N$,

$$-\mathrm{div}(|\nabla u|^{p-2}\nabla u) - \mathrm{div}(\Phi(x,u)) = f - \mathrm{div}(F) \quad \text{in} \quad \Omega,$$

$$u = 0 \quad \text{on} \quad \partial\Omega;$$

2) for $M(t) = t\log(1+t)$ and $a(x,u,\nabla u) = (1+|u|^2)\nabla u \dfrac{\log(1+|\nabla u|)}{|\nabla u|}$, $c(\cdot) \in L^\infty(\Omega)$ and $F = 0$,

$$-\mathrm{div}(a(x,u,\nabla u)) - \mathrm{div}\left(c(x)\exp\left(\frac{\eta}{\|x\|+1}\right)\overline{M}^{-1}M(\alpha_0|u|)\right) = f \quad \text{in} \quad \Omega,$$

$$u = 0 \quad \text{on} \quad \partial\Omega.$$

**Appendix.** First call the following lemma.

**Lemma 6** [6]. *Under assumptions* (4)–(9), *and let* $(z_n)$ *be a sequence in* $W_0^1L_M(\Omega)$ *such that*

$$z_n \rightharpoonup z \quad \textit{for} \quad \sigma(\Pi L_M, \Pi E_{\overline{M}}),$$

$$\{a(z_n,\nabla z_n)\}_n \quad \textit{is bounded in } (L_{\overline{M}}(\Omega))^N,$$

$$\int_\Omega [a(z_n,\nabla z_n) - a(z_n,\nabla z\chi_s)][\nabla z_n - \nabla z\chi_s]dx \to 0$$

*as* $n$ *and* $s$ *tend to* $+\infty$, *and where* $\chi_s$ *is the characteristic function of* $\Omega_s = \{x \in \Omega; |\nabla z| \leq s\}$. *Then*

$$\nabla z_n \to \nabla z \quad \textit{a.e. in} \quad \Omega,$$

$$\lim_{n\to+\infty}\int_\Omega a(z_n,\nabla z_n)\nabla z_n dx = \int_\Omega a(z,\nabla z)\nabla z dx,$$

$$M(|\nabla z_n|) \to M(|\nabla z|) \quad \textit{in } L^1(\Omega).$$

Now, we show that $\nabla u_n \to \nabla u$ a.e. in $\Omega$, where $u_n$ is the solution of the approximate problem (14).

Indeed, we introduce a sequence of increasing $\mathbf{C}^1(\mathbb{R})$-functions $S_m$ such that, for any $m \geq 1$,

$$S_m(r) = 1 \quad \text{for} \quad |r| \leq m,$$

$$S_m(r) = m + 1 - |r| \quad \text{for} \quad m \leq |r| \leq m+1,$$

$$S_m(r) = 0 \quad \text{for} \quad |r| \geq m+1,$$

and we denote by $\varepsilon(n,\eta,j,m)$ the quantities (possibly different) such that

$$\lim_{m\to+\infty}\lim_{j\to+\infty}\lim_{\eta\to+\infty}\lim_{\mu\to+\infty}\lim_{n\to+\infty}\varepsilon(n,\eta,j,m) = 0.$$

Let $\upsilon_j \in \mathcal{D}(\Omega)$ be a sequence such that $\upsilon_j \to u$ in $W_0^1L_M(\Omega)$ for the modular convergence. For fixed $k \geq 0$, let $W_\eta^{n,j} = T_\eta(T_k(u_n) - T_k(\upsilon_j))^+$ and $W_\eta^j = T_\eta(T_k(u) - T_k(\upsilon_j))^+$.

