DOI: 10.37863/umzh.v73i1.889

UDC 517.9

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ACCURATE APPROXIMATED SOLUTION TO THE DIFFERENTIAL INCLUSION BASED ON THE ORDINARY DIFFERENTIAL EQUATION

ТОЧНИЙ НАБЛИЖЕНИЙ РОЗВ'ЯЗОК ДИФЕРЕНЦІАЛЬНОГО ВКЛЮЧЕННЯ НА ОСНОВІ ЗВИЧАЙНОГО ДИФЕРЕНЦІАЛЬНОГО РІВНЯННЯ

Many problems in applied mathematics can be transformed and described by the differential inclusion $\dot{x} \in f(t,x) - N_Q x$ involving $N_Q x$, which is a normal cone to a closed convex set $Q \in \mathbb{R}^n$ at $x \in Q$. The Cauchy problem of this inclusion is studied in the paper. Since the change of x leads to the change of $N_Q x$, solving the inclusion becomes extremely complicated. In this paper, we consider an ordinary differential equation containing a control parameter K. When K is large enough, the studied equation gives a solution approximating to a solution of the inclusion above. The theorem about the approximation of these solutions with arbitrary small error (this error can be controlled by increasing K) is proved in this paper.

Багато задач у прикладній математиці можна трансформувати та описати за допомогою диференціального включення $\dot{x} \in f(t,x) - N_Q x$, в яке входить $N_Q x$, що є нормальним конусом для замкненої опуклої множини $Q \in \mathbb{R}^n$ у точці $x \in Q$. У цій роботі вивчається задача Коші для такого включення. Оскільки зміна x обумовлює зміну $N_Q x$, розв'язання цього включення стає надто складним. Тут розглядається звичайне диференціальне рівняння, яке містить керуючий параметр K. Коли K є достатньо великим, це рівняння дає розв'язок, який наближає розв'язок досліджуваного включення. Доведено теорему про наближення цих розв'язків з будь-якою точністю (відповідна похибка контролюється за допомогою зростання K).

1. Introduction. Theory of differential inclusions has gained great interest in recent decades. Under some assumptions, differential inclusions are equivalent to equations in contigent [1, 2] and equations in paratingent [3]. Particularly, A. F. Filippov [4] provided some research on the following differential inclusion:

$$\dot{x} \in F(t,x), \tag{1}$$

where $t \in [t_0, T]$, $x \in \mathbb{R}^n$ and $F(t,x)$ is a closed convex set of $\mathbb{R}^n$.

A solution of inclusion (1) is a locally absolutely continuous function $x = x(t)$ satisfying (1) for almost all $t \in [t_0, T]$ (see [4, p. 54]). In addition, if $x(t)$, $t \in [t_0, T]$, is a solution of (1), then there exists a vector function $\varphi(t,x) \in F(t,x)$ such that $\dot{x} = \varphi(t,x)$ for almost all $t \in [t_0, T]$.

Using differential inclusions that can be considered as a mathematical model for some problems in the field of circuit theory [5] and general ecosystems [6], a specific problem modeled by differential inclusions can be found in Section 2.

Let $Q \subset \mathbb{R}^n$ be a fixed closed convex set and $x \in Q$. *The normal cone* to Q at x is defined by

$$N_Q x = \big\{ z \in \mathbb{R}^n : (z, \xi - x) \le 0 \ \ \forall \xi \in Q \big\}, \tag{2}$$

where $(\cdot,\cdot)$ denotes the scalar product in $\mathbb{R}^n$. Some properties of $N_Q x$ will be studied in Section 2.

Fix $(t_0, x_0) \in \mathbb{R} \times Q$ and $T > 0$, and let $f \colon [t_0, T] \times Q \to \mathbb{R}^n$ be a continuous function satisfying a Lipschitz condition with some Lipschitz constant $L > 0$ and bounded by a constant C, that is,

$$\big\| f(t,x) - f(t,y) \big\| \le L \|x - y\| \quad \forall x \in Q \quad \forall t \in [t_0, T], \tag{3}$$

$$\big\|f(t,x)\big\| \le C \quad \forall x \in Q \quad \forall t \in [t_0, T], \tag{4}$$

where $\|\cdot\|$ denotes the norm defined by $\|\cdot\| = \sqrt{(\cdot,\cdot)}$.

