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A. Beiranvand, A. G. Ghazanfari (Lorestan Univ., Khoramabad, Iran)

IMPROVED YOUNG AND HEINZ OPERATOR INEQUALITIES WITH KANTOROVICH CONSTANT

ВДОСКОНАЛЕНІ ОПЕРАТОРНІ НЕРІВНОСТІ ЯНГА І ХАЙНЦА З КОНСТАНТОЮ КАНТОРОВИЧА

We present numerous refinements of the Young inequality by the Kantorovich constant. We use these improved inequalities to establish corresponding operator inequalities on a Hilbert space and some new inequalities involving the Hilbert – Schmidt norm of matrices.

Отримано ряд покращень нерівності Янга за допомогою константи Канторовича. Ці покращені нерівності використовуються для встановлення відповідних операторних нерівностей у просторі Гільберта та деяких нових нерівностей, що включають норми Гільберта – Шмідта для матриць.

1. Introduction and preliminaries. Let $M_{m,n}(\mathbb{C})$ be the space of $m \times n$ complex matrices and $M_n(\mathbb{C}) = M_{n,n}(\mathbb{C})$. Let $\|.\|$ denote any unitarily invariant norm on $M_n(\mathbb{C})$. So, $\|UAV\| = \|A\|$ for all $A \in M_n(\mathbb{C})$ and for all unitary matrices $U, V \in M_n(\mathbb{C})$. The Hilbert – Schmidt and trace class norm of $A = [a_{ij}] \in M_n(\mathbb{C})$ are denoted by

$$\|A\|_2 = \left(\sum_{j=1}^{n} s_j^2(A)\right)^{\frac{1}{2}}, \qquad \|A\|_1 = \sum_{j=1}^{n} s_j(A),$$

where $s_1(A) \geq s_2(A) \geq \ldots \geq s_n(A)$ are the singular values of A, which are the eigenvalues of the positive semidefinite matrix $\mid A \mid = (A^*A)^{\frac{1}{2}}$, arranged in decreasing order and repeated according to multiplicity. For Hermitian matrices $A, B \in M_n(\mathbb{C})$, we write that $A \geq 0$ if A is positive semidefinite, $A > 0$ if A is positive definite, and $A \geq B$ if $A - B \geq 0$.

Let $a, b \geq 0$ and $0 \leq \nu \leq 1$. Young's inequality for real numbers states that

$$a^\nu b^{1-\nu} \leq \nu a + (1-\nu) b \tag{1.1}$$

with equality if and only if $a = b$. This inequality has numerous applications in various fields. Young's inequality and its reverse have received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found (see, for example, [1, 2, 8, 9, 15, 17]).

Zhao and Wu in [15] obtained refinements of the Young inequality and its reverses in the following forms:

if $0 < \nu \leq 1/2$, then

$$\begin{aligned} \nu a + (1-\nu) b &\geq a^\nu b^{1-\nu} + \nu\left(\sqrt{a} - \sqrt{b}\right)^2 + r_1\left(\sqrt[4]{ab} - \sqrt{b}\right)^2, \\ \nu a + (1-\nu) b &\leq a^\nu b^{1-\nu} + (1-\nu)\left(\sqrt{a} - \sqrt{b}\right)^2 - r_1\left(\sqrt[4]{ab} - \sqrt{a}\,\right)^2; \end{aligned} \tag{1.2}$$

if $1/2 < \nu < 1$, then

$$\begin{aligned} \nu a+(1-\nu)b &\geq a^{\nu}b^{1-\nu}+(1-\nu)\big(\sqrt{a}-\sqrt{b}\big)^2+r_1\big(\sqrt[4]{ab}-\sqrt{b}\big)^2, \\ \nu a+(1-\nu)b &\leq a^{\nu}b^{1-\nu}+\nu\big(\sqrt{a}-\sqrt{b}\big)^2-r_1\big(\sqrt[4]{ab}-\sqrt{a}\,\big)^2, \end{aligned} \tag{1.3}$$

where $r=\min\{\nu,1-\nu\}$ and $r_1=\min\{2r,1-2r\}$.