Choosing in the approximating equation the test function $\exp(G(u_n))W_\eta^{n,j}S_m(u_n)$ and using (6) and (8), we obtain

$$\int\limits_\Omega a_n(u_n,\nabla u_n)\exp(G(u_n))\nabla(W_\eta^{n,j})S_m(u_n)dx +$$

$$+\int\limits_\Omega a_n(u_n,\nabla u_n)\nabla u_n\exp(G(u_n))W_\eta^{n,j}S'_m(u_n)\,dx -$$

$$-\int\limits_\Omega \Phi_n(u_n)\exp(G(u_n))\nabla(W_\eta^{n,j})S_m(u_n)\,dx -$$

$$-\int\limits_\Omega \Phi_n(u_n)\nabla u_n\exp(G(u_n))W_\eta^{n,j}S'_m(u_n)dx \le$$

$$\le \int\limits_\Omega f_n\exp(G(u_n))W_\eta^{n,j}S_m(u_n)\,dx + \int\limits_\Omega h(x)\exp(G(u_n))W_\eta^{n,j}S_m(u_n)dx +$$

$$+\int\limits_\Omega F\exp(G(u_n))\nabla(W_\eta^{n,j})S_m(u_n)\,dx + \int\limits_\Omega F\nabla u_n\exp(G(u_n))W_\eta^{n,j}S'_m(u_n)dx. \qquad (28)$$

Now we pass to the limit in (28) for $k$ real number fixed.

In order to perform this task we prove below the following results for any fixed $k\ge 0$:

$$\int\limits_\Omega \Phi_n(u_n)S_m(u_n)\exp(G(u_n))\nabla(W_\eta^{n,j})\,dx = \varepsilon(n,j) \quad \text{for any } m\ge 1, \qquad (29)$$

$$\int\limits_\Omega \Phi_n(u_n)\nabla u_n S'_m(u_n)\exp(G(u_n))W_\eta^{n,j}\,dx = \varepsilon(n,j) \qquad \text{for any } m\ge 1, \qquad (30)$$

$$\int\limits_\Omega a_n(u_n,\nabla u_n)\nabla u_n S'_m(u_n)\exp(G(u_n))W_\eta^{n,j}\,dx \le \varepsilon(n,m), \qquad (31)$$

$$\int\limits_\Omega a_n(u_n,\nabla u_n)S_m(u_n)\exp(G(u_n))\nabla(W_\eta^{n,j})\,dx \le C\eta + \varepsilon(n,j,m), \qquad (32)$$

$$\int\limits_\Omega f_nS_m(u_n)\exp(G(u_n))W_\eta^{n,j}\,dx + \int\limits_\Omega h(x)\exp(G(u_n))W_\eta^{n,j}S_m(u_n)\,dx \le C\eta + \varepsilon(n,\eta), \qquad (33)$$

$$\int\limits_\Omega F\exp(G(u_n))\nabla(W_\eta^{n,j})S_m(u_n)\,dx + \int\limits_\Omega F\nabla u_n\exp(G(u_n))W_\eta^{n,j}S'_m(u_n)dx \le$$

$$\le \varepsilon(n,m,j,\eta), \qquad (34)$$

$$\int\limits_{\Omega} [a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(u))]\,[\nabla T_k(u_n) - \nabla T_k(u)]\,dx \to 0. \tag{35}$$

***Proof of*** **(29).** If we take $n > m+1$, we get

$$\Phi_n(u_n)\exp(G(u_n))S_m(u_n) = \Phi(T_{m+1}(u_n))\exp(G(T_{m+1}(u_n)))S_m(T_{m+1}(u_n)).$$

Then $\Phi_n(u_n)\exp(G(u_n))S_m(u_n)$ is bounded in $L_{\overline{M}}(Q)$, thus, by using the pointwise convergence of $u_n$ and Lebesgue's theorem, we obtain

$$\Phi_n(u_n)\exp(G(u_n))S_m(u_n) \to \Phi(u)\exp(G(u))S_m(u) \quad \text{as} \quad n \to +\infty$$

with the modular convergence.

