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# INEQUALITIES OF THE EDMUNDSON – LAH – RIBARIČ TYPE FOR $n$-CONVEX FUNCTIONS WITH APPLICATIONS*

# НЕРІВНОСТІ ТИПУ ЕДМУНДСОНА – ЛАХА – РИБАРИЧА ДЛЯ $n$-ОПУКЛИХ ФУНКЦІЙ ТА ЇХ ЗАСТОСУВАННЯ

We derive some Edmundson – Lah – Ribarič type inequalities for positive linear functionals and $n$-convex functions. Main results are applied to the generalized $f$-divergence functional. Examples with Zipf – Mandelbrot law are used to illustrate the results.

Отримано нерівності типу Едмундсона – Лаха – Рибарича для додатних лінійних функціоналів та $n$-опуклих функцій. Основні результати застосовуються до узагальнених $f$-дивергентних функціоналів. Наведено приклади, в яких використовується закон Зіпфа – Мандельброта.

**1. Introduction.** Let $E$ be a nonempty set and let $L$ be a vector space of real-valued functions $f : E \to \mathbb{R}$ having the properties:

$(\mathrm{L}_1)$ $f, g \in L \Rightarrow (af + bg) \in L$ for all $a, b \in \mathbb{R}$;

$(\mathrm{L}_2)$ $\mathbf{1} \in L$, i.e., if $f(t) = 1$ for every $t \in E$, then $f \in L$.

We also consider positive linear functionals $A\colon L \to \mathbb{R}$. That is, we assume that:

$(\mathrm{A}_1)$ $A(af + bg) = aA(f) + bA(g)$ for $f, g \in L$ and $a, b \in \mathbb{R}$;

$(\mathrm{A}_2)$ $f \in L,\ f(t) \geq 0$ for every $t \in E \Rightarrow A(f) \geq 0$ ($A$ is positive).

Since it was proved, the famous Jensen inequality and its converses have been extensively studied by many authors and have been generalized in numerous directions. Jessen [17] gave the following generalization of Jensen's inequality for convex functions (see also [30, p.47]).

**Theorem 1.1** [17]. *Let $L$ satisfy properties $(\mathrm{L}_1)$ and $(\mathrm{L}_2)$ on a nonempty set $E$, and assume that $f$ is a continuous convex function on an interval $I \subset \mathbb{R}$. If $A$ is a positive linear functional with $A(1) = 1$, then for all $g \in L$ such that $f(g) \in L$ we have $A(g) \in I$ and*

$$f(A(g)) \leq A(f(g)). \tag{1.1}$$

The following result is one of the most famous converses of the Jensen inequality known as the Edmundson – Lah – Ribarič inequality, and it was proved in [3] by Beesack and Pečarić (see also [30, p.98]).

**Theorem 1.2** [3]. *Let $f$ be convex on the interval $I = [a, b]$ such that $-\infty < a < b < \infty$. Let $L$ satisfy conditions $(\mathrm{L}_1)$ and $(\mathrm{L}_2)$ on $E$ and let $A$ be any positive linear functional on $L$ with $A(1) = 1$. Then for every $g \in L$ such that $f(g) \in L$ (so that $a \leq g(t) \leq b$ for all $t \in E$) we have*

$$A(f(g)) \leq \frac{b - A(g)}{b - a} f(a) + \frac{A(g) - a}{b - a} f(b). \tag{1.2}$$

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For some recent results on the converses of the Jensen inequality, the reader is referred to [7, 19, 20, 27, 29, 31].

Unlike the results from the above mentioned papers, which require convexity of the involved functions, the main objective of this paper is to obtain inequalities of the Edmundson – Lah – Ribarič type that hold for $n$-convex functions, which will also be a generalization of the results from [24, 25].

Definition of $n$-convex functions is characterized by $n$th order divided differences. The $n$th order divided difference of a function $f\colon [a,b] \to \mathbb{R}$ at mutually distinct points $t_0, t_1, \ldots, t_n \in [a,b]$ is defined recursively by

$$[t_i]f = f(t_i), \quad i = 0, \ldots, n,$$

$$[t_0, \ldots, t_n]f = \frac{[t_1, \ldots, t_n]f - [t_0, \ldots, t_{n-1}]f}{t_n - t_0}.$$

The value $[t_0, \ldots, t_n]f$ is independent of the order of the points $t_0, \ldots, t_n$.

Definition of divided differences can be extended to include the cases in which some or all the points coincide (see, e.g., [2, 30]):

$$f[\underbrace{a, \ldots, a}_{n \text{ times}}] = \frac{1}{(n-1)!} f^{(n-1)}(a), \quad n \in \mathbb{N}.$$

A function $f\colon [a,b] \to \mathbb{R}$ is said to be $n$-convex ($n \geq 0$) if and only if for all choices of $(n+1)$ distinct points $t_0, t_1, \ldots, t_n \in [a,b]$, we have $[t_0, \ldots, t_n]f \geq 0$.

The results in this paper are obtained by utilizing Hermite's interpolating polynomial, so first we need to give a definition and some properties (see [2]).

Let $-\infty < a < b < \infty$ and let $a \leq a_1 < a_2 < \ldots < a_r \leq b$, where $r \geq 2$, be given points. For $f \in \mathcal{C}^n([a,b])$ there exists a unique polynomial $P_H(t)$, called Hermite's interpolating polynomial, of degree $(n-1)$ fulfilling Hermite's conditions

$$P_H^{(i)}(a_j) = f^{(i)}(a_j)\colon \quad 0 \leq i \leq k_j, \quad 1 \leq j \leq r, \qquad \sum_{j=1}^{r} k_j + r = n.$$

Among other special cases, these conditions include type $(m, n-m)$ conditions, which will be of special interest to us:

$$(r = 2,\ 1 \leq m \leq n-1,\ k_1 = m-1,\ k_2 = n-m-1)$$

$$P_{mn}^{(i)}(a) = f^{(i)}(a), \quad 0 \leq i \leq m-1,$$

$$P_{mn}^{(i)}(b) = f^{(i)}(b), \quad 0 \leq i \leq n-m-1.$$

To give a development of the interpolating polynomial in terms of divided differences, first let us assume that the function $f$ is also defined at a point $t \neq a_j,\ 1 \leq j \leq r$. In [2] it is shown that

$$f(t) = P(t) + R(t), \tag{1.3}$$

where

$$P(t) = f(a_1) + (t - a_1) f[a_1, a_2] + (t - a_1)(t - a_2) f[a_1, a_2, a_3] + \ldots$$
$$\ldots + (t - a_1) \ldots (t - a_{r-1}) f[a_1, \ldots, a_r] \tag{1.4}$$

and

$$R(t) = (t - a_1) \dots (t - a_r) f[t, a_1, \dots, a_r]. \tag{1.5}$$

In case of $(m, n - m)$ conditions, (1.4) and (1.5) become

$$P_{mn}(t) = f(a) + (t - a) f[a, a] + \dots + (t - a)^{m-1} f[\underbrace{a, \dots, a}_{m \text{ times}}] +$$
$$+ (t - a)^m f[\underbrace{a, \dots, a}_{m \text{ times}}; b] + (t - a)^m (t - b) f[\underbrace{a, \dots, a}_{m \text{ times}}; b, b] + \dots$$
$$\dots + (t - a)^m (t - b)^{n-m-1} f[\underbrace{a, \dots, a}_{m \text{ times}}; \underbrace{b, b, \dots, b}_{(n-m) \text{ times}}] \tag{1.6}$$

and

$$R_m(t) = (t - a)^m (t - b)^{n-m} f[t; \underbrace{a, \dots, a}_{m \text{ times}}; \underbrace{b, b, \dots, b}_{(n-m) \text{ times}}]. \tag{1.7}$$

This paper is organized as follows. Main results, that are inequalities of the Edmundson – Lah – Ribarič type for $n$-convex functions, are given in Section 2. Application of the main results to the generalized $f$-divergence functional is given in Section 3. Finally, in Section 4 the results for the generalized $f$-divergence are applied to Zipf – Mandelbrot law.

**2. Results.** Throughout this paper, whenever mentioning the interval $[a, b]$, we assume that $-\infty < a < b < \infty$ holds.

Let $L$ satisfy conditions $(\mathrm{L}_1)$ and $(\mathrm{L}_2)$ on a nonempty set $E$, let $A$ be any positive linear functional on $L$ with $A(\mathbf{1}) = 1$, and let $g \in L$ be any function such that $g(E) \subseteq [a, b]$. For a given function $f: [a, b] \to \mathbb{R}$ denote

$$LR(f, g, a, b, A) = A(f(g)) - \frac{b - A(g)}{b - a} f(a) - \frac{A(g) - a}{b - a} f(b). \tag{2.1}$$

Following representations of the left-hand side in the Edmundson – Lah – Ribarič inequality are obtained by using Hermite's interpolating polynomials in terms of divided differences (1.6).