A multiple-term refinement of Young's inequality presented in [12] as follows:

$$a^{\nu}b^{1-\nu}+S_N(\nu;a,b)\leq \nu a+(1-\nu)b, \tag{1.4}$$

where $S_N(N;a,b)$ is the following nonnegative function:

$$\sum_{j=1}^{N} s_j(\nu)\Big(\sqrt[2^j]{b^{2^{j-1}-k_j(\nu)}a^{k_j(\nu)}}-\sqrt[2^j]{a^{k_j(\nu)+1}b^{2^{j-1}-k_j(\nu)-1}}\Big)^2.$$

The Kantorovich constant is defined as

$$K(t,2)=\frac{(t+1)^2}{4t}\quad \text{for}\quad t>0.$$

Zuo et al. in [17] improved the classical Young's inequality (1.1) via the Kantorovich constant as follows:

$$a\nabla_{\nu}b=\nu a+(1-\nu)b\geq K(h,2)^{r}a^{\nu}b^{1-\nu} \tag{1.5}$$

for all $\nu\in[0,1]$, where $r=\min\{\nu,1-\nu\}$ and $h=\dfrac{b}{a}$.

Liao and Wu in [9] gave refinements of inequalities (1.2) and (1.3) with the Kantorovich constant:

if $0<\nu\leq 1/2$, then

$$\nu a+(1-\nu)b\geq \nu\big(\sqrt{a}-\sqrt{b}\big)^2+r_1\big(\sqrt[4]{ab}-\sqrt{b}\big)^2+K\big(\sqrt[4]{h},2\big)^{\hat{r}_1}a^{\nu}b^{1-\nu}; \tag{1.6}$$

if $1/2<v<1$, then

$$\nu a+(1-\nu)b\geq (1-\nu)\big(\sqrt{a}-\sqrt{b}\big)^2+r_1\big(\sqrt[4]{ab}-\sqrt{b}\big)^2+K\big(\sqrt[4]{h},2\big)^{\hat{r}_1}a^{\nu}b^{1-\nu}, \tag{1.7}$$

where $h=\dfrac{b}{a}$, $r=\min\{\nu,1-\nu\}$, $r_1=\min\{2r,1-2r\}$ and $\hat{r}_1=\min\{2r_1,1-2r_1\}$. Using the Kantorovich constant a refinement of (1.4) given in [13] as follows:

$$K\left(\sqrt[2^N]{\frac{b}{a}},2\right)^{\beta_N(\nu)}a^{\nu}b^{1-\nu}+S_N(\nu;a,b)\leq \nu a+(1-\nu)b, \tag{1.8}$$

where $\beta_N(\nu)$ is a special function which defined therein.

The Heinz means are defined as follows:

$$H_{\nu}(a,b)=\frac{a^{\nu}b^{1-\nu}+a^{1-\nu}b^{\nu}}{2}.$$

It is easy to show that the Heinz means interpolate between the geometric mean and the arithmetic mean:

$$\sqrt{ab} \le H_\nu(a,b) \le \frac{a+b}{2}. \tag{1.9}$$

The second inequality of (1.9) is know as the Heinz inequality for nonnegative real numbers.

Let $A, B \in B(H)$ be two positive operators, $\nu \in [0,1]$. The ν-weighted arithmetic mean of A and B, denoted by $A\nabla_\nu B$, is defined as $A\nabla_\nu B = (1-\nu)A + \nu B$. If A is invertible, the ν-geometric mean of A and B, denoted by $A\sharp_\nu B$, is defined as

$$A\sharp_\nu B = A^{\frac{1}{2}}(A^{\frac{-1}{2}} B A^{\frac{-1}{2}})^\nu A^{\frac{1}{2}}.$$

The Heinz operator mean is defined by

$$H_\nu(A,B) = \frac{A\sharp_\nu B + A\sharp_{1-\nu} B}{2},$$

where A and B are two invertible positive operators in $B(H)$.

Zuo et al. in [17] show that the inequality (1.5) admits an operator extension

$$A\nabla_\nu B \ge K(h,2)^r A\sharp_\nu B$$

for positive operators A, B on a Hilbert space.

Bakherad et al. in [1] proved that if $\nu \ge 0$ or $\nu \le -1$, then

$$A\nabla_{-\nu} B \le A\sharp_{-\nu} B.$$

In [9], the authors have presented operator versions of inequalities (1.6) and (1.7) on a Hilbert space and corresponding inequalities with the Hilbert – Schmidt norm.

Let A be positive operator acting on a Hilbert space H and $\lambda \in [0,1]$. Hölder – McCarthy inequality states that

$$\langle Ax, x\rangle^\lambda \ge \langle A^\lambda x, x\rangle \quad \text{for all unit vectors} \quad x \in H.$$

It is known that the Hölder – McCarthy inequality and the Young inequality are equivalent, e.g., [5] (§ 3.1.3).