Then $\Phi_n(u_n)\exp(G(u_n))S_m(u_n) \to \Phi(u)\exp(G(u))S_m(u)$ for $\sigma(\Pi L_{\overline{M}}, \Pi L_M)$.

In the other hand, $\nabla W_\eta^{n,j} = \nabla T_k(u_n) - \nabla(T_k(\upsilon_j))$, for $|T_k(u_n) - (T_k(\upsilon_j)| \le \eta$, converges to $\nabla T_k(u) - \nabla(T_k(\upsilon_j))$ weakly in $(L_M(\Omega))^N$, then

$$\int\limits_{\Omega} \Phi_n(u_n)\exp(G(u_n))S_m(u_n)\nabla W_\eta^{n,j}\,dx \to \int\limits_{\Omega} \Phi(u)S_m(u)\exp(G(u))\nabla W_\eta^{j}\,dx$$

as $n \to +\infty$.

By using the modular convergence of $W_\eta^j$ as $j \to +\infty$ and letting $\mu$ tends to infinity, we get (29).

***Proof of*** **(30).** For $n > m+1 > k$ , we have $\nabla u_n S'_m(u_n) = \nabla T_{m+1}(u_n)$ a.e. in $\Omega$. By the almost every where convergence of $u_n$, we have $\exp(G(u_n))W_\eta^{n,j} \to \exp(G(u))W_\eta^j$ in $L^\infty(\Omega)$ weak-* and since the sequence $(\Phi_n(T_{m+1}(u_n)))_n$ converges strongly in $E_{\overline{M}}(\Omega)$, then

$$\Phi_n(T_{m+1}(u_n))\exp(G(u_n))\;W_\eta^{n,j} \to \Phi(T_{m+1}(u))\;\exp(G(u))W_\eta^j$$

converges strongly in $E_{\overline{M}}(\Omega)$ as $n \to +\infty$. By virtue of $\nabla T_{m+1}(u_n) \to \nabla T_{m+1}(u)$ weakly in $(L_M(\Omega))^N$, we have

$$\int\limits_{\{m\le|u_n|\le m+1\}} \Phi_n(T_{m+1}(u_n))\nabla u_n S'_m(u_n)\exp(G(u_n))W_\eta^{n,j}\,dx \to$$

$$\to \int\limits_{\{m\le|u|\le m+1\}} \Phi(u)\nabla u\exp(G(u))W_\eta^j\,dx \quad \text{as} \quad n \to +\infty.$$

With the modular convergence of $W_\eta^j$ as $j \to +\infty$ and letting $\mu \to +\infty$, we get (30).

***Proof of*** **(31).** For (31), we have

$$\int\limits_{\Omega} a_n(u_n, \nabla u_n)\nabla u_n \exp(G(u_n))W_\eta^{n,j}S'_m(u_n)\,dx =$$

$$= \int\limits_{\{m\le|u_n|\le m+1\}} a_n(u_n, \nabla u_n)\nabla u_n \exp(G(u_n))W_\eta^{n,j}S'_m(u_n)\,dx \le$$

$$\leq \eta C \int\limits_{\{m\leq|u_n|\leq m+1\}} a_n(u_n,\nabla u_n)\nabla u_n\,dx.$$

By using (25), we get

$$\int\limits_{\Omega} a_n(u_n,\nabla u_n)\nabla u_n \exp(G(u_n))W_\eta^{n,j}S'_m(u_n)\,dx \leq \varepsilon(n,m).$$

***Proof of* (33).** Since $S_m(r)\leq 1$ and $W_\eta^{n,j}\leq\eta$, we get

$$\int\limits_{\Omega} f_n S_m(u_n)\exp(G(u_n))W_\eta^{n,j}dx \leq \varepsilon(n,\eta)$$

and

$$\int\limits_{\Omega} h(x)\exp(G(u_n))W_\eta^{n,j}S_m(u_n)dx \leq C\eta.$$

***Proof of* (34).** We obtain

$$\int\limits_{\Omega} F\exp(G(u_n))\nabla(W_\eta^{n,j})S_m(u_n)\,dx + \int\limits_{\Omega} F\nabla u_n\exp(G(u_n))W_\eta^{n,j}S'_m(u_n)dx = K_{F,1}+K_{F,2}.$$