**Lemma 2.1.** *Let $L$ satisfy conditions $(\mathrm{L}_1)$ and $(\mathrm{L}_2)$ on a nonempty set $E$ and let $A$ be any positive linear functional on $L$ with $A(\mathbf{1}) = 1$. Let $f \in \mathcal{C}^n([a, b])$ and let $g \in L$ be any function such that $f \circ g \in L$. Then the following identities hold:*

$$LR(f, g, a, b, A) = \sum_{k=2}^{n-1} f\big[a; \underbrace{b, \dots, b}_{k \text{ times}}\big] A\big[(g - a\mathbf{1})(g - b\mathbf{1})^{k-1}\big] + A(R_1(g)), \tag{2.2}$$

$$LR(f, g, a, b, A) = f[a, a; b] A[(g - a\mathbf{1})(g - b\mathbf{1})] +$$
$$+ \sum_{k=2}^{n-2} f\big[a, a; \underbrace{b, \dots, b}_{k \text{ times}}\big] A\big[(g - a\mathbf{1})^2 (g - b\mathbf{1})^{k-1}\big] + A(R_2(g)), \tag{2.3}$$

$$LR(f, g, a, b, A) = (A(g) - a)\,(f[a, a] - f[a, b]) + \sum_{k=2}^{m-1} \frac{f^{(k)}(a)}{k!} A\big[(g - a\mathbf{1})^k\big] +$$
$$+ \sum_{k=1}^{n-m} f\big[\underbrace{a, \dots, a}_{m \text{ times}}; \underbrace{b, \dots, b}_{k \text{ times}}\big] A\big[(g - a\mathbf{1})^m (g - b\mathbf{1})^{k-1}\big] + A(R_m(g)), \tag{2.4}$$

*where $m \geq 3$ and $R_m(\cdot)$ is defined in* (1.7).

***Proof.*** From representation (1.3) of every function $f \in \mathcal{C}^n([a,b])$ and its Hermite interpolating polynomial of type $(m, n-m)$ conditions in terms of divided differences (1.6) we have

$$\begin{aligned} f(t) = f(a) + (t-a)f[a,a] + \ldots + (t-a)^{m-1} f[\underbrace{a,\ldots,a}_{m \text{ times}}] + \\ +(t-a)^m f[\underbrace{a,\ldots,a}_{m \text{ times}}; b] + (t-a)^m (t-b) f[\underbrace{a,\ldots,a}_{m \text{ times}}; b, b] + \ldots \\ \ldots + (t-a)^m (t-b)^{n-m-1} f[\underbrace{a,\ldots,a}_{m \text{ times}};\ \underbrace{b,b,\ldots,b}_{(n-m) \text{ times}}] + R_m(t), \end{aligned} \tag{2.5}$$

where $R_m(\cdot)$ is defined in (1.7). After some straightforward calculations, for different choices of $1 \le m \le n-1$, from (2.5) we get the following:

for $m = 1$ it holds

$$\begin{aligned} LR(f, \mathbf{1}, a, b, \mathrm{id}) = (t-a)(t-b)f[a;b,b] + (t-a)(t-b)^2 f[a;b,b,b] + \ldots \\ \ldots + (t-a)(t-b)^{n-2} f\big[a; \underbrace{b,b,\ldots,b}_{(n-1) \text{ times}}\big] + R_1(t), \end{aligned} \tag{2.6}$$

for $m = 2$ it holds

$$\begin{aligned} LR(f, \mathbf{1}, a, b, \mathrm{id}) = (t-a)(t-b)f[a,a;b] + (t-a)^2(t-b) f[a,a;b,b] + \ldots \\ \ldots + (t-a)^2(t-b)^{n-3} f\big[a,a; \underbrace{b,b,\ldots,b}_{(n-2) \text{ times}}\big] + R_2(t), \end{aligned} \tag{2.7}$$

for $3 \le m \le n-1$ it holds

$$\begin{aligned} LR(f, \mathbf{1}, a, b, \mathrm{id}) = (t-a)\left(f[a,a] - f[a,b]\right) + \ldots + (t-a)^{m-1} f[\underbrace{a,\ldots,a}_{m \text{ times}}] + \\ +(t-a)^m f[\underbrace{a,\ldots,a}_{m \text{ times}}; b] + (t-a)^m (t-b) f[\underbrace{a,\ldots,a}_{m \text{ times}}; b, b] + \ldots \\ \ldots + (t-a)^m (t-b)^{n-m-1} f[\underbrace{a,\ldots,a}_{m \text{ times}};\ \underbrace{b,b,\ldots,b}_{(n-m) \text{ times}}] + R_m(t). \end{aligned} \tag{2.8}$$

Since $f \circ g \in L$ it holds $g(E) \subseteq [a,b]$, so we can replace $t$ with $g(t)$ in (2.6), (2.7) and (2.8), and obtain

$$LR(f, g, a, b, \mathrm{id}) = \sum_{k=2}^{n-1} (g(t)-a)(g(t)-b)^{k-1} f\big[a; \underbrace{b,\ldots,b}_{k \text{ times}}\big] + R_1(g(t)),$$

$$\begin{aligned} LR(f, g, a, b, \mathrm{id}) = (g(t)-a)(g(t)-b)f[a,a;b] + \\ + \sum_{k=2}^{n-2} (g(t)-a)^2 (g(t)-b)^{k-1} f\big[a,a; \underbrace{b,\ldots,b}_{k \text{ times}}\big] + R_2(g(t)) \end{aligned}$$

and

$$LR(f,g,a,b,\mathrm{id}) = (g(t)-a)\left(f[a,a]-f[a,b]\right) + \sum_{k=3}^{m}(g(t)-a)^{k-1} f[\underbrace{a,\ldots,a}_{k \text{ times}}] +$$

$$+\sum_{k=1}^{n-m}(g(t)-a)^m(g(t)-b)^{k-1} f\big[\underbrace{a,\ldots,a}_{m \text{ times}};\underbrace{b,\ldots,b}_{k \text{ times}}\big] + R_m(g(t)).$$

Identities (2.2), (2.3) and (2.4) follow by applying positive normalized linear functional $A$ to the previous equalities, respectively.

Lemma 2.1 is proved.

**Lemma 2.2.** *Let $L$ satisfy conditions* $(\mathrm{L}_1)$ *and* $(\mathrm{L}_2)$ *on a nonempty set $E$ and let $A$ be any positive linear functional on $L$ with $A(\mathbf{1}) = 1$. Let $f \in \mathcal{C}^n([a,b])$ and let $g \in L$ be any function such that $f \circ g \in L$. Then the following identities hold:*

$$LR(f,g,a,b,A) = \sum_{k=2}^{n-1} f\big[b;\underbrace{a,\ldots,a}_{k \text{ times}}\big] A\big[(g-b\mathbf{1})(g-a\mathbf{1})^{k-1}\big] + A(R_1^*(g)), \tag{2.9}$$

$$LR(f,g,a,b,A) = f[b,b;a]A\big[(g-b\mathbf{1})(g-a\mathbf{1})\big] +$$

$$+\sum_{k=2}^{n-2} f\big[b,b;\underbrace{a,\ldots,a}_{k \text{ times}}\big] A\big[(g-b\mathbf{1})^2(g-a\mathbf{1})^{k-1}\big] + A(R_2^*(g)), \tag{2.10}$$

$$LR(f,g,a,b,A) = (b-A(g))\big(f[a,b]-f[b,b]\big) + \sum_{k=2}^{m-1}\frac{f^{(k)}(b)}{k!}A[(g-b\mathbf{1})^k] +$$

$$+\sum_{k=1}^{n-m} f\big[\underbrace{b,\ldots,b}_{m \text{ times}};\underbrace{a,\ldots,a}_{k \text{ times}}\big] A[(g-b\mathbf{1})^m(g-a\mathbf{1})^{k-1}] + A(R_m^*(g)), \tag{2.11}$$

*where $m \geq 3$ and*

$$A(R_m^*(g)) = A\Big[f\big[g;\underbrace{b\mathbf{1},\ldots,b\mathbf{1}}_{m \text{ times}};\underbrace{a\mathbf{1},\ldots,a\mathbf{1}}_{(n-m) \text{ times}}\big](g-b\mathbf{1})^m(g-a\mathbf{1})^{n-m}\Big]. \tag{2.12}$$