Fujii and Nakamoto in [4] proved that for $A \ge 0$ and $0 \le \mu,\ \nu \le 1$ the following refinement of the Young inequality:

$$\mu A + 1 - \mu - A^\mu \ge \min\left\{\frac{1-\mu}{1-\nu}, \frac{\mu}{\nu}\right\}(\nu A + 1 - \nu - A^\nu)$$

is also equivalent to the following refinement of the Hölder – McCarthy inequality

$$1 - \frac{\langle A^\mu x, x\rangle}{\langle Ax, x\rangle^\mu} \ge \min\left\{\frac{1-\mu}{1-\nu}, \frac{\mu}{\nu}\right\}\left(1 - \frac{\langle A^\nu x, x\rangle}{\langle Ax, x\rangle^\nu}\right) \quad \text{for unit vector} \quad x.$$

For more information on the equivalent between Hölder – McCarthy inequality and Young inequality, the reader is referred to [4] and the references therein.

Sababheh and Moslehian in [13] obtained several multiterm refinements of Young type inequalities for both real numbers and operators. They also proved the following operator inequality:

$$K\left(\sqrt[2^N]{\frac{M}{m}}, 2\right)^{\beta_N(\nu)} A\sharp_\nu B +$$

$$+ \sum_{j=1}^{N} s_j(\nu)\big(A\sharp_{\alpha_j(\nu)} B + A\sharp_{2^{1-j}\alpha_j(\nu)} B - 2A\sharp_{2^{-j}\alpha_j(\nu)} B\big) \le A\nabla_\nu B$$

for the positive operators $mI \le A,\ B \le MI$, where $s_j(\nu)$ and $\alpha_j(\nu)$ are special functions which have been defined therein.

In this paper, we present numerous refinements of the Young inequality by the Kantorovich constant that improve several known results. We use these improved inequalities to obtain corresponding operator inequalities on a Hilbert space. Moreover, some new Young type inequalities involving the Hilbert – Schmidt norm are established.

2. Main results. ***2.1. Several refinements of the Young inequality.*** First of all, we state a refinement of the weighted arithmetic-geometric mean inequality for n positive numbers, which was shown by Pečarić et al., see [10, p. 717] (Theorem 1) and [3].

Lemma 1. *Let $x_1, \ldots, x_n$ belong to a closed interval $I = [a, b]$, $a < b$, $p_1, \ldots, p_n \ge 0$ with $\sum_{i=1}^n p_i = 1$ and $\lambda = \min\{p_1, \ldots, p_n\}$. If f is a convex function on I, then*

$$\sum_{i=1}^n p_i f(x_i) - f\left(\sum_{i=1}^n p_i x_i\right) \ge n\lambda \sum_{i=1}^n \frac{1}{n} f(x_i) - f\left(\frac{1}{n}\sum_{i=1}^n x_i\right).$$

Before exposing the main results, we state the following corollary from Lemma 1, which we will use in what follows.

***Corollary* 1.** *If $x_i \in [a, b]$, $0 < a < b$, $p_1, \ldots, p_n \ge 0$ with $\sum_{i=1}^n p_i = 1$ and $\lambda = \min\{p_1, \ldots, p_n\}$, then*

$$\frac{\sum_{i=1}^n p_i x_i}{\prod_{i=1}^n x_i^{p_i}} \ge \left(\frac{\frac{1}{n}\sum_{i=1}^n x_i}{\prod_{i=1}^n x_i^{\frac{1}{n}}}\right)^{n\lambda}.$$

Suppose that f is a real convex (concave) function on $[0, 1]$ and $n \in \mathbb{N} \cup \{0\}$.

Let $A_{0,0} = [0, 1]$, $A_{n,i} = [2^{-n}i, 2^{-n}(i+1))$ for $n = 1, 2, 3, \ldots$, $i = 0, 1, \ldots, 2^n - 1$, and

$$f_n(\nu) = \sum_{i=0}^{2^n-1} \left[(i + 1 - 2^n\nu) f(2^{-n}i) + (2^n\nu - i) f(2^{-n}(i+1))\right]\chi_{A_{n,i}}(\nu). \tag{2.1}$$

It can be easily shown that f_n is continuous on $[0, 1]$ for every $n \in \mathbb{N}$, and $\{f_n\}$ is a decreasing (increasing) sequence that converges pointwise to f. An example of such functions f_n for $n = 2$ given in [16] (Theorem 2.1). Recently the authors in [14] extended this result for any integer $n \ge 2$.

Theorem 1. *Suppose that $0 \le \nu \le 1$, $a, b \ge 0$, with the assumption* (2.1) *for convex function $f(\nu) = a^\nu b^{1-\nu}$. Then*

$$a^\nu b^{1-\nu} \le K\left(\sqrt[2^n]{\frac{b}{a}}, 2\right)^{\lambda_n} a^\nu b^{1-\nu} \le f_n(\nu) \le \nu a + (1-\nu) b, \tag{2.2}$$

where $\lambda_n = \sum_{i=0}^{2^n-1} \min\{i + 1 - 2^n\nu, 2^n\nu - i\}\chi_{A_{n,i}}(\nu)$.