For the first integral, we have

$$K_{F,1}\leq \exp\left(\frac{\|\rho\|_{L^1(I\!R)}}{\alpha'}\right)\int\limits_{\Omega} F\,\nabla W_\eta^{n,j}dx \leq \varepsilon(\eta).$$

Since $T_k(u_n)$ and $T_k(v_j)$ converges weakly in $W_0^1L_M(\Omega)$, we deduce

$$K_{F,1}\leq\varepsilon(n,j,\eta).$$

For the second integral, we know that $\nabla u_n S'_m(u_n)=\nabla T_{m+1}(u_n)$ and using (6), we get

$$K_{F,2}\leq \exp\left(\frac{\|\rho\|_{L^1(I\!R)}}{\alpha'}\right)\left[\varepsilon_1\int\limits_{\Omega}\overline{M}\left(\frac{F}{\varepsilon_1}\right)W_\eta^{n,j}dx + \varepsilon_1\eta\int\limits_{m\leq|u_n|\leq m+1} a_n(u_n,\nabla u_n)\nabla u_n dx\right]\leq$$

$$\leq\varepsilon(n,m,j,\eta).$$

***Proof of* (32).** We obtain

$$\int\limits_{\Omega} a_n(u_n,\nabla u_n)S_m(u_n)\exp(G(u_n))\nabla W_\eta^{n,j}\,dx =$$

$$= \int\limits_{\{|u_n|\leq k\}\cap\{0\leq T_k(u_n)-T_k(v_j)\leq\eta\}} a_n(T_k(u_n),\nabla T_k(u_n))S_m(u_n)\exp(G(u_n))\times$$

$$\times(\nabla T_k(u_n) - \nabla T_k(\upsilon_j))\,dx -$$

$$- \int\limits_{\{|u_n|>k\}\cap\{0\le T_k(u_n)-T_k(\upsilon_j)\le\eta\}} a_n(u_n,\nabla u_n)\nabla T_k(\upsilon_j)\exp(G(u_n))S_m(u_n)\,dx. \qquad (36)$$

Since $a_n(T_{k+\eta}(u_n),\nabla T_{k+\eta}(u_n))$ is bounded in $(L_{\overline{M}}(\Omega))^N$, there exist some $\varpi_{k+\eta}\in(L_{\overline{M}}(\Omega))^N$ such that $a_n(T_{k+\eta}(u_n),\nabla T_{k+\eta}(u_n))\to\varpi_{k+\eta}$ weakly in $(L_{\overline{M}}(\Omega))^N$. Consequently,

$$\int\limits_{\{|u_n|>k\}\cap\{0\le T_k(u_n)-T_k(\upsilon_j)\le\eta\}} a_n(u_n,\nabla u_n)\nabla T_k(\upsilon_j)\exp(G(u_n))S_m(u_n)\,dx =$$

$$= \int\limits_{\{|u|>k\}\cap\{0\le T_k(u)-T_k(\upsilon_j)\le\eta\}} \varpi_{k+\eta}.\nabla T_k(\upsilon_j)S_m(u)\exp(G(u))\,dx + \varepsilon(n), \qquad (37)$$

where we have used the fact that

$$S_m(u_n)\exp(G(u_n))\nabla T_k(\upsilon_j)\chi_{\{|u_n|>k\}\cap\{0\le T_k(u_n)-T_k(\upsilon_j)\le\eta\}}\to$$

$$\to S_m(u)\exp(G(u))\nabla T_k(\upsilon_j)\chi_{\{|u|>k\}\cap\{0\le T_k(u)-T_k(\upsilon_j)\le\eta\}}$$

strongly in $(E_M(\Omega))^N$.