***Proof.*** Let us define an auxiliary function $F\colon [a,b] \to \mathbb{R}$ with

$$F(t) = f(a+b-t).$$

Since $f \in \mathcal{C}^n([a,b])$ we immediately have $F \in \mathcal{C}^n([a,b])$, so we can apply (2.6), (2.7) and (2.8) to $F$ and obtain respectively

$$LR(F,\mathbf{1},a,b,\mathrm{id}) = \sum_{k=2}^{n-1} F\big[a;\underbrace{b,\ldots,b}_{k \text{ times}}\big](t-a)(t-b)^{k-1} + R_1(t), \tag{2.13}$$

$$LR(F,\mathbf{1},a,b,\mathrm{id}) = F[a,a;b](t-a)(t-b) +$$

$$+\sum_{k=2}^{n-2} F\big[a,a;\underbrace{b,\ldots,b}_{k \text{ times}}\big](t-a)^2(t-b)^{k-1} + R_2(t), \tag{2.14}$$

$$LR(F,\mathbf{1},a,b,\mathrm{id}) = (t-a)\left(F[a,a]-F[a,b]\right) + \sum_{k=2}^{m-1}\frac{F^{(k)}(a)}{k!}(t-a)^k +$$

$$+\sum_{k=1}^{n-m} F\big[\underbrace{a,\dots,a}_{m\text{ times}};\underbrace{b,\dots,b}_{k\text{ times}}\big](t-a)^m(t-b)^{k-1} + R_m(t). \tag{2.15}$$

We can calculate divided differences of the function $F$ in terms of divided differences of the function $f$:

$$F\big[\underbrace{a,\dots,a}_{k\text{ times}};\underbrace{b,\dots,b}_{i\text{ times}}\big] = (-1)^{k+i-1} f\big[\underbrace{b,\dots,b}_{k\text{ times}};\underbrace{a,\dots,a}_{i\text{ times}}\big].$$

Now (2.13), (2.14) and (2.15) become

$$LR(F,\mathbf{1},a,b,\mathrm{id}) = \sum_{k=2}^{n-1}(-1)^k f\big[b;\underbrace{a,\dots,a}_{k\text{ times}}\big](t-a)(t-b)^{k-1} + \bar{R}_1(t), \tag{2.16}$$

$$LR(F,\mathbf{1},a,b,\mathrm{id}) = (-1)^2 f[b,b;a](t-a)(t-b)+$$

$$+\sum_{k=2}^{n-2}(-1)^{k+1} f\big[b,b;\underbrace{a,\dots,a}_{k\text{ times}}\big](t-a)^2(t-b)^{k-1} + \bar{R}_2(t), \tag{2.17}$$

$$LR(F,\mathbf{1},a,b,\mathrm{id}) = (t-a)\left(-f[b,b]+f[a,b]\right) + \sum_{k=2}^{m-1}\frac{(-1)^k f^{(k)}(b)}{k!}(t-a)^k +$$

$$+\sum_{k=1}^{n-m}(-1)^{m+k-1} f\big[\underbrace{b,\dots,b}_{m\text{ times}};\underbrace{a,\dots,a}_{k\text{ times}}\big](t-a)^m(t-b)^{k-1} + \bar{R}_m(t), \tag{2.18}$$

where

$$\bar{R}_m(t) = (t-a)^m(t-b)^{n-m}(-1)^n f\big[a+b-t;\underbrace{b,\dots,b}_{m\text{ times}};\underbrace{a,a,\dots,a}_{(n-m)\text{ times}}\big].$$

Let $g \in L$ be any function such that $f\circ g \in L$, that is, $a \le g(t) \le b$ for every $t \in E$. Let us define a function $\bar{g}(t) = a+b-g(t)$. Trivially, we have $a \le \bar{g}(t) \le b$ and $\bar{g} \in L$. Since

$$LR(F,\bar{g},a,b,\mathrm{id}) = f(a+b-(a+b-g(t))) - \frac{b-(a+b-g(t))}{b-a}f(a+b-a)-$$

$$-\frac{a+b-g(t)-a}{b-a}f(a+b-b) = LR(f,g,a,b,\mathrm{id}),$$

after putting $\bar{g}(t)$ in (2.16), (2.17) and (2.18) instead of $t$, we get

$$LR(f,g,a,b,\mathrm{id}) = \sum_{k=2}^{n-1}(-1)^k f\big[b;\underbrace{a,\dots,a}_{k\text{ times}}\big](b-g(t))(a-g(t))^{k-1} + \bar{R}_1(a+b-g(t)),$$

$$LR(f,g,a,b,\mathrm{id}) = (-1)^2 f[b,b;a](b-g(t))(a-g(t))+$$

$$+\sum_{k=2}^{n-2}(-1)^{k+1}f\big[b,b;\underbrace{a,\ldots,a}_{k\text{ times}}\big](b-g(t))^2(a-g(t))^{k-1}+\bar{R}_2(a+b-g(t)),$$

$$LR(f,g,a,b,\mathrm{id})=(b-g(t))\left(-f[b,b]+f[a,b]\right)+\sum_{k=2}^{m-1}\frac{(-1)^k f^{(k)}(b)}{k!}(b-g(t))^k+$$

$$+\sum_{k=1}^{n-m}(-1)^{m+k-1}f\big[\underbrace{b,\ldots,b}_{m\text{ times}};\underbrace{a,\ldots,a}_{k\text{ times}}\big](b-g(t))^m(a-g(t))^{k-1}+\bar{R}_m(a+b-g(t)).$$

Identities (2.9), (2.10) and (2.11) follow after applying a normalized positive linear functional $A$ to previous equalities, respectively.

Lemma 2.2 is proved.

Our first result is an upper bound for the difference in the Edmundson – Lah – Ribarič inequality, expressed by Hermite's interpolating polynomials in terms of divided differences.

**Theorem 2.1.** *Let $L$ satisfy conditions* $(\mathrm{L}_1)$ *and* $(\mathrm{L}_2)$ *on a nonempty set $E$ and let $A$ be any positive linear functional on $L$ with $A(\mathbf{1})=1$. Let $f\in\mathcal{C}^n([a,b])$ and let $g\in L$ be any function such that $f\circ g\in L$. If the function $f$ is $n$-convex and if $n$ and $m\geq 3$ are of different parity, then*

$$LR(f,g,a,b,A)\leq (A(g)-a)\left(f[a,a]-f[a,b]\right)+\sum_{k=2}^{m-1}\frac{f^{(k)}(a)}{k!}A\big[(g-a\mathbf{1})^k\big]+$$

$$+\sum_{k=1}^{n-m}f\big[\underbrace{a,\ldots,a}_{m\text{ times}};\underbrace{b,\ldots,b}_{k\text{ times}}\big]A\big[(g-a\mathbf{1})^m(g-b\mathbf{1})^{k-1}\big]. \tag{2.19}$$

*Inequality* (2.19) *also holds when the function $f$ is $n$-concave and $n$ and $m$ are of equal parity. In case when the function $f$ is $n$-convex and $n$ and $m$ are of equal parity, or when the function $f$ is $n$-concave and $n$ and $m$ are of different parity, the inequality sign in* (2.19) *is reversed.*

***Proof.*** We start with the representation of the left-hand side in the Edmundson – Lah – Ribarič inequality (2.4) with a special focus on the last term:

$$A(R(g))=A\left((g-a\mathbf{1})^m(g-b\mathbf{1})^{n-m}f\big[g;\underbrace{a\mathbf{1},\ldots,a\mathbf{1}}_{m\text{ times}};\underbrace{b\mathbf{1},\ldots,b\mathbf{1}}_{(n-m)\text{ times}}\big]\right).$$

Since $A$ is positive, it preserves the sign, so we need to study the sign of the expression

$$(g(t)-a)^m(g(t)-b)^{n-m}f\big[g(t);\underbrace{a,\ldots,a}_{m\text{ times}};\underbrace{b,b,\ldots,b}_{(n-m)\text{ times}}\big].$$

Since $a\leq g(t)\leq b$ for every $t\in E$, we have $(g(t)-a)^m\geq 0$ for every $t\in E$ and any choice of $m$. For the same reason we have $(g(t)-b)\leq 0$. Trivially it follows that $(g(t)-b)^{n-m}\leq 0$ when $n$ and $m$ are of different parity, and $(g(t)-b)^{n-m}\geq 0$ when $n$ and $m$ are of equal parity.

If the function $f$ is $n$-convex, then $f\big[g(t);\underbrace{a,\ldots,a}_{m\text{ times}};\underbrace{b,b,\ldots,b}_{(n-m)\text{ times}}\big]\geq 0$, and if the function $f$ is $n$-concave, then $f\big[g(t);\underbrace{a,\ldots,a}_{m\text{ times}};\underbrace{b,b,\ldots,b}_{(n-m)\text{ times}}\big]\leq 0$.

Now (2.19) easily follows from (2.1).

Theorem 2.1 is proved.

Following result provides us with a similar upper bound for the difference in the Edmundson – Lah – Ribarič inequality, and it is obtained from Lemma 2.2.