Proof. If $\nu \in [2^{-n}i, 2^{-n}(i+1))$, by substitution $x_1 = f(2^{-n}i)$, $x_2 = f(2^{-n}(i+1))$, $p_1 = i + 1 - 2^n\nu$ and $p_2 = 2^n\nu - i$ in Corollary 1, we get

$$f(2^{-n}i)^{i+1-2^n\nu} f(2^{-n}(i+1))^{2^n\nu - i} \left(\frac{\frac{1}{2}\left(f(2^{-n}i) + f(2^{-n}(i+1))\right)}{f(2^{-n}i)^{\frac{1}{2}} f(2^{-n}(i+1))^{\frac{1}{2}}}\right)^{2\lambda_n} \le$$

$$\leq (i+1-2^n\nu)f(2^{-n}i) + (2^n\nu - i)f(2^{-n}(i+1)).$$

This implies that

$$a^\nu b^{1-\nu}\left(\frac{\left(\left(\dfrac{b}{a}\right)^{2^{-n}}+1\right)^2}{4\left(\dfrac{b}{a}\right)^{2^{-n}}}\right)^{\lambda_n} \leq (i+1-2^n\nu)f(2^{-n}i) + (2^n\nu - i)f(2^{-n}(i+1))$$

or

$$K(h,2)^{\lambda_n}a^\nu b^{1-\nu} \leq (i+1-2^n\nu)f(2^{-n}i) + (2^n\nu - i)f(2^{-n}(i+1)). \tag{2.3}$$

Using Young's inequality (1.1), we obtain

$$(i+1-2^n\nu)f(2^{-n}i) + (2^n\nu - i)f(2^{-n}(i+1)) \leq$$

$$\leq (i+1-2^n\nu)\big(2^{-n}ia + (1-2^{-n}i)b\big) +$$

$$+(2^n\nu - i)\big(2^{-n}(i+1)a + (1-2^{-n}(i+1))b\big) = a\nabla_\nu b. \tag{2.4}$$

From inequalities (2.3) and (2.4), we deduce the desired inequalities (2.2).

Theorem 1 is proved.

***Remark* 1.** Let $0 \leq \nu \leq 1$, $n \in \mathbb{N}$ and $r_0 = \min\{\nu, 1-\nu\}$. For $n = 1, 2, \ldots$ and $i = 0, 1, \ldots, 2^{n-1}-1$ let

$$E_{n,i} = \big[2^{-n}i, 2^{-n}(i+1)\big) \cup \big(1-2^{-n}(i+1), 1-2^{-n}i\big].$$

If $a, b \geq 0$, since $H_\nu(a,b)$ is symmetric with respect to $\nu = 1/2$, that is, $H_\nu(a,b) = H_{1-\nu}(a,b)$, $\nu \in [0,1]$, then the following inequalities hold:

$$K\left(\sqrt[2^n]{\frac{b}{a}}, 2\right)^{\lambda_n} H_\nu(a,b) \leq \sum_{i=0}^{2^{n-1}-1}\big[(i+1-2^n r_0)H_{2^{-n}i}(a,b) +$$

$$+(2^n r_0 - i)H_{2^{-n}(i+1)}(a,b)\big]\chi_{E_{n,i}}(\nu) \leq$$

$$\leq \frac{a+b}{2}, \tag{2.5}$$

where $\lambda_n = \sum_{i=0}^{2^{n-1}-1}\min\{i+1-2^n r_0, 2^n r_0 - i\}\chi_{E_{n,i}}(\nu)$. Clearly, the inequalities (2.5) are refinements of inequalities (2.2) in [14].

By the same argument used in the proof of Theorem 1, we give new inequalities as to the Young inequality in the following theorem.

Theorem 2. *Suppose that* $0 \leq \nu \leq 1$, $a, b \geq 0$ *with the assumption* (2.1) *for concave function* $g(\nu) = \sqrt{\nu a + (1-\nu)b}$. *Then the following inequalities hold*:

$$a^\nu b^{1-\nu} \leq g_n^2(\nu) = \sum_{i=0}^{2^n-1}\Big[(i+1-2^n\nu)\sqrt{a\nabla_{2^{-n}i}b} +$$

$$+(2^n\nu - i)\sqrt{a\nabla_{2^{-n}(i+1)}b}\,\Big]^2\chi_{A_{n,i}}(\nu) \leq \nu a + (1-\nu)b. \tag{2.6}$$