Letting $j\to+\infty$, we obtain

$$\int\limits_{\{|u|>k\}\cap\{0\le T_k(u)-T_k(\upsilon_j)\le\eta\}} S_m(u)\exp(G(u))\varpi_{k+\eta}.\nabla T_k(\upsilon_j)\,dx =$$

$$= \int\limits_{\{|u|>k\}\cap\{0\le T_k(u)-T_k(u)\le\eta\}} S_m(u)\exp(G(u))\varpi_{k+\eta}.\nabla T_k(u)\,dx + \varepsilon(n,j).$$

One easily has,

$$\int\limits_{\{|u|>k\}\cap\{0\le T_k(u)-T_k(u)\le\eta\}} S_m(u)\exp(G(u))\varpi_{k+\eta}.\nabla T_k(u)\,dx = \varepsilon(n,j,\mu).$$

By (28) – (33), (36) and (37), we have

$$\int\limits_{\{|u_n|\le k\}\cap\{0\le T_k(u_n)-T_k(\upsilon_j)\le\eta\}} S_m(u_n)\exp(G(u_n))a_n(T_k(u_n),\nabla T_k(u_n))\times$$

$$\times(\nabla T_k(u_n) - \nabla T_k(\upsilon_j))\,dx \le$$

$$\le C\eta + \varepsilon(n,j,\mu,m).$$

We know that $\exp(G(u_n))\ge 1$ and $S_m(u_n)=1$ for $|u_n|\le k$, then

$$\int\limits_{\{|u_n|\le k\}\cap\{0\le T_k(u_n)-T_k(v_j)\le\eta\}} a_n(x,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n)-\nabla T_k(v_j))\,dx \le$$

$$\le C\eta+\varepsilon(n,j,\mu,m). \tag{38}$$

***Proof of* (35).** Setting for $s>0$, $\Omega^s=\{x\in\Omega\colon|\nabla T_k(u)|\le s\}$ and $\Omega^s_j=\{x\in\Omega\colon |\nabla T_k(v_j)|\le s\}$ and denoting by $\chi^s$ and $\chi^s_j$ the characteristic functions of $\Omega^s$ and $\Omega^s_j$, respectively, we deduce that letting $0<\delta<1$, define

$$\Theta_{n,k}=(a(T_k(u_n),\nabla T_k(u_n))-a(T_k(u_n),\nabla T_k(u)))(\nabla T_k(u_n)-\nabla T_k(u)).$$

For $s>0$, we have

$$0\le\int\limits_{\Omega^s}\Theta^\delta_{n,k}\,dx=\int\limits_{\Omega^s}\Theta^\delta_{n,k}\chi_{\{0\le T_k(u_n)-T_k(v_j)\le\eta\}}\,dx+\int\limits_{\Omega^s}\Theta^\delta_{n,k}\chi_{\{T_k(u_n)-T_k(v_j)>\eta\}}\,dx.$$

The first term of the right-hand side with the Hölder inequality

$$\int\limits_{\Omega^s}\Theta^\delta_{n,k}\chi_{\{0\le T_k(u_n)-T_k(v_j)\le\eta\}}\,dx\le\left(\int\limits_{\Omega^s}\Theta_{n,k}\chi_{\{0\le T_k(u_n)-T_k(v_j)\le\eta\}}\,dx\right)^\delta\left(\int\limits_{\Omega^s}dx\right)^{1-\delta}\le$$

$$\le C_1\left(\int\limits_{\Omega^s}\Theta_{n,k}\chi_{\{0\le T_k(u_n)-T_k(v_j)\le\eta\}}\,dx\right)^\delta.$$