**Theorem 2.2.** *Let $L$ satisfy conditions* $(\mathrm{L}_1)$ *and* $(\mathrm{L}_2)$ *on a nonempty set $E$ and let $A$ be any positive linear functional on $L$ with $A(\mathbf{1})=1$. Let $f\in\mathcal{C}^n([a,b])$ and let $g\in L$ be any function such that $f\circ g\in L$. If the function $f$ is $n$-convex and if $m\geq 3$ is odd, then*

$$LR(f,g,a,b,A)\leq (b-A(g))\big(f[a,b]-f[b,b]\big)+\sum_{k=2}^{m-1}\frac{f^{(k)}(b)}{k!}A[(g-b\mathbf{1})^k]+$$

$$+\sum_{k=1}^{n-m} f\big[\underbrace{b,\ldots,b}_{m\text{ times}};\underbrace{a,\ldots,a}_{k\text{ times}}\big]A\big[(g-b\mathbf{1})^m(g-a\mathbf{1})^{k-1}\big]. \tag{2.20}$$

*Inequality* (2.20) *also holds when the function $f$ is $n$-concave and $m$ is even. In case when the function $f$ is $n$-convex and $m$ is even, or when the function $f$ is $n$-concave and $m$ is odd, the inequality sign in* (2.20) *is reversed.*

***Proof.*** Similarly as in the proof of the previous theorem, we start with the representation of the left-hand side in the Edmundson – Lah – Ribarič inequality (2.11) with a special focus on the last term:

$$A(R_m^*(g))=A\left(f\big[g;\underbrace{b\mathbf{1},\ldots,b\mathbf{1}}_{m\text{ times}};\underbrace{a\mathbf{1},\ldots,a\mathbf{1}}_{(n-m)\text{ times}}\big](g-b\mathbf{1})^m(g-a\mathbf{1})^{n-m}\right).$$

As before, because of the positivity of the linear functional $A$, we only need to study the sign of the expression:

$$(g(t)-b)^m(g(t)-a)^{n-m}f\big[g(t);\underbrace{b,\ldots,b}_{m\text{ times}};\underbrace{a,a,\ldots,a}_{(n-m)\text{ times}}\big].$$

Since $a\leq g(t)\leq b$ for every $t\in E$, we have $(g(t)-a)^{n-m}\geq 0$ for every $t\in E$ and any choice of $m$. For the same reason we have $(g(t)-b)\leq 0$. Trivially it follows that $(g(t)-b)^m\leq 0$ when $m$ is odd, and $(g(t)-b)^m\geq 0$ when $m$ is even.

If the function $f$ is $n$-convex, then its $n$th order divided differences are greater of equal to zero, and if the function $f$ is $n$-concave, then its $n$th order divided differences are less or equal to zero.

Now (2.20) easily follows from Lemma 2.2.

Theorem 2.2 is proved.

***Corollary* 2.1.** *Let $L$ satisfy conditions* $(\mathrm{L}_1)$ *and* $(\mathrm{L}_2)$ *on a nonempty set $E$ and let $A$ be any positive linear functional on $L$ with $A(\mathbf{1})=1$. Let $n$ be an odd number, let $f\in\mathcal{C}^n([a,b])$, and let $g\in L$ be any function such that $f\circ g\in L$. If the function $f$ is $n$-convex and if $m\geq 3$ is odd, then*

$$(A(g)-a)\left(f[a,a]-f[a,b]\right)+\sum_{k=2}^{m-1}\frac{f^{(k)}(a)}{k!}A\big[(g-a\mathbf{1})^k\big]+$$

$$+\sum_{k=1}^{n-m} f\big[\underbrace{a,\ldots,a}_{m\text{ times}};\underbrace{b,\ldots,b}_{k\text{ times}}\big]A\big[(g-a\mathbf{1})^m(g-b\mathbf{1})^{k-1}\big]\leq$$

$$\leq LR(f,g,a,b,A) \leq (b - A(g))\big(f[a,b] - f[b,b]\big) + \sum_{k=2}^{m-1} \frac{f^{(k)}(b)}{k!} A[(g - b\mathbf{1})^k] +$$

$$+ \sum_{k=1}^{n-m} f\big[\underbrace{b,\ldots,b}_{m \text{ times}};\underbrace{a,\ldots,a}_{k \text{ times}}\big] A[(g-b\mathbf{1})^m (g - a\mathbf{1})^{k-1}]. \tag{2.21}$$

*Inequality* (2.21) *also holds when the function $f$ is $n$-concave and $m$ is even. In case when the function $f$ is $n$-convex and $m$ is even, or when the function $f$ is $n$-concave and $m$ is odd, the inequality signs in* (2.21) *are reversed.*

***Remark* 2.1.** In [25] (Theorem 2.3) is proved that for a 3-convex functions we have

$$(A(g) - a)\left[f'(a) - \frac{f(b) - f(a)}{b - a}\right] + \frac{f''(a)}{2} A[(g - a\mathbf{1})^2] \leq$$

$$\leq LR(f,g,a,b,A) \leq (b - A(g))\left[\frac{f(b) - f(a)}{b - a} - f'(b)\right] + \frac{f''(b)}{2} A[(b\mathbf{1} - g)^2]$$

and if the function $f$ is 3-concave, then the inequality signs are reversed. It is obvious that inequalities (2.21) from Corollary 2.1 provide us with a generalization of the result stated above.

Next result gives us an upper and a lower bound for the difference in the Edmundson – Lah – Ribarič inequality expressed by Hermite's interpolating polynomials in terms of divided differences, and it is obtained from Lemma 2.1.

**Theorem 2.3.** *Let $L$ satisfy conditions* $(\mathrm{L}_1)$ *and* $(\mathrm{L}_2)$ *on a nonempty set $E$ and let $A$ be any positive linear functional on $L$ with $A(\mathbf{1}) = 1$. Let $f \in C^n([a,b])$ and let $g \in L$ be any function such that $f \circ g \in L$. If the function $f$ is $n$-convex and if $n$ is odd, then*

$$\sum_{k=2}^{n-1} f\big[a;\underbrace{b,\ldots,b}_{k \text{ times}}\big] A\big[(g - a\mathbf{1})(g - b\mathbf{1})^{k-1}\big] \leq LR(f,g,a,b,A) \leq$$

$$\leq f[a,a;b] A[(g - a\mathbf{1})(g - b\mathbf{1})] + \sum_{k=2}^{n-2} f\big[a,a;\underbrace{b,\ldots,b}_{k \text{ times}}\big] A\big[(g - a\mathbf{1})^2 (g - b\mathbf{1})^{k-1}\big]. \tag{2.22}$$

*Inequalities* (2.22) *also hold when the function $f$ is $n$-concave and $n$ is even. In case when the function $f$ is $n$-convex and $n$ is even, or when the function $f$ is $n$-concave and $n$ is odd, the inequality signs in* (2.22) *are reversed.*

***Proof.*** From the discussion about positivity and negativity of the term $A(R_m(g))$ in the proof of Theorem 2.1, for $m = 1$ it follows that

$A(R_1(g)) \geq 0$ when the function $f$ is $n$-convex and $n$ is odd, or when $f$ is $n$-concave and $n$ even;

$A(R_1(g)) \leq 0$ when the function $f$ is $n$-concave and $n$ is odd, or when $f$ is $n$-convex and $n$ even.

Now the identity (2.2) gives us

$$LR(f,g,a,b,A) \geq f[a;b,b] A[(g - a\mathbf{1})(g - b\mathbf{1})] + f[a;b,b,b] A[(g - a\mathbf{1})(g - b\mathbf{1})^2] + \ldots$$

$$\ldots + f\big[a;\underbrace{b,b,\ldots,b}_{(n-1) \text{ times}}\big] A\big[(g - a\mathbf{1})(g - b\mathbf{1})^{n-2}\big]$$

for $A(R_1(g)) \geq 0$, and in case $A(R_1(g)) \leq 0$ the inequality sign is reversed.

In the same manner, for $m = 2$ it follows that

$A(R_2(g)) \leq 0$ when the function $f$ is $n$-convex and $n$ is odd, or when $f$ is $n$-concave and $n$ even;

$A(R_2(g)) \geq 0$ when the function $f$ is $n$-concave and $n$ is odd, or when $f$ is $n$-convex and $n$ even.

In this case the identity (2.3) for $A(R_2(g)) \leq 0$ gives us

$$LR(f, g, a, b, A) \leq f[a, a; b]A[(g - a\mathbf{1})(g - b\mathbf{1})] + f[a, a; b, b]A[(g - a\mathbf{1})^2(g - b\mathbf{1})] + \ldots$$
$$\ldots + f\big[a, a; \underbrace{b, b, \ldots, b}_{(n-2)\ \text{times}}\big]A\big[(g - a\mathbf{1})^2(g - b\mathbf{1})^{n-3}\big]$$

and in case $A(R_2(g)) \geq 0$ the inequality sign is reversed.

When we combine the two results from above, we get exactly (2.22).