Proof. Similarly as in Theorem 1, for concave function $g(\nu) = \sqrt{\nu a + (1-\nu)b}$ there exists a sequence

$$g_n(\nu) = \sum_{i=0}^{2^n-1} \left[(i+1-2^n\nu)\sqrt{a\nabla_{2^{-n}i}b} + (2^n\nu - i)\sqrt{a\nabla_{2^{-n}(i+1)}b}\,\right]\chi_{A_{n,i}}(\nu),$$

such that $g_n(\nu) \leq \sqrt{\nu a + (1-\nu)b}$. Using Young's inequality (1.1), we obtain

$$(i+1-2^n\nu)g(2^{-n}i) + (2^n\nu - i)g(2^{-n}(i+1)) \geq$$

$$\geq (i+1-2^n\nu)\big(a^{2^{-n-1}i}b^{\frac{1}{2}-2^{-n-1}i}\big) + (2^n\nu - i)\big(a^{2^{-n-1}(i+1)}b^{\frac{1}{2}-2^{-n-1}(i+1)}\big) \geq$$

$$\geq \sqrt{a^\nu b^{1-\nu}}.$$

Theorem 2 is proved.

Furthermore, converging $g_n(\nu)$ to $\sqrt{\nu a + (1-\nu)b}$ and inequality (1.8), imply that, for any $N \in \mathbb{N}$, there exists a positive integer n_1 such that, for every $n > n_1$,

$$K\left(\sqrt[2^N]{\frac{b}{a}}, 2\right)^{\beta_N(\nu)} a^\nu b^{1-\nu} + S_N(\nu; a, b) \leq g_n^2(\nu) \leq \nu a + (1-\nu)b. \tag{2.7}$$

Hence (2.7) is a refinement of (1.8). Moreover, a benefit of (2.6) is that it has an explicit formula which doesn't depend on certain functions.

2.2. Some matrix versions of Young and Heinz inequalities. Let $A, B, X \in M_n(\mathbb{C})$ be such that A and B are positive semidefinite and $0 \leq \nu \leq 1$. Hirzallah and Kittaneh in [6] proved that

$$\big\|A^\nu X B^{1-\nu}\big\|_2^2 + r_0^2\big\|AX - XB\big\|_2^2 \leq \big\|\nu AX + (1-\nu)\big\|_2^2,$$

where $r_0 = \min\{\nu, 1-\nu\}$.

Zou and Jiang [16] obtained refinements of the Heinz inequality for matrices in the following forms:

Theorem 3. *Let $A, B, X \in M_n(\mathbb{C})$ such that A and B are positive semidefinite and suppose that*

$$\phi(\nu) = \big\|A^\nu X B^{1-\nu} + A^{1-\nu} X B^\nu\big\|_2$$

for $\nu \in [0,1]$. Then

$$\phi(\nu) \leq \begin{cases} (1-4r_0)\phi(0) + 4r_0\phi\left(\frac{1}{4}\right), & \nu \in \left[0, \frac{1}{4}\right] \cup \left[\frac{3}{4}, 1\right], \\ (4r_0-1)\phi\left(\frac{1}{2}\right) + 2(1-2r_0)\phi\left(\frac{1}{4}\right), & \nu \in \left[\frac{1}{4}, \frac{3}{4}\right], \end{cases}$$

where $r_0 = \min\{\nu, 1-\nu\}$.

Krnić in [8] proved that

$$\big\|A^\nu X B^{1-\nu} + A^{1-\nu} X B^\nu\big\|_2^2 + 4\nu(1-\nu)\|AX - XB\|_2^2 \leq \|AX + XB\|_2^2. \tag{2.8}$$

In the following theorem we give some refinements of the Young inequality for the Hilbert–Schmidt norm based on the inequality (2.2).

Theorem 4. *Let* $n \in \mathbb{N} \cup \{0\}$ *and* $A, B, X \in M_m(\mathbb{C})$ *such that* A *and* B *are positive definite. Let* $\operatorname{Sp}(A) = \{\lambda_1, \ldots, \lambda_m\}$ *be the spectrum of* A, $\operatorname{Sp}(B) = \{\mu_1, \ldots, \mu_m\}$ *and let*

$$K_n = \min\left\{K\left(\left(\frac{\lambda_j}{\mu_k}\right)^{2^{-n}}, 2\right) : k, j = 1, \ldots, m\right\}.$$

Then the following inequalities hold:

$$K_n^{\gamma_n} \|A^\nu X B^{1-\nu}\|_2 \leq \sum_{i=0}^{2^n-1} \Big\|((i+1-2^n\nu) A^{2^{-n}i} X B^{1-2^{-n}i} +$$

$$+(-i+2^n\nu) A^{2^{-n}(i+1)} X B^{1-2^{-n}(i+1)}\Big\|_2 \chi_{A_{n,i}} \leq \big\|\nu AX + (1-\nu)XB\big\|_2,$$

where $\gamma_n = \sum_{i=0}^{2^n-1} \min\{i+1-2^n\nu, 2^n\nu - i\}\chi_{A_{n,i}}$.