Under the conditions (3), (4) we consider the differential inclusion

$$\dot{x} \in f(t,x) - N_Q x, \qquad t \in [t_0, T], \quad x(t) \in Q. \tag{5}$$

Both existence and uniqueness of solutions of the Cauchy problem

$$\begin{aligned} &\dot{x} \in f(t,x) - N_Q x, \\ &x(t_0) = x_0 \in Q \end{aligned} \tag{6}$$

will be shown in Section 4. For this, we recall a solution of (6) as an absolutely continuous function $x = x(t)$ which satisfies (5) for almost all $t \in [t_0, T]$ and the initial condition $x(t_0) = x_0 \in Q$.

Obviously, we see that differential inclusions are usually more difficult and complicated than differential equations. Therefore, we propose a form of an ordinary differential equation whose right-hand side is a continuous function in order to find a solution approximated to a solution of (5). This is the main objective of the paper.

Now, let $Q \subset \mathbb{R}^n$ be a convex closed set and $y \in \mathbb{R}^n$. An element $\overline{y} \in Q$ is called point of best approximation to y in Q and denoted $\overline{y} = P_Q(y)$ if

$$\|y - \overline{y}\| = \operatorname{dist}(y, Q) := \inf\big\{\|y - z\| : z \in Q\big\}. \tag{7}$$

The element $\overline{y}$ is called *projection* of y onto Q.

With the mentioned goal above, we consider the following ordinary differential equation, where the right-hand side is a continuous function involving a sufficiently large control parameter K:

$$\begin{aligned} &\dot{y} = f(t, \overline{y}) - K(y - \overline{y}), \\ &y(t_0) = x_0 \in Q. \end{aligned} \tag{8}$$

Obviously, for each K the Cauchy problem (8) has at most one continuously differentiable solution $y_K(t)$ on $[t_0, T]$. The contribution of this paper is to evaluate the error between the solution obtained by solving (8) and (6). We show that an arbitrary small error can be obtained by increasing the control parameter K to a sufficiently large value.

2. Preliminaries. In $\mathbb{R}^n$ let us consider cones in the sense of the following definition.

Definition 1. *A subset $S \subset \mathbb{R}^n$ is called a cone, if*

$$x, y \in S \qquad \textit{and} \qquad s, t \ge 0 \Rightarrow sx + ty \in S.$$

Besides, the adjoint cone of a cone $S \subset \mathbb{R}^n$ is a special normal cone, which is defined as follows.

Definition 2. *Let $S \subset \mathbb{R}^n$ be a cone. The adjoint cone to S is defined by*

$$S^* := N_S 0 = \big\{z \in \mathbb{R}^n : (z, y) \le 0 \ \ \forall y \in S\big\}.$$

Let $Q \subset \mathbb{R}^n$ be a closed convex set and $x \in Q$, the normal cone $N_Q(x)$ to Q at x is defined by (2). *Next, the tangent cone to Q at x is defined by*

$$T_Q x = \{z \in \mathbb{R}^n : (z, \xi) \le 0 \ \ \forall \xi \in N_Q x\}. \tag{9}$$

A comparison of (2) and (9) shows that

$$T_Q x = (N_Q x)^*.$$

Now, the important properties of N_Q are described in two following propositions.

Proposition 1. *Let $Q \subset \mathbb{R}^n$ be a closed convex set. Then the normal cone $N_Q x$, $x \in Q$, is a cone in the sense of Definition* 1, *and it is a closed set.*

Proof. The proof relies on the linearity and continuity of the scalar product. Given $\xi \in Q$, $z_1, z_2 \in N_Q x$, and $t_1, t_2 \geq 0$, we have

$$(t_1 z_1 + t_2 z_2, \xi - x) = t_1(z_1, \xi - x) + t_2(z_2, \xi - x) \leq 0.$$

Hence, $t_1 z_1 + t_2 z_2 \in N_Q x$.