Theorem 2.3 is proved.

By utilizing Lemma 2.2 we can get similar bounds for the difference in the Edmundson – Lah – Ribarič inequality that hold for all $n \in \mathbb{N}$, not only the odd ones.

**Theorem 2.4.** *Let $L$ satisfy conditions* $(\mathrm{L}_1)$ *and* $(\mathrm{L}_2)$ *on a nonempty set $E$ and let $A$ be any positive linear functional on $L$ with $A(\mathbf{1}) = 1$. Let $f \in \mathcal{C}^n([a, b])$ and let $g \in L$ be any function such that $f \circ g \in L$. If the function $f$ is $n$-convex, then*

$$f[b, b; a]A\big[(g - b\mathbf{1})(g - a\mathbf{1})\big] + \sum_{k=2}^{n-2} f\big[b, b; \underbrace{a, \ldots, a}_{k\ \text{times}}\big]A\big[(g - b\mathbf{1})^2(g - a\mathbf{1})^{k-1}\big] \leq$$
$$\leq LR(f, g, a, b, A) \leq \sum_{k=1}^{n-1} f\big[b; \underbrace{a, \ldots, a}_{k\ \text{times}}\big]A\big[(g - b\mathbf{1})(g - a\mathbf{1})^{k-1}\big]. \tag{2.23}$$

*If the function $f$ is $n$-concave, the inequality signs in* (2.23) *are reversed.*

***Proof.*** We return to the discussion about positivity and negativity of the term $A(R^*_m(g))$ in the proof of Theorem 2.2. For $m = 1$ we have

$$(g(t) - b)^1(g(t) - a)^{n-1} \leq 0 \quad \text{for every} \quad t \in E,$$

so $A(R^*_1(g)) \geq 0$ when the function $f$ is $n$-concave, and $A(R^*_1(g)) \leq 0$ when the function $f$ is $n$-convex. Now the identity (2.9) for a $n$-convex function $f$ gives us

$$LR(f, g, a, b, A) \geq f[b, b; a]A\big[(g - b\mathbf{1})(g - a\mathbf{1})\big] + f[b, b; a, a]A[(g - b\mathbf{1})^2(g - a\mathbf{1})] + \ldots$$
$$\ldots + f\big[b, b; \underbrace{a, a, \ldots, a}_{(n-2)\ \text{times}}\big]A[(g - b\mathbf{1})^2(g - a\mathbf{1})^{n-3}]$$

and if the function $f$ is $n$-concave, the inequality sign is reversed.

Similarly, for $m = 2$ we have

$$(g(t) - b)^2(g(t) - a)^{n-2} \geq 0 \quad \text{for every} \quad t \in E,$$

so $A(R^*_2(g)) \geq 0$ when the function $f$ is $n$-convex, and $A(R^*_2(g)) \leq 0$ when the function $f$ is $n$-concave. In this case the identity (2.10) for a $n$-convex function $f$ gives us

$$LR(f,g,a,b,A) \leq f[b;a,a]A\big[(g-b\mathbf{1})(g-a\mathbf{1})\big] + f[b;a,a,a]A\big[(g-b\mathbf{1})(g-a\mathbf{1})^2\big] + \ldots$$
$$\ldots + f\big[b;\underbrace{a,a,\ldots,a}_{(n-1)\ \text{times}}\big]A\big[(g-b\mathbf{1})(g-a\mathbf{1})^{n-2}\big]$$

and if the function $f$ is $n$-concave, the inequality sign is reversed.

When we combine the two results from above, we get exactly (2.23).

Theorem 2.4 is proved.

***Remark* 2.2.** Since

$$f[a;b,b] = \frac{1}{b-a}\left(f'(b) - \frac{f(b)-f(a)}{b-a}\right),$$
$$f[a,a;b] = \frac{1}{b-a}\left(f'(b) - \frac{f(b)-f(a)}{b-a}\right),$$

when we take $n=3$ in (2.22) or (2.23), we get that

$$\frac{A[(g-a\mathbf{1})(g-b\mathbf{1})]}{b-a}\left(f'(b) - \frac{f(b)-f(a)}{b-a}\right) \leq$$
$$\leq LR(f,g,a,b,A) \leq \frac{A[(g-a\mathbf{1})(g-b\mathbf{1})]}{b-a}\left(f'(b) - \frac{f(b)-f(a)}{b-a}\right) \tag{2.24}$$

holds for a 3-convex function, and for a 3-concave function the inequality signs are reversed. Inequalities (2.24) are proved in [25] (Theorem 2.1), so it follows that Theorem 2.3 and Theorem 2.4 give a generalization of a result from [25].

**3. Applications to Csiszár divergence.** Let us denote the set of all finite discrete probability distributions by $\mathbb{P}$, that is we say $\boldsymbol{p} = (p_1,\ldots,p_r) \in \mathbb{P}$ if $p_i \in [0,1]$ for $i = 1,\ldots,r$ and $\sum_{i=1}^{r} p_i = 1$.

Numerous theoretic divergence measures between two probability distributions have been introduced and comprehensively studied. Their applications can be found in the analysis of contingency tables [13], in approximation of probability distributions [8, 22], in signal processing [18], and in pattern recognition [4, 6].

Csiszár [9 – 10] introduced the $f$-divergence functional as

$$D_f(\boldsymbol{p},\boldsymbol{q}) = \sum_{i=1}^{r} q_i f\left(\frac{p_i}{q_i}\right), \tag{3.1}$$

where $f\colon [0,+\infty\rangle$ is a convex function, and it represent a “distance function” on the set of probability distributions $\mathbb{P}$.

A great number of theoretic divergences are special cases of Csiszár $f$-divergence for different choices of the function $f$.

As in Csiszár [10], we interpret undefined expressions by

$$f(0) = \lim_{t\to 0^+} f(t), \qquad 0\cdot f\left(\frac{0}{0}\right) = 0,$$
$$0\cdot f\left(\frac{a}{0}\right) = \lim_{\epsilon\to 0^+} \epsilon\cdot f\left(\frac{a}{\epsilon}\right) = a\cdot\lim_{t\to\infty}\frac{f(t)}{t}.$$

In this section our intention is to derive mutual bounds for the generalized $f$-divergence functional in described setting. In such a way, we will obtain some new reverse relations for the generalized $f$-divergence functional that correspond to the class of $n$-convex functions. It is a generalization of the results obtained in [25]. Throughout this section, when mentioning the interval $[a,b]$, we assume that $[a,b] \subseteq \mathbb{R}_+$. For a $n$-convex function $f\colon [a,b] \to \mathbb{R}$ we give the following definition of generalized $f$-divergence functional:

$$\tilde{D}_f(\boldsymbol{p},\boldsymbol{q}) = \sum_{i=1}^{r} q_i f\left(\frac{p_i}{q_i}\right). \tag{3.2}$$

The first result in this section is carried out by virtue of our Theorem 2.1.

**Theorem 3.1.** *Let $[a,b] \subset \mathbb{R}$ be an interval such that $a \le 1 \le b$. Let $f \in \mathcal{C}^n([a,b])$ and let $\boldsymbol{p} = (p_1,\ldots,p_r)$ and $\boldsymbol{q} = (q_1,\ldots,q_r)$ be probability distributions such that $p_i/q_i \in [a,b]$ for every $i = 1,\ldots,r$. If the function $f$ is $n$-convex and if $n$ and $3 \le m \le n-1$ are of different parity, then*

$$\frac{b-1}{b-a} f(a) + \frac{1-a}{b-a} f(b) - \tilde{D}_f(\boldsymbol{p},\boldsymbol{q}) \le$$
$$\le (1-a)\left(f[a,a] - f[a,b]\right) + \sum_{k=2}^{m-1} \frac{f^{(k)}(a)}{k!} \sum_{i=1}^{r} \frac{(p_i - aq_i)^k}{q_i^{k-1}} +$$
$$+ \sum_{k=1}^{n-m} f\big[\underbrace{a,\ldots,a}_{m \text{ times}};\underbrace{b,\ldots,b}_{k \text{ times}}\big] \sum_{i=1}^{r} \frac{(p_i - aq_i)^m (p_i - aq_i)^{k-1}}{q_i^{m+k-2}}. \tag{3.3}$$

*Inequality* (3.3) *also holds when the function $f$ is $n$-concave and $n$ and $m$ are of equal parity. In case when the function $f$ is $n$-convex and $n$ and $m$ are of equal parity, or when the function $f$ is $n$-concave and $n$ and $m$ are of different parity, the inequality sign in* (3.3) *is reversed.*

***Proof.*** Let $\boldsymbol{x} = (x_1,\ldots,x_r)$ be such that $x_i \in [a,b]$ for $i = 1,\ldots,r$. In the relation (2.19) we can replace

$$g \longleftrightarrow \boldsymbol{x} \qquad \text{and} \qquad A(\boldsymbol{x}) = \sum_{i=1}^{r} p_i x_i.$$

In that way we get

$$\frac{b-\bar{x}}{b-a} f(a) + \frac{\bar{x}-a}{b-a} f(b) - \sum_{i=1}^{r} p_i f(x_i) \le$$
$$\le (\bar{x}-a)\left(f[a,a] - f[a,b]\right) + \sum_{k=2}^{m-1} \frac{f^{(k)}(a)}{k!} \sum_{i=1}^{r} p_i (x_i - a)^k +$$
$$+ \sum_{k=1}^{n-m} f\big[\underbrace{a,\ldots,a}_{m \text{ times}};\underbrace{b,\ldots,b}_{k \text{ times}}\big] \sum_{i=1}^{r} p_i (x_i - a)^m (x_i - b)^{k-1},$$

where $\bar{x} = \sum_{i=1}^{n} p_i x_i$. In the previous relation we can set

$$p_i = q_i \qquad \text{and} \qquad x_i = \frac{p_i}{q_i},$$

and after calculating

$$\bar{x} = \sum_{i=1}^{n} q_i \frac{p_i}{q_i} = \sum_{i=1}^{n} p_i = 1$$

we get (3.3).