Proof. Since A and B are positive definite, it follows by the spectral theorem that there exist unitary matrices $U, V \in M_m(\mathbb{C})$ such that

$$A = U\Lambda_1 U^* \qquad \text{and} \qquad B = V\Lambda_2 V^*,$$

where

$$\Lambda_1 = \operatorname{diag}(\lambda_1, \ldots, \lambda_m), \qquad \Lambda_2 = \operatorname{diag}(\mu_1, \ldots, \mu_m), \qquad \lambda_k, \mu_k > 0, \quad k, j = 1, \ldots.$$

Let

$$Y = U^* X V = [y_{kj}].$$

We have

$$A^\nu X B^{1-\nu} = (U\Lambda_1 U^*)^\nu X (V\Lambda_2 V^*)^{1-\nu} = U\Lambda_1^\nu Y \Lambda_2^{1-\nu} V^*.$$

Therefore,

$$K_n^{2\gamma_n} \|A^\nu X B^{1-\nu}\|_2^2 = K_n^{2\gamma_n} \|\Lambda_1^\nu Y \Lambda_2^{1-\nu}\|_2^2 = K_0^{2\gamma} \sum_{k,j=1}^{m} \left(\lambda_j^\nu \mu_k^{1-\nu}\right)^2 |y_{kj}|^2,$$

$$\nu AX + (1-\nu)XB = U(\nu\Lambda_1 Y + (1-\nu)Y\Lambda_2)V^*,$$

and

$$\sum_{i=0}^{2^n-1} \Big\|((i+1-2^n\nu) A^{2^{-n}i} X B^{1-2^{-n}i} + (-i+2^n\nu) A^{2^{-n}(i+1)} X B^{1-2^{-n}(i+1)}\Big\|_2^2 \chi_{A_{n,i}} =$$

$$= \sum_{i=0}^{2^n-1} \sum_{k,j=1}^{m} \left((i+1-2^n\nu)\lambda_j^{2^{-n}i} \mu_k^{1-2^{-n}i} + (-i+2^n\nu)\lambda_j^{2^{-n}(i+1)} \mu_k^{1-2^{-n}(i+1)}\right)^2 |y_{kj}|^2 \chi_{A_{n,i}}.$$

Now, from inequalities (2.2), we deduce

$$K_n^{2\gamma_n}\|A^\nu XB^{1-\nu}\|_2^2 = K_n^{2\gamma_n}\sum_{k,j=1}^{m}\left(\lambda_j^\nu\mu_k^{1-\nu}\right)^2|y_{kj}|^2 =$$

$$= \sum_{i=0}^{2^n-1}\sum_{k,j=1}^{m}\left((i+1-2^n\nu)\lambda_j^{2^{-n}i}\mu_k^{1-2^{-n}i} + (-i+2^n\nu)\lambda_j^{2^{-n}(i+1)}\mu_k^{1-2^{-n}(i+1)}\right)^2|y_{kj}|^2\chi_{A_{n,i}} \leq$$

$$\leq \sum_{k,j=1}^{m}\left(\nu\lambda_j + (1-\nu)\mu_k\right)^2|y_{kj}|^2 = \|\nu AX + (1-\nu)XB\|_2^2.$$

Theorem 4 is proved.

Since for every unitarily invariant norm $|||\cdot|||$, the function $f(\nu) = \left|\left|\left|A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu\right|\right|\right|$ is convex, using the same strategy as in the proof of Theorem 4, we can present new refinements of matrix versions of the Heinz inequality.

Corollary 2. *Let $A, B, X \in M_m(\mathbb{C})$ such that A and B are positive definite. Let* $\mathrm{Sp}(A) = \{\lambda_1, \ldots, \lambda_m\}$ *be the spectrum of A,* $\mathrm{Sp}(B) = \{\mu_1, \ldots, \mu_m\}$ *be the spectrum of B and let*

$$K_n = \min\left\{K\left(\left(\frac{\lambda_j}{\mu_k}\right)^{2^{-n}}, 2\right) : k, j = 1, \ldots, m\right\}.$$