Now, if $(z_k)_{k=1}^{+\infty}$ is a sequence in $N_Q x$ converging to $z \in \mathbb{R}^n$, we get

$$(z, \xi - x) = \lim_{k \to +\infty} (z_k, \xi - x) \leq 0$$

for all $\xi \in Q$. Hence, $z \in N_Q x$.

Proposition 1 is proved.

Proposition 2. *Let $Q \subset \mathbb{R}^n$ be a closed convex set. Then the multivalued map $N_Q : x \mapsto N_Q(x)$ is closed. That is, if $(x_k)_{k=1}^{+\infty}$ is any sequence in Q and $z_k \in N_Q(x_k)$, then the relations*

$$\lim_{k \to +\infty} x_k = x, \qquad \lim_{k \to +\infty} z_k = z$$

for some $z \in \mathbb{R}^n$ imply that $z \in N_Q(x)$.

Proof. The relation $z_k \in N_Q(x_k)$ means that

$$(z_k, \xi - x_k) \leq 0 \quad \forall \xi \in Q.$$

Passing in this inequality to the limit as $k \to +\infty$ yields

$$(z, \xi - x) \leq 0 \quad \forall \xi \in Q$$

which means that $z \in N_Q(x)$.

Proposition 2 is proved.

Based on these concepts we study a problem that is modeled by the differential inclusion of the form (5). Let us consider the electrical circuit on Fig. 1. Recall that the inputs L (the inductance), R (the resistance), and E (the electromotive force source) of the circuit are related by the equations

$$L \frac{di_L}{dt} = u_L, \qquad R i_R = u_R, \qquad u_E = -e(t).$$

The parameters L (measured in Henry) and R (measured in Ohm) are given constants, while the known time-dependent function $e(t)$ is measured in Volt. The functions i (measured in Ampere) and u (measured in Volt) with corresponding subscripts are the current and voltage, respectively; their direction is indicated in Fig. 1 by the arrow.

An ideal diode D is an element which leads the current in the direction of the arrow (i.e., from the *anode* to the *cathode*), but not in the reverse direction. For negative voltage ($u_D < 0$), the current is zero ($i_D = 0$), while for vanishing voltage ($u_D = 0$), the current is positive or zero ($i_D \geq 0$); a

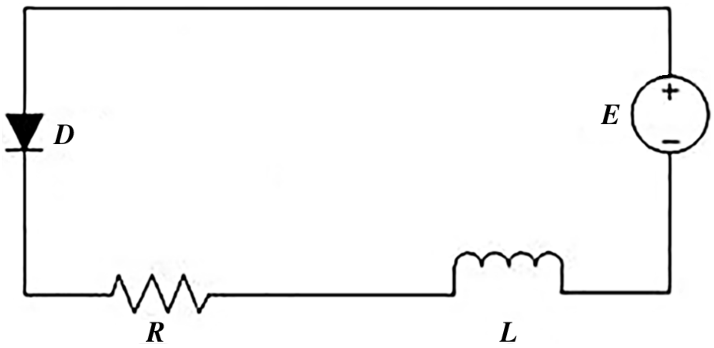

Fig. 1. An example of electrical circuits.

positive voltage is not possible. So the dependence of the current on the voltage in an ideal diode may be described formally as a set of the following relations:

$$i_D \geq 0, \qquad u_D \leq 0, \qquad i_D \cdot u_D = 0. \tag{10}$$

From (10) it follows $i_D \in Q := \big[0, +\infty\big)$ and

$$u_D \in U := \begin{cases} \{0\}, & \text{if} \quad i_D > 0, \\ (-\infty, 0], & \text{if} \quad i_D = 0. \end{cases}$$