Theorem 3.1 is proved.

By utilizing Theorem 2.2 in the analogous way as above, we get an Edmundson – Lah – Ribarič type inequality for the generalized $f$-divergence functional (3.2) which does not depend on parity of $n,$ and it is given in the following theorem.

**Theorem 3.2.** *Let $[a,b] \subset \mathbb{R}$ be an interval such that $a \leq 1 \leq b$. Let $f \in \mathcal{C}^n([a,b])$ and let $\boldsymbol{p} = (p_1, \ldots, p_r)$ and $\boldsymbol{p} = (q_1, \ldots, q_r)$ be probability distributions such that $p_i/q_i \in [a,b]$ for every $i = 1, \ldots, r$. If the function $f$ is $n$-convex and if $3 \leq m \leq n-1$ is odd, then*

$$\frac{b-1}{b-a} f(a) + \frac{1-a}{b-a} f(b) - \tilde{D}_f(\boldsymbol{p}, \boldsymbol{q}) \leq$$

$$\leq (b-1)\big(f[a,b] - f[b,b]\big) + \sum_{k=2}^{m-1} \frac{f^{(k)}(b)}{k!} \sum_{i=1}^{r} \frac{(p_i - bq_i)^k}{q_i^{k-1}} +$$

$$+ \sum_{k=1}^{n-m} f\big[\underbrace{b, \ldots, b}_{m \text{ times}}; \underbrace{a, \ldots, a}_{k \text{ times}}\big] \sum_{i=1}^{r} \frac{(p_i - bq_i)^m (p_i - aq_i)^{k-1}}{q_i^{m+k-2}}. \qquad (3.4)$$

*Inequality* (3.4) *also holds when the function $f$ is $n$-concave and $m$ is even. In case when the function $f$ is $n$-convex and $m$ is even, or when the function $f$ is $n$-concave and $m$ is odd, the inequality sign in* (3.4) *is reversed.*

Another generalization of the Edmundson – Lah – Ribarič inequality, which provides us with a lower and an upper bound for the generalized $f$-divergence functional, is given in the following theorem.

**Theorem 3.3.** *Let $[a,b] \subset \mathbb{R}$ be an interval such that $a \leq 1 \leq b$. Let $f \in \mathcal{C}^n([a,b])$ and let $\boldsymbol{p} = (p_1, \ldots, p_r)$ and $\boldsymbol{p} = (q_1, \ldots, q_r)$ be probability distributions such that $p_i/q_i \in [a,b]$ for every $i = 1, \ldots, r$. If the function $f$ is $n$-convex and if $n$ is odd, then we have*

$$\sum_{k=2}^{n-1} f[a; \underbrace{b, b, \ldots, b}_{k \text{ times}}] \sum_{i=1}^{r} \frac{(p_i - aq_i)(p_i - bq_i)^{k-1}}{q_i^{k-1}} \leq \frac{b-1}{b-a} f(a) + \frac{1-a}{b-a} f(b) - \tilde{D}_f(\boldsymbol{p}, \boldsymbol{q}) \leq$$

$$\leq f[a,a;b] \sum_{i=1}^{r} \frac{(p_i - aq_i)(p_i - bq_i)}{q_i} + \sum_{k=2}^{n-2} f\big[a, a; \underbrace{b, \ldots, b}_{k \text{ times}}\big] \sum_{i=1}^{r} \frac{(p_i - aq_i)^2 (p_i - bq_i)^{k-1}}{q_i^k}. \qquad (3.5)$$

*Inequalities* (3.5) *also hold when the function $f$ is $n$-concave and $n$ is even. In case when the function $f$ is $n$-convex and $n$ is even, or when the function $f$ is $n$-concave and $n$ is odd, the inequality signs in* (3.5) *are reversed.*

***Proof.*** We start with inequalities (2.22), and follow the steps from the proof of Theorem 3.1.

By utilizing Theorem 2.4 in an analogue way, we can get similar bounds for the generalized $f$-divergence functional that hold for all $n \in \mathbb{N}$, not only the odd ones.

**Theorem 3.4.** *Let $[a,b] \subset \mathbb{R}$ be an interval such that $a \leq 1 \leq b$. Let $f \in \mathcal{C}^n([a,b])$ and let $\boldsymbol{p} = (p_1,\ldots,p_r)$ and $\boldsymbol{p} = (q_1,\ldots,q_r)$ be probability distributions such that $p_i/q_i \in [a,b]$ for every $i = 1,\ldots,r$. If the function $f$ is $n$-convex, then we have*

$$f[b,b;a]\sum_{i=1}^{r}\frac{(p_i-aq_i)(p_i-bq_i)}{q_i}+\sum_{k=2}^{n-2}f[b,b;\underbrace{a,a,\ldots,a}_{k\text{ times}}]\sum_{i=1}^{r}\frac{(p_i-aq_i)^{k-1}(p_i-bq_i)^2}{q_i^k}\leq$$

$$\leq\frac{b-1}{b-a}f(a)+\frac{1-a}{b-a}f(b)-\tilde{D}_f(\boldsymbol{p},\boldsymbol{q})\leq\sum_{k=2}^{n-1}f\big[b;\underbrace{a,\ldots,a}_{k\text{ times}}\big]\sum_{i=1}^{r}\frac{(p_i-aq_i)^{k-1}(p_i-bq_i)}{q_i^{k-1}}. \tag{3.6}$$

*If the function $f$ is $n$-concave, the inequality signs in* (3.6) *are reversed.*

***Example* 3.1.** Let $\boldsymbol{p} = (p_1,\ldots,p_r)$ and $\boldsymbol{p} = (q_1,\ldots,q_r)$ be probability distributions.

**Kullback – Leibler divergence** of the probability distributions $\boldsymbol{p}$ and $\boldsymbol{q}$ is defined as

$$D_{KL}(\boldsymbol{p},\boldsymbol{q}) = \sum_{i=1}^{r} q_i \log\frac{q_i}{p_i},$$

and the corresponding generating function is $f(t) = t\log t,\ t>0$. We can calculate

$$f^{(n)}(t) = (-1)^n(n-2)!t^{-(n-1)}.$$

It is clear that this function is $(2n-1)$-concave and $(2n)$-convex for any $n\in\mathbb{N}$.

**Hellinger divergence** of the probability distributions $\boldsymbol{p}$ and $\boldsymbol{q}$ is defined as

$$D_H(\boldsymbol{p},\boldsymbol{q}) = \frac{1}{2}\sum_{i=1}^{n}(\sqrt{q_i}-\sqrt{p_i})^2,$$

and the corresponding generating function is $f(t) = \dfrac{1}{2}(1-\sqrt{t})^2, t>0$. We see that

$$f^{(n)}(t) = (-1)^n\frac{(2n-3)!!}{2^n}t^{-\frac{2n-1}{2}},$$

so function $f$ is $(2n-1)$-concave and $(2n)$-convex for any $n\in\mathbb{N}$.

**Harmonic divergence** of the probability distributions $\boldsymbol{p}$ and $\boldsymbol{q}$ is defined as

$$D_{Ha}(\boldsymbol{p},\boldsymbol{q}) = \sum_{i=1}^{n}\frac{2p_iq_i}{p_i+q_i},$$

and the corresponding generating function is $f(t) = \dfrac{2t}{1+t}$. We can calculate

$$f^{(n)}(t) = 2(-1)^{n+1}n!(1+t)^{-(n+1)}.$$

Two cases need to be considered:

if $t<-1$, then the function $f$ is $n$-convex for every $n\in\mathbb{N}$;

if $t>-1$, then the function $f$ is $(2n)$-concave and $(2n-1)$-convex for any $n\in\mathbb{N}$.