Also, by using the Hölder inequality and second term of the right-hand side, we have

$$\int\limits_{\Omega^s}\Theta^\delta_{n,k}\chi_{\{T_k(u_n)-T_k(v_j)>\eta\}}\,dx\le\left(\int\limits_{\Omega^s}\Theta_{n,k}\,dx\right)^\delta\left(\int\limits_{\{T_k(u_n)-T_k(v_j)>\eta\}}dx\right)^{1-\delta}.$$

Since $\{a(T_k(u_n),\nabla T_k(u_n))\}_n$ is bounded in $(L_{\overline{M}}(\Omega))^N$, while $\{\nabla T_k(u_n)\}_n$ is bounded in $(L_M(\Omega))^N$, then

$$\int\limits_{\Omega^s}\Theta^\delta_{n,k}\chi_{\{T_k(u_n)-T_k(v_j)>\eta\}}\,dx\le C_2\left(\text{meas}\{x\in\Omega\colon T_k(u_n)-T_k(v_j)>\eta\}\right)^{1-\delta}.$$

We obtain

$$\int\limits_{\Omega^s}\Theta^\delta_{n,k}\,dx\le C_1\left(\int\limits_{\Omega^s}\Theta_{n,k}\chi_{\{0\le T_k(u_n)-T_k(v_j)\le\eta\}}\,dx\right)^\delta+$$

$$+\,C_2\left(\text{meas}\{x\in\Omega\colon T_k(u_n)-T_k(v_j)>\eta\}\right)^{1-\delta}.$$

On the other hand,

$$\int\limits_{\Omega^s} \Theta_{n,k}\chi_{\{0\le T_k(u_n)-T_k(\upsilon_j)\le\eta\}}\,dx \le$$

$$\le \int\limits_{\{0\le T_k(u)-T_k(\upsilon_j)\le\eta\}} (a(T_k(u_n),\nabla T_k(u_n)) - a(T_k(u_n),\nabla T_k(u)\chi_s))\times$$

$$\times(\nabla T_k(u_n) - \nabla T_k(u)\chi_s)\,dx.$$

For each $s>r$, $r>0$, one has

$$0 \le \int\limits_{\Omega^r\cap\{0\le T_k(u_n)-T_k(\upsilon_j)\le\eta\}} (a(T_k(u_n),\nabla T_k(u_n)) - a(T_k(u_n),\nabla T_k(u)))\times$$

$$\times(\nabla T_k(u_n) - \nabla T_k(u))\,dx \le$$

$$\le \int\limits_{\Omega^s\cap\{0\le T_k(u_n)-T_k(\upsilon_j)\le\eta\}} (a(T_k(u_n),\nabla T_k(u_n)) - a(T_k(u_n),\nabla T_k(u)))\times$$

$$\times(\nabla T_k(u_n) - \nabla T_k(u))\,dx =$$

$$= \int\limits_{\Omega^s\cap\{0\le T_k(u_n)-T_k(\upsilon_j)\le\eta\}} (a(T_k(u_n),\nabla T_k(u_n)) - a(T_k(u_n),\nabla T_k(u)\chi_s))\times$$

$$\times(\nabla T_k(u_n) - \nabla T_k(u)\chi_s)\,dx \le$$

$$\le \int\limits_{\Omega\cap\{0\le T_k(u_n)-T_k(\upsilon_j)\le\eta\}} (a(T_k(u_n),\nabla T_k(u_n)) - a(T_k(u_n),\nabla T_k(u)\chi^s))\times$$

$$\times(\nabla T_k(u_n) - \nabla T_k(u)\chi^s)\,dx =$$

$$= \int\limits_{\{0\le T_k(u_n)-T_k(\upsilon_j)\le\eta\}} (a(T_k(u_n),\nabla T_k(u_n)) - a(T_k(u_n),\nabla T_k(\upsilon_j)\chi_j^s))\times$$