By using (2), we get

$$u_D \in N_Q i_D. \tag{11}$$

For such a circuit the second Kirchhoff rule states that

$$u_L + u_R + u_D + u_E = 0,$$

i.e., the sum of the voltages of all elements is zero. Using the fact that the current of all elements in a non-ramified circuit (a circuit is called non-ramified (or non-bifurcating) if every knot joins exactly two electrical elements) is the same $(i_L = i_R = i_D =: i)$, we arrive at the differential equation

$$L\frac{di}{dt} + Ri = e(t) - u$$

for the unknown functions i and $u := u_D$. For a complete description of the circuit we also have to take into account the relations (10) between u and i (or, equivalently, the affirmation in (11)). Then the mathematical model leads to the differential inclusion

$$L\frac{di}{dt} \in e(t) - Ri - N_Q i.$$

It is convenient to rewrite this differential inclusion in the form

$$\frac{di}{dt} \in g(t) - N_Q i, \tag{12}$$

where $g(t) = \dfrac{e(t)}{L} - \dfrac{R}{L}i.$

Thus, the mathematical model (12) for the reseach circuit is differential inclusion of form (5).

This is an illustration for a simple problem, and much more complex problems of electrical circuit theory can be modelled by using differential inclusion (5).

3. Auxiliary lemma. Let us denote $E = N_Q x$ and $\overline{E} = T_Q x$.

Lemma 1. *For each $c \in \mathbb{R}^n$, the following two relations are equivalent*:

$$c = a + b, \qquad a \in E, \quad b \in \overline{E}, \qquad (a, b) = 0, \tag{13}$$

$$a = P_E c, \qquad b = P_{\overline{E}} c. \tag{14}$$

Proof. (13) $\Rightarrow$ (14). Suppose that the relation (13) holds, we need to prove that (14) holds also. By using (13) and Definition 2, we obtain

$$(c - a, \xi - a) = (b, \xi) \le 0 \quad \forall \xi \in E,$$

and, consequently, $\|c - a\| \le \|c - \xi\|$ for all $\xi \in E$, because $\|c - \xi\|$ is the length of a side of the triangle. This is opposite to right angle or obtuse angle of the triangle, while the length of the other two sides are $\|c - a\|$ and $\|\xi - a\|$. From here and (7) directly implies $a = P_E c$. Similarly, we can easily prove $b = P_{\overline{E}} c$. Hence, (14) is proved.

(14) $\Rightarrow$ (13). From (14) and (7) we will have $a \in E$, $b \in \overline{E}$ and

$$(c - a, \xi - a) \le 0 \quad \forall \xi \in E, \tag{15}$$

$$(c - b, \mu - b) \le 0 \quad \forall \mu \in \overline{E}. \tag{16}$$

Indeed, we consider a function $z(t) = t\xi + (1 - t)a$, where $t \in [0, 1]$, and according to Proposition 1 we have $z(t) \in E$. Let

$$\varphi(t) = \|c - z(t)\|^2.$$

It is easy to see that the function $\varphi(t)$ has the smallest value at $t = 0$, so $\dot{\varphi}(0) \ge 0$. On the other-hand, we have

$$\dot{\varphi}(t) = 2\big(c - z(t), -\dot{z}(t)\big) = 2\big(c - t\xi - (1 - t)a, a - \xi\big) =$$
$$= -2(c - a, \xi - a) + 2t\|\xi - a\|$$

which gives

$$\dot{\varphi}(0) = -2(c - a, \xi - a) \ge 0.$$

Therefore, the inequality (15) is proved. Similarly, we can prove the condition (16) is also true.

Now, suppose $b_2 = c - a$ and since the set $E \subset \mathbb{R}^n$ is a cone in the sense of Definition 1, so $0 \in E$. By virtue of (15) we get

$$(b_2, a) \ge 0. \tag{17}$$

Moreover, since $a = P_E c \in E$, $2a \in E$. Then

$$(b_2, a) \le 0. \tag{18}$$

By virtue of (17) and (18) we can obtain $(b_2, a) = 0$. From here and (15) it immediately follows that

$$(b_2, \xi) = (c - a, \xi - a) \leq 0 \quad \forall \xi \in E.$$

Next, by using (9), we get $b_2 \in \overline{E}$. In addition, from Definition 2 it follows that

$$(c - b_2, \mu - b_2) = (a, \mu) \leq 0 \quad \forall \mu \in \overline{E}.$$

Thus, the inequality $\|c - b_2\| \leq \|c - \mu\|$ holds for all $\mu \in \overline{E}$, that is, $\|c - b_2\|$ is the smallest length from c to the set $\overline{E}$. And, by virtue of (7) we easily get $b_2 = P_{\overline{E}} c$.