**Jeffreys divergence** of the probability distributions $\boldsymbol{p}$ and $\boldsymbol{q}$ is defined as

$$D_J(\boldsymbol{p},\boldsymbol{q}) = \frac{1}{2}\sum_{i=1}^{n}(q_i - p_i)\log\frac{q_i}{p_i},$$

and the corresponding generating function is $f(t) = (1-t)\log\dfrac{1}{t}$, $t > 0$. After calculating, we see that

$$f^{(n)}(t) = (-1)^{n+1}t^{-n}(n-1)!\,(1+nt).$$

Obviously, this function is $(2n-1)$-convex and $(2n)$-concave for any $n \in \mathbb{N}$.

It is clear that all of the results from this section can be applied to the special types of divergences mentioned in this example.

**4. Examples with Zipf and Zipf – Mandelbrot law.** Zipf's law [33, 34] has a significant application in a wide variety of scientific disciplines — from astronomy to demographics to software structure to economics to zoology, and even to warfare [12]. It is one of the basic laws in information science and bibliometrics, but it is also often used in linguistics. Typically one is dealing with integer-valued observables (numbers of objects, people, cities, words, animals, corpses) and the frequency of their occurrence.

Probability mass function of Zipf's law with parameters $N \in \mathbb{N}$ and $s > 0$ is

$$f(k; N, s) = \frac{1/k^s}{H_{N,s}}, \qquad \text{where} \quad H_{N,s} = \sum_{i=1}^{N}\frac{1}{i^s}.$$

Benoit Mandelbrot in 1966 gave an improvement of Zipf law for the count of the low-rank words. Various scientific fields use this law for different purposes, for example information sciences use it for indexing [11, 32], ecological field studies in predictability of ecosystem [26], in music it is used to determine aesthetically pleasing music [23].

Zipf – Mandelbrot law is a discrete probability distribution with parameters $N \in \mathbb{N}$, $q, s \in \mathbb{R}$ such that $q \geq 0$ and $s > 0$, possible values $\{1, 2, \ldots, N\}$ and probability mass function

$$f(i; N, q, s) = \frac{1/(i+q)^s}{H_{N,q,s}}, \qquad \text{where} \quad H_{N,q,s} = \sum_{i=1}^{N}\frac{1}{(i+q)^s}. \tag{4.1}$$

Let $\boldsymbol{p}$ and $\boldsymbol{q}$ be Zipf – Mandelbrot laws with parameters $N \in \mathbb{N}$, $q_1$, $q_2 \geq 0$ and $s_1$, $s_2 > 0$, respectively, and let us denote

$$\begin{gathered} H_{N,q_1,s_1} = H_1, \qquad H_{N,q_2,s_2} = H_2, \\ a_{\boldsymbol{p},\boldsymbol{q}} := \min\left\{\frac{p_i}{q_i}\right\} = \frac{H_2}{H_1}\min\left\{\frac{(i+q_2)^{s_2}}{(i+q_1)^{s_1}}\right\}, \\ b_{\boldsymbol{p},\boldsymbol{q}} := \max\left\{\frac{p_i}{q_i}\right\} = \frac{H_2}{H_1}\max\left\{\frac{(i+q_2)^{s_2}}{(i+q_1)^{s_1}}\right\}. \end{gathered} \tag{4.2}$$

In this section we utilize the results regarding Csiszár divergence from the previous section in order to obtain different inequalities for the Zipf – Mandelbrot law. The following results are special cases of Theorems 3.1, 3.2, 3.3 and 3.4, respectively, and they gives us Edmundson – Lah – Ribarič type inequality for the generalized $f$-divergence of the Zipf – Mandelbrot law.

***Corollary* 4.1.** *Let* $\boldsymbol{p}$ *and* $\boldsymbol{q}$ *be Zipf–Mandelbrot laws with parameters* $N \in \mathbb{N}$, $q_1,\ q_2 \geq 0$ *and* $s_1,\ s_2 > 0$, *respectively, and let* $H_1,\ H_2,\ a_{\boldsymbol{p},\boldsymbol{q}}$ *and* $a_{\boldsymbol{p},\boldsymbol{q}}$ *be defined in* (4.2). *Let* $f \in \mathcal{C}^n([a_{\boldsymbol{p},\boldsymbol{q}}, b_{\boldsymbol{p},\boldsymbol{q}}])$ *be a* $n$*-convex function. If* $n$ *and* $3 \leq m \leq n-1$ *are of different parity, then*

$$\frac{b_{\boldsymbol{p},\boldsymbol{q}}-1}{b_{\boldsymbol{p},\boldsymbol{q}}-a_{\boldsymbol{p},\boldsymbol{q}}} f(a_{\boldsymbol{p},\boldsymbol{q}}) + \frac{1-a_{\boldsymbol{p},\boldsymbol{q}}}{b_{\boldsymbol{p},\boldsymbol{q}}-a_{\boldsymbol{p},\boldsymbol{q}}} f(b_{\boldsymbol{p},\boldsymbol{q}}) - \tilde{D}_f(\boldsymbol{p},\boldsymbol{q}) \leq$$

$$\leq (1-a_{\boldsymbol{p},\boldsymbol{q}})\left(f'(a_{\boldsymbol{p},\boldsymbol{q}}) - f[a_{\boldsymbol{p},\boldsymbol{q}}, b_{\boldsymbol{p},\boldsymbol{q}}]\right) + \sum_{k=2}^{m-1} \frac{f^{(k)}(a_{\boldsymbol{p},\boldsymbol{q}})}{H_2 k!} \sum_{i=1}^{r} \frac{\left(\frac{H_2(i+q_2)^{s_2}}{H_1(i+q_1)^{s_1}} - a_{\boldsymbol{p},\boldsymbol{q}}\right)^k}{(i+q_2)^{s_2}} +$$

$$+ \sum_{k=1}^{n-m} f[\underbrace{a_{\boldsymbol{p},\boldsymbol{q}}, \ldots, a_{\boldsymbol{p},\boldsymbol{q}}}_{m \text{ times}}; \underbrace{b_{\boldsymbol{p},\boldsymbol{q}}, \ldots, b_{\boldsymbol{p},\boldsymbol{q}}}_{k \text{ times}}] \sum_{i=1}^{r} \frac{\left(\frac{H_2(i+q_2)^{s_2}}{H_1(i+q_1)^{s_1}} - a_{\boldsymbol{p},\boldsymbol{q}}\right)^m \left(\frac{H_2(i+q_2)^{s_2}}{H_1(i+q_1)^{s_1}} - b_{\boldsymbol{p},\boldsymbol{q}}\right)^{k-1}}{H_2(i+q_2)^{s_2}}.$$

*This inequality also holds when the function* $f$ *is* $n$*-concave and* $n$ *and* $m$ *are of equal parity. In case when the function* $f$ *is* $n$*-convex and* $n$ *and* $m$ *are of equal parity, or when the function* $f$ *is* $n$*-concave and* $n$ *and* $m$ *are of different parity, the inequality sign is reversed.*

***Corollary* 4.2.** *Let* $\boldsymbol{p}$ *and* $\boldsymbol{q}$ *be Zipf–Mandelbrot laws with parameters* $N \in \mathbb{N}$, $q_1,\ q_2 \geq 0$ *and* $s_1,\ s_2 > 0$, *respectively, and let* $H_1,\ H_2,\ a_{\boldsymbol{p},\boldsymbol{q}}$ *and* $a_{\boldsymbol{p},\boldsymbol{q}}$ *be defined in* (4.2). *Let* $f \in \mathcal{C}^n([a_{\boldsymbol{p},\boldsymbol{q}}, b_{\boldsymbol{p},\boldsymbol{q}}])$ *be a* $n$*-convex function and let* $3 \leq m \leq n-1$ *be of different parity. Then*

$$\frac{b_{\boldsymbol{p},\boldsymbol{q}}-1}{b_{\boldsymbol{p},\boldsymbol{q}}-a_{\boldsymbol{p},\boldsymbol{q}}} f(a_{\boldsymbol{p},\boldsymbol{q}}) + \frac{1-a_{\boldsymbol{p},\boldsymbol{q}}}{b_{\boldsymbol{p},\boldsymbol{q}}-a_{\boldsymbol{p},\boldsymbol{q}}} f(b_{\boldsymbol{p},\boldsymbol{q}}) - \tilde{D}_f(\boldsymbol{p},\boldsymbol{q}) \leq$$

$$\leq (b_{\boldsymbol{p},\boldsymbol{q}}-1)\left(f[a_{\boldsymbol{p},\boldsymbol{q}}, b_{\boldsymbol{p},\boldsymbol{q}}] - f'(b_{\boldsymbol{p},\boldsymbol{q}})\right) + \sum_{k=2}^{m-1} \frac{f^{(k)}(b_{\boldsymbol{p},\boldsymbol{q}})}{H_2 k!} \sum_{i=1}^{r} \frac{\left(\frac{H_2(i+q_2)^{s_2}}{H_1(i+q_1)^{s_1}} - b_{\boldsymbol{p},\boldsymbol{q}}\right)^k}{(i+q_2)^{s_2}} +$$

$$+ \sum_{k=1}^{n-m} f\Big[\underbrace{b_{\boldsymbol{p},\boldsymbol{q}}, \ldots, b_{\boldsymbol{p},\boldsymbol{q}}}_{m \text{ times}}; \underbrace{a_{\boldsymbol{p},\boldsymbol{q}}, \ldots, a_{\boldsymbol{p},\boldsymbol{q}}}_{k \text{ times}}\Big] \sum_{i=1}^{r} \frac{\left(\frac{H_2(i+q_2)^{s_2}}{H_1(i+q_1)^{s_1}} - b_{\boldsymbol{p},\boldsymbol{q}}\right)^m \left(\frac{H_2(i+q_2)^{s_2}}{H_1(i+q_1)^{s_1}} - a_{\boldsymbol{p},\boldsymbol{q}}\right)^{k-1}}{H_2(i+q_2)^{s_2}}.$$