$$\times(\nabla T_k(u_n) - \nabla T_k(\upsilon_j)\chi_j^s)\,dx +$$

$$+ \int\limits_{\{0\le T_k(u_n)-T_k(\upsilon_j)\le\eta\}} a(T_k(u_n),\nabla T_k(u_n))(\nabla T_k(\upsilon_j)\chi_j^s - \nabla T_k(u)\chi^s)\,dx +$$

$$+ \int\limits_{\{0\le T_k(u_n)-T_k(\upsilon_j)\le\eta\}} (a(T_k(u_n),\nabla T_k(\upsilon_j)\chi_j^s) - a(T_k(u_n),\nabla T_k(u)\chi^s))\nabla T_k(u_n)\,dx -$$

$$-\int\limits_{\{0\le T_k(u_n)-T_k(\upsilon_j)\le\eta\}} a(T_k(u_n),\nabla T_k(\upsilon_j)\chi_j^s)\nabla T_k(\upsilon_j)\chi_j^s\,dx+$$

$$+\int\limits_{\{0\le T_k(u_n)-T_k(\upsilon_j)\le\eta\}} a(T_k(u_n),\nabla T_k(u)\chi^s)\nabla T_k(u)\chi^s\,dx=$$

$$=I_1(n,j,s)+I_2(n,j)+I_3(n,j)+I_4(n,j)+I_5(n).$$

We will go to the limit as $n,\ j,\ \mu$ and $s\to+\infty$:

$$I_1=\int\limits_{\{0\le T_k(u_n)-T_k(\upsilon_j)\le\eta\}} a(T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n)-\nabla T_k(\upsilon_j))\,dx-$$

$$-\int\limits_{\{0\le T_k(u_n)-T_k(\upsilon_j)\le\eta\}} a(T_k(u_n),\nabla T_k(u_n))(\nabla T_k(\upsilon_j)\chi_j^s-\nabla T_k(\upsilon_j))\,dx-$$

$$-\int\limits_{\{0\le T_k(u_n)-T_k(\upsilon_j)\le\eta\}} a(T_k(u_n),\nabla T_k(\upsilon_j)\chi_j^s)(\nabla T_k(u_n)-\nabla T_k(\upsilon_j)\chi_j^s)\,dx.$$

Using (38) and the first term of the right-hand side, we get

$$\int\limits_{\{0\le T_k(u_n)-T_k(\upsilon_j)\le\eta\}} a(T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n)-\nabla T_k(\upsilon_j))\,dx\le$$

$$\le C\eta+\varepsilon(n,m,j,s)-\int\limits_{\{|u|>k\}\cap\{0\le T_k(u)-T_k(\upsilon_j)\le\eta\}} a(T_k(u),0)\nabla T_k(\upsilon_j)\,dx\le$$

$$\le C\eta+\varepsilon(n,m,j,\mu).$$

The second term of the right-hand side tends to

$$\int\limits_{\{0\le T_k(u)-T_k(\upsilon_j)\le\eta\}} \varpi_k(\nabla T_k(\upsilon_j)\chi_j^s-\nabla T_k(\upsilon_j))\,dx.$$

Since $\{a(T_k(u_n),\nabla T_k(u_n))\}_n$ is bounded in $(L_{\overline{M}}(\Omega))^N$, there exists $\varpi_k\in(L_{\overline{M}}(\Omega))^N$ such that (for a subsequence still denoted by $u_n$)

$$a(T_k(u_n),\nabla T_k(u_n))\to\varpi_k\quad\text{in}\quad(L_M(\Omega))^N\quad\text{for}\quad\sigma(\Pi L_{\overline{M}},\Pi E_M).$$