To finish the proof of Lemma 1, we have to show $b = b_2$. Indeed, since $b, b_2 \in \overline{E}$ and using (16) we get that the vectors b, b_2 are satisfied such that

$$(c - b, b_2 - b) \leq 0 \qquad \text{and} \qquad (c - b_2, b - b_2) \leq 0. \tag{19}$$

Besides, we can see that

$$(c - b, b_2 - b) = -(c - b_2, b - b_2) + (b_2 - b, b_2 - b) \geq 0. \tag{20}$$

Now, from (19) and (20) it follows that $b = b_2$.

Lemma 1 is proved.

4. Main result.

Theorem 1. *Let $Q \subset \mathbb{R}^n$ be a closed bounded convex set, $(t_0, x_0) \in \mathbb{R} \times Q$, $T > t_0$ and $f \colon [t_0, T] \times Q \to \mathbb{R}^n$ a continuous function satisfying conditions* (3), (4). *Then the Cauchy problem* (6) *has a unique solution $x(t)$. Moreover,*

$$\big\|x(t) - y_K(t)\big\| \leq \frac{Ce^{L(T-t_0)}}{\sqrt{L}} \frac{1}{\sqrt{K}}, \quad t \in [t_0, T], \tag{21}$$

where $y_K(t)$ is a solution of problem (8) *for a fixed parameter K.*

Proof. First, we prove that when K is large enough, solution $y_K(t)$ of (8) can not go far beyond the set Q, namely $y_K(t)$ satisfies

$$\big\|y_K - \overline{y}_K\big\| \leq \frac{C}{K}, \tag{22}$$

where C is defined by (4) and $\overline{y}_K = P_Q(y_K)$.

To prove (22), we put

$$\psi(t) = \big\|y_K(t) - \overline{y}_K(t)\big\|^2, \quad t \in [t_0, T].$$

We have

$$\dot{\psi}(t) = 2\big(y_K - \overline{y}_K, \dot{y}_K - \dot{\overline{y}}_K\big) = 2\Big(y_K - \overline{y}_K, f(t, \overline{y}_K) - K(y_K - \overline{y}_K) - \dot{\overline{y}}_K\Big),$$

consequently,

$$\dot{\psi}(t) = 2\big(y_K - \overline{y}_K, f(t, \overline{y}_K)\big) - 2K\|y_K - \overline{y}_K\|^2 - 2\big(y_K - \overline{y}_K, \dot{\overline{y}}_K\big). \tag{23}$$

Moreover, we can prove that $\big(y_K - \overline{y}_K, \dot{\overline{y}}_K\big) = 0$. Indeed, assuming the contrary, we conclude that there exists $t_1 \in [t_0, T]$ such that

$$\big(y_K(t_1) - \overline{y}_K(t_1), \dot{\overline{y}}_K(t_1)\big) = \alpha \neq 0. \tag{24}$$

If $\alpha > 0$, then with $\Delta t > 0$ small enough, we obtain

$$\overline{y}_K(t_1 + \Delta t) - \overline{y}_K(t_1) = \dot{\overline{y}}_K(t_1) \cdot \Delta t + o(\Delta t).$$

From (24) it follows that

$$\left(y_K(t_1) - \overline{y}_K(t_1), \frac{\overline{y}_K(t_1 + \Delta t) - \overline{y}_K(t_1)}{\Delta t} \right) = \alpha + \left(y_K(t_1) - \overline{y}_K(t_1), \frac{o(\Delta t)}{\Delta t} \right). \tag{25}$$

When $\Delta t \to 0^+$, the right-hand side of (25) will go to $\alpha > 0$, while using inequality (15) the left-hand side is smaller than 0. We thus arrive at a contradiction.