Then the following inequalities hold:

$$K_n^{\gamma_n}\left|\left|\left|A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu\right|\right|\right| \leq$$

$$\leq \sum_{i=0}^{2^{n-1}-1}\left((i+1-2^n r_0)\left|\left|\left|A^{2^{-n}i}XB^{1-2^{-n}i} + A^{1-2^{-n}i}XB^{2^{-n}i}\right|\right|\right| +\right.$$

$$\left.+(2^n r_0 - i)\left|\left|\left|A^{2^{-n}(i+1)}XB^{1-2^{-n}(i+1)} + A^{1-2^{-n}(i+1)}XB^{2^{-n}(i+1)}\right|\right|\right|\right)\chi_{E_{n,i}} \leq$$

$$\leq |||AX + XB|||, \tag{2.9}$$

where $\gamma_n = \sum_{i=0}^{2^{n-1}-1}\min\{i+1-2^n r_0, 2^n r_0 - i\}\chi_{E_{n,i}}$.

Clearly, inequalities (2.9) are refinements of inequality (2.10) in [7].

2.3. Some operator versions of Young and Heinz inequalities. In this section, we give an operator version of the inequalities (2.2). To reach inequalities for bounded self-adjoint operators on Hilbert space, we shall use the following monotonicity property for operator functions:

if $X \in B_h(H)$ with a spectrum $\mathrm{Sp}(X)$ and f, g are continuous real-valued functions on $\mathrm{Sp}(X)$, then

$$f(t) \geq g(t), \quad t \in \mathrm{Sp}(X) \Rightarrow f(X) \geq g(X). \tag{2.10}$$

For more details about this property, the reader is referred to [11].

Theorem 5. *Let $n \in \mathbb{N} \cup \{0\}$ and $0 \leq \nu \leq 1$. If A, B are two invertible positive operators in $B(H)$ and h a positive real number such that either $A < hA \leq B$ or $A > hA \geq B$, then*

$$\left(K(h^{2^{-n}}, 2)\right)^{\lambda_n} A\sharp_\nu B \leq \sum_{i=0}^{2^n-1}\left[(i+1-2^n\nu)A\sharp_{2^{-n}i}B + (2^n\nu - i)A\sharp_{2^{-n}(i+1)}B\right]\chi_{A_{n,i}} \leq A\nabla_\nu B, \tag{2.11}$$

where $\lambda_n = \sum_{i=0}^{2^n-1} \min\{i+1-2^n\nu, 2^n\nu - i\}\chi_{An,i}$.

Proof. For $\nu \in [2^{-n}i, 2^{-n}(i+1))$ and $A < hA \le B$, then it is clear that $I < hI \le A^{\frac{-1}{2}} B A^{\frac{-1}{2}}$ and $1 < h$. Since the function $K(t,2)$ is continuous and monotone increasing on, therefore, for all real numbers t such that $1 < h \le t$, we have

$$\big(K(h^{2^{-n}},2)\big)^{\lambda_n} \le \big(K(t^{2^{-n}},2)\big)^{\lambda_n}. \tag{2.12}$$

Inequalities (2.2), for $b = 1$, become

$$\big(K(a^{2^{-n}},2)\big)^{\lambda_n} a^{\nu} \le (i+1-2^n\nu) a^{2^{-n}i} + (2^n\nu - i) a^{2^{-n}(i+1)} \le$$
$$\le \nu a + (1-\nu). \tag{2.13}$$

According to (2.10), we can insert $X = A^{\frac{-1}{2}} B A^{\frac{-1}{2}}$ in (2.12) to get

$$\big(K(h^{2^{-n}},2)\big)^{\lambda_n} I \le \big(K(X^{2^{-n}},2)\big)^{\lambda_n}. \tag{2.14}$$

Multiplying both sides of inequality (2.14) by $X^{\frac{\nu}{2}}$ on the left and right, we have

$$\big(K(h^{2^{-n}},2)\big)^{\lambda_n} X^{\nu} \le X^{\nu/2}\big(K(X^{2^{-n}},2)\big)^{\lambda_n} X^{\nu/2} = \big(K(X^{2^{-n}},2)\big)^{\lambda_n} X^{\nu}. \tag{2.15}$$

We also can insert $X = A^{\frac{-1}{2}} B A^{\frac{-1}{2}}$ in (2.13) to deduce

$$\big(K(X^{2^{-n}},2)\big)^{\lambda_n} X^{\nu} \le (i+1-2^n\nu) X^{2^{-n}i} + (2^n\nu - i) X^{2^{-n}(i+1)} \le$$
$$\le \nu X + (1-\nu) I. \tag{2.16}$$

From inequalities (2.15), (2.16), we obtain

$$\big(K(h^{2^{-n}},2)\big)^{\lambda_n} X^{\nu} \le (i+1-2^n\nu) X^{2^{-n}i} + (2^n\nu - i) X^{2^{-n}(i+1)} \le$$
$$\le \nu X + (1-\nu) I. \tag{2.17}$$

Finally, if we multiply inequalities (2.17) by $A^{\frac{1}{2}}$ on the left- and right-sides, we get the desired inequalities (2.11).