*The inequality above also holds when the function* $f$ *is* $n$*-concave and* $m$ *is even. In case when the function* $f$ *is* $n$*-convex and* $m$ *is even, or when the function* $f$ *is* $n$*-concave and* $m$ *is odd, the inequality sign is reversed.*

***Corollary* 4.3.** *Let* $\boldsymbol{p}$ *and* $\boldsymbol{q}$ *be Zipf–Mandelbrot laws with parameters* $N \in \mathbb{N}$, $q_1,\ q_2 \geq 0$ *and* $s_1,\ s_2 > 0$, *respectively, and let* $H_1,\ H_2,\ a_{\boldsymbol{p},\boldsymbol{q}}$ *and* $a_{\boldsymbol{p},\boldsymbol{q}}$ *be defined in* (4.2). *Let* $f \in \mathcal{C}^n([a_{\boldsymbol{p},\boldsymbol{q}}, b_{\boldsymbol{p},\boldsymbol{q}}])$ *be a* $n$*-convex function. If* $n$ *is odd, then we have*

$$\sum_{k=2}^{n-1} f\Big[a_{\boldsymbol{p},\boldsymbol{q}}; \underbrace{b_{\boldsymbol{p},\boldsymbol{q}}, \ldots, b_{\boldsymbol{p},\boldsymbol{q}}}_{k \text{ times}}\Big] \sum_{i=1}^{r} \frac{\left(\frac{H_2(i+q_2)^{s_2}}{H_1(i+q_1)^{s_1}} - a_{\boldsymbol{p},\boldsymbol{q}}\right) \left(\frac{H_2(i+q_2)^{s_2}}{H_1(i+q_1)^{s_1}} - b_{\boldsymbol{p},\boldsymbol{q}}\right)^{k-1}}{H_2(i+q_2)^{s_2}} \leq$$

$$\leq \frac{b_{\boldsymbol{p},\boldsymbol{q}}-1}{b_{\boldsymbol{p},\boldsymbol{q}}-a_{\boldsymbol{p},\boldsymbol{q}}} f(a_{\boldsymbol{p},\boldsymbol{q}}) + \frac{1-a_{\boldsymbol{p},\boldsymbol{q}}}{b_{\boldsymbol{p},\boldsymbol{q}}-a_{\boldsymbol{p},\boldsymbol{q}}} f(b_{\boldsymbol{p},\boldsymbol{q}}) - \tilde{D}_f(\boldsymbol{p},\boldsymbol{q}) \leq$$

$$\leq f[a_{\boldsymbol{p},\boldsymbol{q}},a_{\boldsymbol{p},\boldsymbol{q}};b_{\boldsymbol{p},\boldsymbol{q}}]\sum_{i=1}^{r}\frac{\left(\dfrac{H_2(i+q_2)^{s_2}}{H_1(i+q_1)^{s_1}}-a_{\boldsymbol{p},\boldsymbol{q}}\right)\left(\dfrac{H_2(i+q_2)^{s_2}}{H_1(i+q_1)^{s_1}}-b_{\boldsymbol{p},\boldsymbol{q}}\right)}{H_2(i+q_2)^{s_2}}+$$

$$+\sum_{k=2}^{n-2}f\Big[a_{\boldsymbol{p},\boldsymbol{q}},a_{\boldsymbol{p},\boldsymbol{q}};\underbrace{b_{\boldsymbol{p},\boldsymbol{q}},\ldots,b_{\boldsymbol{p},\boldsymbol{q}}}_{k\text{ times}}\Big]\sum_{i=1}^{r}\frac{\left(\dfrac{H_2(i+q_2)^{s_2}}{H_1(i+q_1)^{s_1}}-a_{\boldsymbol{p},\boldsymbol{q}}\right)^2\left(\dfrac{H_2(i+q_2)^{s_2}}{H_1(i+q_1)^{s_1}}-b_{\boldsymbol{p},\boldsymbol{q}}\right)^{k-1}}{H_2(i+q_2)^{s_2}}.$$

*Stated inequalities also hold when the function $f$ is $n$-concave and $n$ is even. In case when the function $f$ is $n$-convex and $n$ is even, or when the function $f$ is $n$-concave and $n$ is odd, the inequality signs are reversed.*

***Corollary* 4.4.** *Let $\mathbf{p}$ and $\boldsymbol{q}$ be Zipf – Mandelbrot laws with parameters $N\in\mathbb{N}$, $q_1,\ q_2\geq 0$ and $s_1,\ s_2>0$, respectively, and let $H_1,\ H_2,\ a_{\boldsymbol{p},\boldsymbol{q}}$ and $a_{\boldsymbol{p},\boldsymbol{q}}$ be defined in* (4.2). *Let $f\in\mathcal{C}^n([a_{\boldsymbol{p},\boldsymbol{q}},b_{\boldsymbol{p},\boldsymbol{q}}])$ be a $n$-convex function. Then we have*

$$f[b_{\boldsymbol{p},\boldsymbol{q}},b_{\boldsymbol{p},\boldsymbol{q}};a_{\boldsymbol{p},\boldsymbol{q}}]\sum_{i=1}^{r}\frac{\left(\dfrac{H_2(i+q_2)^{s_2}}{H_1(i+q_1)^{s_1}}-a_{\boldsymbol{p},\boldsymbol{q}}\right)\left(\dfrac{H_2(i+q_2)^{s_2}}{H_1(i+q_1)^{s_1}}-b_{\boldsymbol{p},\boldsymbol{q}}\right)}{H_2(i+q_2)^{s_2}}+$$

$$+\sum_{k=2}^{n-2}f\Big[b_{\boldsymbol{p},\boldsymbol{q}},b_{\boldsymbol{p},\boldsymbol{q}};\underbrace{a_{\boldsymbol{p},\boldsymbol{q}},\ldots,a_{\boldsymbol{p},\boldsymbol{q}}}_{k\text{ times}}\Big]\sum_{i=1}^{r}\frac{\left(\dfrac{H_2(i+q_2)^{s_2}}{H_1(i+q_1)^{s_1}}-a_{\boldsymbol{p},\boldsymbol{q}}\right)^{k-1}\left(\dfrac{H_2(i+q_2)^{s_2}}{H_1(i+q_1)^{s_1}}-b_{\boldsymbol{p},\boldsymbol{q}}\right)^2}{H_2(i+q_2)^{s_2}}\leq$$

$$\leq\frac{b-1}{b-a}f(a)+\frac{1-a}{b-a}f(b)-\tilde{D}_f(\boldsymbol{p},\boldsymbol{q})\leq$$

$$\leq\sum_{k=2}^{n-1}f\Big[b_{\boldsymbol{p},\boldsymbol{q}};\underbrace{a_{\boldsymbol{p},\boldsymbol{q}},\ldots,a_{\boldsymbol{p},\boldsymbol{q}}}_{k\text{ times}}\Big]\sum_{i=1}^{r}\frac{\left(\dfrac{H_2(i+q_2)^{s_2}}{H_1(i+q_1)^{s_1}}-a_{\boldsymbol{p},\boldsymbol{q}}\right)^{k-1}\left(\dfrac{H_2(i+q_2)^{s_2}}{H_1(i+q_1)^{s_1}}-b_{\boldsymbol{p},\boldsymbol{q}}\right)}{H_2(i+q_2)^{s_2}}.$$

*If the function $f$ is $n$-concave, the inequality signs are reversed.*

***Remark* 4.1.** By taking into consideration Example 3.1 one can see that general results from this section can easily be applied to any of the following divergences: Kullback – Leibler divergence, Hellinger divergence, harmonic divergence or Jeffreys divergence.

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