If $\alpha < 0$, then with $\Delta t < 0$ small enough, we have

$$\overline{y}_K(t_1 + \Delta t) - \overline{y}_K(t_1) = -\dot{\overline{y}}_K(t_1)(-\Delta t) + o(\Delta t),$$

consquently,

$$\left(y_K(t_1) - \overline{y}_K(t_1), \frac{\overline{y}_K(t_1 + \Delta t) - \overline{y}_K(t_1)}{-\Delta t} \right) = -\alpha + \left(y_K(t_1) - \overline{y}_K(t_1), \frac{o(\Delta t)}{-\Delta t} \right).$$

Arguing as in case $\alpha > 0$, the contradiction is derived.

Therefore, from the definition of function $\psi(t)$ and (23) we have

$$\dot{\psi}(t) \leq 2C\sqrt{\psi(t)} - 2K\psi(t). \tag{26}$$

Setting $u(t) = \sqrt{\psi(t)}$, we get

$$\dot{u} \leq C - Ku. \tag{27}$$

From (27) we assert that $u(t) = \left\| y_K(t) - \overline{y}_K(t) \right\|$ satisfies

$$\dot{u} = -Ku + C - a(t), \tag{28}$$

where $a(t)$ a continuous, non-negative function. Also, since $u(t_0) = \sqrt{\psi(t_0)} = 0$, we have

$$u(t) = \int\limits_{t_0}^{t} e^{K(s-t)} \big[C - a(s)\big] ds.$$

For all $t \in [t_0, T]$, we have

$$\left\| y_K - \overline{y}_K \right\| = \|u\| \leq C \int\limits_{t_0}^{t} e^{K(s-t)} ds \leq \frac{C}{K},$$

which proves (22). Consequently, when K is sufficiently large, the solution $y_K(t)$ of (8) is inside of set Q_1 defined by

$$Q_1 = \left\{ z \in \mathbb{R}^n : \ \mathrm{dist}(z, Q) \leq 1 \right\}.$$

Then the solution $y_K(t)$ is bounded on $[t_0, T]$.

Moreover, from (4) and (22) it follows that

$$\left\|\dot{y}_K\right\| \leq 2C. \tag{29}$$

The important point is now that the conditions (22) and (29) show that the set of all solutions (y_K) of (8) is equicontinuous and bounded such that the Arzela compactness criterion implies that there is a subsequence of (y_K) (which for simplicity is denoted by $(y_K)_n$) converging uniformly on $[t_0, T]$ to some continuous x.

We are now going to prove that the limit function x takes its values in Q, and therefore it is a solution of the system (6) in $[t_0, T]$. Using (22) and the fact that

$$\|\overline{y}_K - x\| \leq \|\overline{y}_K - y_K\| + \|y_K - x\|,$$

the subsequence $(\overline{y}_K)$ converges uniformly on $[t_0, T]$ to the function x. From $\overline{y}_K = P_Q(y_K) \in Q$ it directly follows $x \in Q$.

Moreover, from (29) it follows that the solution $x(t)$ satisfies the Lipschitz condition, consequently, $x(t)$ is an absolutely continuous function in $[t_0, T]$. Therefore, $\dot{x}(t)$ exists for almost all $t \in [t_0, T]$ and we have

$$\int\limits_{t_0}^{t} \dot{x}(s)ds = x(t) - x_0, \quad t \in [t_0, T].$$

Next, it can be easily seen that $\dot{y}_K$ converges weakly to $\dot{x}$ for $K \to +\infty$. Indeed,

$$\int\limits_{t_0}^{t} \dot{y}_K(s)ds = y_K(t) - x_0 \longrightarrow x(t) - x_0 = \int\limits_{t_0}^{t} \dot{x}(s)ds \quad \text{for} \quad K \to +\infty.$$

Besides, by using (15), we have

$$y_K - \overline{y}_K \in N_Q(\overline{y}_K),$$

and, by using the properties $N_Q(\cdot)$ in Proposition 1, we get

$$(y_K - \overline{y}_K) \in N_Q(\overline{y}_K).$$

Hence,

$$\dot{y}_K \in f(t, \overline{y}_K) - N_Q(\overline{y}_K). \tag{30}$$

By virtue of Proposition 2 and passing in (30) to the limit as $K \to +\infty$, we can obtain that the function x satisfies (5).