Theorem 5 is proved.

The assumptions of Theorem 5 are weaker than the assumptions of Theorem 7 in [17]. Because if m, m', M, M' are positive real numbers such that $0 < m'I \le A \le mI < MI \le B \le M'I$, then $A < \dfrac{M}{m} A \le B$.

By the same method used in the proof of Theorem 5, we give new refinements of operator versions of the Heinz inequality.

***Corollary* 3.** *Let* A, B *are two invertible positive operators in* $B(H)$ *and* h *a positive real number such that either* $A < hA \le B$ *or* $A > hA \ge B$. *Then*

$$\big(K(h^{2^{-n}},2)\big)^{\lambda_n} (A\sharp_{\nu} B + A\sharp_{1-\nu} B) \le$$
$$\le \sum_{i=0}^{2^{n-1}-1} \Big[(i+1-2^n r_0)(A\sharp_{2^{-n}i} B + A\sharp_{1-2^{-n}i} B) +$$

$$+(2^n r_0 - i)(A\sharp_{2^{-n}(i+1)}B + A\sharp_{1-2^{-n}(i+1)}B)\Big]\chi_{E_{n,i}} \leq$$

$$\leq \frac{A+B}{2},$$

where $\lambda_n = \sum_{i=0}^{2^{n-1}-1} \min\{i+1-2^n r_0, 2^n r_0 - i\}\chi_{En,i}$.

References

1. M. Bakherad, M. Krnić, M. S. Moslehian, *Reverse Young-type inequalities for matrices and operators*, Rocky Mountain J. Math., **46**, № 4, 1089 – 1105 (2016).
2. C. Conde, *Young type inequalities for positive operators*, Ann. Funct. Anal., **4**, № 2, 144 – 152 (2013).
3. J. I. Fujii, M. Fujii, Y. Seo, Z. Hongliang, *Recent developments of matrix versions of the arithmetic-geometric mean inequality*, Ann. Funct. Anal., **7**, № 1, 102 – 117 (2016).
4. M. Fujii, R. Nakamoto, *Refinements of Hölder – McCarthy inequality and Young inequality*, Adv. Oper. Theory, **1**, № 2, 184 – 188 (2016).
5. T. Furuta, *Invitation to linear operators. From matrices to bounded linear operators on a Hilbert space*, Taylor & Francis, Ltd., London (2001).
6. O. Hirzallah, F. Kittaneh, *Matrix Young inequalities for the Hilbert – Schmidt norm*, Linear Algebra and Appl., **308**, 77 – 84 (2000).
7. F. Kittaneh, *On the convexity of the Heinz means*, Integral. Equat. and Oper. Theory, **68**, 519 – 527 (2010).
8. M. Krnić, *More accurate Young, Heinz, and Hölder inequalities for matrices*, Period Math. Hungar., **71**, 78 – 91 (2015).
9. W. Liao, J. Wu, *Improved Young and Heinz inequalities with the Kantorovich constant*, J. Math. Inequal., **10**, № 2, 559 – 570 (2016).
10. D. S. Mitrinović, J. Pečarić, A. M. Fink, *Classical and new inequalities in analysis*, Kluwer Acad. Publ., Dordrecht etc. (1993).
11. J. E. Pečarić, T. Furuta, J. Mićić Hot, Y. Seo, *Mond – Pečarić method in operator inequalities*, Element, Zagreb (2005).
12. M. Sababheh, D. Choi, *A complete refinement of Young's inequality*, J. Math. Anal. and Appl., **440**, 379 – 393 (2016).
13. M. Sababheh, M. S. Moslehian, *Advanced refinements of Young and Heinz inequalities*, J. Number Theory, **172**, 178 – 199 (2017).
14. M. Shafiei, A. G. Ghazanfari, *Numerous refinements of Pólya and Heinz operator inequalities*, Linear and Multilinear Algebra, **66**, № 4, 852 – 860 (2018).
15. J. G. Zhao, J. L. Wu, *Operator inequalities involving improved Young and its reverse inequalities*, J. Math. Anal. and Appl., **421**, № 2, 1779 – 1789 (2015).
16. L. Zou, Y. Jiang, *Improved Heinz inequality and its application*, J. Inequal. and Appl., **113** (2012); DOI:10.1186/1029-242X-2012-113.
17. H. Zuo, G. Shi, M. Fujii, *Refined Young inequality with Kantorovich constant*, J. Math. Inequal., **5**, 551 – 556 (2011).

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