In view of the fact that

$$(\nabla T_k(\upsilon_j)\chi_j^s-\nabla T_k(\upsilon_j))\chi_{\{0\le T_k(u_n)-T_k(\upsilon_j)\le\eta\}}\to$$

$$\to(\nabla T_k(\upsilon_j)\chi_j^s-\nabla T_k(\upsilon_j))\chi_{\{0\le T_k(u)-T_k(\upsilon_j)\le\eta\}}$$

strongly in $(E_M(\Omega))^N$ as $n \to +\infty$. The third term of the right-hand side tends to

$$\int\limits_{\{0 \le T_k(u) - T_k(\upsilon_j) \le \eta\}} a(T_k(u), \nabla T_k(\upsilon_j)\chi_j^s)(\nabla T_k(u) - \nabla T_k(\upsilon_j)\chi_j^s)\, dx.$$

Since

$$a(T_k(u_n), \nabla T_k(\upsilon_j)\chi_j^s)\chi_{\{0 \le T_k(u_n) - T_k(\upsilon_j) \le \eta\}} \to a(T_k(u), \nabla T_k(\upsilon_j)\chi_j^s)\chi_{\{0 \le T_k(u) - T_k(\upsilon_j) \le \eta\}}$$

in $(E_{\overline{M}}(\Omega))^N$ while $(\nabla T_k(u_n) - \nabla T_k(\upsilon_j)\chi_j^s) \rightharpoonup (\nabla T_k(u) - \nabla T_k(\upsilon_j)\chi_j^s)$ in $(L_M(\Omega))^N$ for $\sigma(\Pi L_M, \Pi E_{\overline{M}})$. Passing to limit as $j \to +\infty$ and $\mu \to +\infty$ and using Lebesgue's theorem, we have

$$I_2 = \varepsilon(n, j).$$

Similar ways as above give

$$I_3 = \varepsilon(n, j),$$

$$I_4 = \int\limits_{\{0 \le T_k(u) - T_k(\upsilon_j) \le \eta\}} a(T_k(u), \nabla T_k(u))\nabla T_k(u)\, dx + \varepsilon(n, j, \mu, s, m),$$

$$I_5 = \int\limits_{\{0 \le T_k(u) - T_k(\upsilon_j) \le \eta\}} a(T_k(u), \nabla T_k(u))\nabla T_k(u)\, dx + \varepsilon(n, j, \mu, s, m).$$

Finally, we obtain

$$\int\limits_{\Omega^s} \Theta_{n,k}\, dx\, dt \le C_1 \left(C\eta + \varepsilon(n, \mu, \eta, m)\right)^\delta + C_2(\varepsilon(n))^{1-\delta},$$

which yields, by passing to the limit supremum over $n$, $j$, $\mu$, $s$ and $\eta$,

$$\int\limits_{\{T_\eta(T_k(u_n) - T_k(\upsilon_j)) \ge 0\} \cap \Omega^r} \Big[(a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(u)))\times$$

$$\times(\nabla T_k(u_n) - \nabla T_k(u))\Big]^\delta dx = \varepsilon(n). \tag{39}$$

On the other hand, taking the functions $W_\eta^{n,j} = T_\eta(T_k(u_n) - T_k(\upsilon_j))^-$ and $W_\eta^j = T_\eta(T_k(u) - - T_k(\upsilon_j))^-$. Choosing in the approximate equation the test function $\exp(G(u_n))W_\eta^{n,j} S_m(u_n)$, we obtain

$$\int\limits_{\{T_\eta(T_k(u_n) - T_k(\upsilon_j)) \le 0\} \cap \Omega^r} \Big[(a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(u)))\times$$

$$\times(\nabla T_k(u_n) - \nabla T_k(u))\Big]^\delta dx = \varepsilon(n). \tag{40}$$

By (39) and (40), we get

$$\int\limits_{\Omega^r} \Big[(a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(u)))(\nabla T_k(u_n) - \nabla T_k(u))\Big]^\delta \, dx = \varepsilon(n).$$

Thus, passing to a subsequence if necessary, $\nabla u_n \to \nabla u$ a.e. in $\Omega^r$, and since $r$ is arbitrary,

$$\nabla u_n \to \nabla u \quad \text{a.e. in} \quad \Omega.$$

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