Now, we are going to prove the uniqueness of solution of problem (6). Suppose that $x_1 = x_1(t)$ and $x_2 = x_2(t)$ are two solutions of the differential inclusion

$$\dot{x} \in f(t, x) - N_Q(x), \quad t \in [t_0, T].$$

The function $t \mapsto \mu(t) := \left\|x_1(t) - x_2(t)\right\|^2$ is absolutely continuous. At any point $t \in [t_0, t_1]$ of differentiability of μ we get

$$\dot{\mu}(t) = 2\big(x_1(t) - x_2(t),\ \dot{x}_1(t) - \dot{x}_2(t)\big),$$

consequently,

$$\dot{\mu}(t) = 2\big(x_1(t) - x_2(t),\ f(t, x_1(t)) - f(t, x_2(t))\big) - 2\big(x_1(t) - x_2(t),\ p_1 - p_2\big) \tag{31}$$

for suitable points $p_1 \in N_Q(x_1(t))$ and $p_2 \in N_Q(x_2(t))$. Moreover, we have

$$\big(x_1(t) - x_2(t), p_1 - p_2\big) = -\big(p_1, x_2(t) - x_1(t)\big) - \big(p_2, x_1(t) - x_2(t)\big) \geq 0.$$

Dropping the last term $\big(x_1(t) - x_2(t), p_1 - p_2\big)$ of (31) we get

$$\dot{\mu}(t) \leq 2\big\|x_1(t) - x_2(t)\big\|\ \big\|f(t, x_1(t)) - f(t, x_2(t))\big\|,$$

and using the Lipschitz condition (3) we obtain

$$\dot{\mu}(t) \leq 2L\mu(t).$$

Then there exists a continuous function $b(t) \leq 0$ such that $\mu(t)$ satisfies the differential equation

$$\dot{\mu}(t) = 2L\mu(t) + b(t).$$

So, we get

$$\mu(t) = \mu(t_0)e^{2L(t-t_0)} + e^{2Lt}\int\limits_{t_0}^{t} e^{-2Ls}b(s)\,ds \leq z(t_0)e^{2L(t-t_0)},$$

since $e^{at} > 0,\ t \geq t_0$, and $e^{-as}b(s) \leq 0$.

Hence $\mu(t_0) := \big\|x_1(t_0) - x_2(t_0)\big\|^2 = 0$, we obtain $x_1(t) \equiv x_2(t)$ for $t_0 \leq t \leq T$, which show the uniqueness of solution.

Now, we are going to prove the estimate (21). For this let us set

$$\rho(t) = \frac{1}{2}\big\|x(t) - y_K(t)\big\|^2, \quad t \in [t_0, T].$$

Since $x(t)$ is the solution of (6), there exists a vector $v(t) \in N_Q x(t)$ such that

$$\dot{x}(t) = f(t, x) - v(t), \quad t \in \big[t_0, T\big]. \tag{32}$$

Then we have

$$\dot{\rho}(t) = \big(\dot{x} - \dot{y}_K, x - y_K\big) = \Big(f(t, x) - f(t, \overline{y}_K) - v + K(y_K - \overline{y}_K), x - y_K\Big),$$

consequently,

$$\rho(t) = \big(f(t, x) - f(t, \overline{y}_K), x - y_K\big) + \big(v, y_K - \overline{y}_K\big) +$$

$$+\big(v, \overline{y}_K - x\big) + K\big(y_K - \overline{y}_K, x - \overline{y}_K\big) + K\big(y_K - \overline{y}_K, \overline{y}_K - y_K\big). \tag{33}$$

By virtue of (15) we get

$$\big(y_K - \overline{y}_K, x - \overline{y}_K\big) \le 0.$$

From (2) it follows that the function $\rho(t)$ can be estimated by

$$\rho(t) \le \big(f(t,x) - f(t,\overline{y}_K), x - y_K\big) + \big(v, y_K - \overline{y}_K\big).$$

Combining this with condition (3) gives

$$\rho(t) \le L\|x - y_K\|^2 + \|v\|\|y_K - \overline{y}_K\| \tag{34}$$

(here we can use the inequality $\|x - \overline{y}_K\| \le \|x - y_K\|$, because$\|x - y_K\|$ is the length of a side of the triangle, which is opposite to right angle or obtuse angle of the triangle).

In addition, since y_K is the solution of equation (8), if $y_K \in Q$, then $\rho(t)$ is estimated as follows:

$$\rho(t) \le L\|x - y_K\|^2 \tag{35}$$

(because $\overline{y}_K = y_K$).

We now consider the value of $\rho(t)$ in the case $y_K \notin Q$.

As the next step, to evaluate the function $\rho(t)$ we evaluate $\|v\|$ in the right-hand side of (34). We first show that

$$\dot{x} \in T_Q x(t) \qquad \text{and} \qquad (\dot{x}, v) = 0 \quad \text{for almost all} \quad t. \tag{36}$$

Let t be a point in $[t_0, T]$ where $\dot{x}(t)$ exists. Given any $\varepsilon > 0$ small enough and $0 \le s < \varepsilon$, we have

$$x(t+s) \in Q \Rightarrow \dot{x}_+(t) \in T_Q x(t), \tag{37}$$

$$x(t-s) \in Q \Rightarrow -\dot{x}_-(t) \in T_Q x(t). \tag{38}$$

Indeed, since

$$\dot{x}_+(t) = \lim_{\delta \to 0^+} \frac{x(t+\delta) - x(t)}{\delta}, \tag{39}$$

by (2) and for every $\xi \in N_Q x(t)$ we obtain

$$\big(x(t+\delta) - x(t), \xi\big) \le 0. \tag{40}$$

Then, from (39), (40) and (9) it follows condition (37) and also $\dot{x}(t) \in T_Q x(t)$. Similarly, (38) is also proved. Combining (37), (38) and Definition 2 of adjoint cone, we obtain: on the one hand $(\dot{x}, v) = \big(\dot{x}_+(t), v\big) \le 0$, on the other hand $(-\dot{x}, v) = -\big(\dot{x}_-(t), v\big) \le 0$, and, consquently, $(\dot{x}, v) = 0$, that is, (36) is proved.

By using (32), (36) and Lemma 1, we obtain that the vector $\dot{x}$ is the projections of $f(t,x)$ on the set $T_Q x(t)$. Since $v(t) = f(t,x) - \dot{x}(t)$ (see (32)), $0 \in T_Q(x(t)$ and by virtue of (7), we get

$$\|v\| = \|f - \dot{x}\| \le \|f(t,x)\|.$$

From this fact and the condition (4) it follows

$$\|v\| \le C. \tag{41}$$

Therefore, from (34), (22) and (41) it follows that

$$\rho(t) \leq L\|x - y_K\|^2 + \frac{C^2}{K}. \tag{42}$$

From (35) and (42) we have (42) for two case: $y \in Q$ and $y \notin Q$.

Since (42), the absolutely continuous function $\gamma(t) := \left\|x(t) - y_K(t)\right\|^2$ satisfies the equation

$$\dot{\gamma}(t) = 2L\gamma(t) + 2\frac{C^2}{K} - p(t),$$

where $p(t)$ is a non-negative continuous function. From the initial condition in (6) and (8) we have $\gamma(t_0) = 0$. Then we obtain

$$\gamma(t) = 2\int_{t_0}^{t} e^{2L(t-s)}\left[\frac{C^2}{K} - \frac{p(s)}{2}\right]ds.$$

Finally, with $t \in [t_0, T]$ we get

$$\left\|x(t) - y_K(t)\right\|^2 \leq \frac{2C^2}{K}\int_{t_0}^{t} e^{2L(t-s)}ds \leq \frac{C^2}{KL}e^{2L(T-t_0)},$$

which leads to the estimate (21).

Theorem 1 is proved.

5. Conclusions. Some of technical problems have been mathematically modeled by the differential inclusion (6). Nevertheless, finding exact solutions to the differential inclusion (6) is very difficult. Therefore, studying the ordinary differential equation in form (8) has high scientific interest. Firstly, we can use the ordinary differential equation (8) to obtain the approximated solutions of the differential inclusion. Secondly, we can find and present the solutions of (8) by using such software as Mathematica, Matlab, and Maple.

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Received 27.01.18,
after revision — 20.08.19