DOI: 10.37863/umzh.v73i2.94

UDC 517.5

S. Yalçın (Bursa Uludag Univ., Turkey),
A. Ebadian (Urmia Univ., Iran),
S. Azizi (Payame Noor Univ., Tehran, Iran)

A PROOF OF A CONJECTURE ON CONVOLUTION OF HARMONIC MAPPINGS AND SOME RELATED PROBLEMS

ДОВЕДЕННЯ ГІПОТЕЗИ ПРО ЗГОРТКУ ГАРМОНІЧНИХ ВІДОБРАЖЕНЬ ТА ДЕЯКІ ПОВ'ЯЗАНІ ЗАДАЧІ

Recently, Kumar et al. proposed a conjecture concerning the convolution of a generalized right half-plane mapping with a vertical strip mapping. They have verified the above conjecture for $n = 1, 2, 3$ and 4. Also, it has been proved only for $\beta = \pi/2$. In this paper, by using of a new method, we settle this conjecture in the affirmative for all $n \in \mathbb{N}$ and $\beta \in (0, \pi)$. Moreover, we will use this method to prove some results on convolution of harmonic mappings. This new method simplifies calculations and shortens the proof of results remarkably.

Нещодавно Kumar та ін. запропонували гіпотезу щодо згортки узагальнених відображень правої півплощини з відображеннями вертикальної смуги. Вони перевірили цю гіпотезу для $n = 1, 2, 3$ та 4. Крім цього, гіпотезу було доведено тільки для $\beta = \pi/2$. Використовуючи новий метод, ми доводимо цю гіпотезу для всіх $n \in \mathbb{N}$ та $\beta \in (0, \pi)$. Більш того, за допомогою цього методу ми отримали деякі результати щодо згортки гармонічних відображень. Новий метод спрощує обчислення та значно скорочує доведення результатів.

1. Introduction. Let $\mathcal{H}$ denote the class of all complex-valued harmonic functions $f = h + \overline{g}$ in the unit disk $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$, where h and g are analytic in $\mathbb{D}$ and normalized by $h(0) = g(0) = 0 = h'(0) - 1$. We call h and g, the analytic and the co-analytic parts of f, respectively, and have the following power series representation:

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \text{ and } g(z) = \sum_{n=1}^{\infty} b_n z^n, \qquad z \in \mathbb{D}.$$

A function $f \in \mathcal{H}$ is locally univalent and sense-preserving in $\mathbb{D}$ if $J_f(z) > 0$ for all z in $\mathbb{D}$, where the Jacobian of $f = h + \overline{g}$ is given by

$$J_f(z) = |h'(z)|^2 - |g'(z)|^2.$$

Using a result of Lewy [12] and the inverse function theorem, one obtains that $J_f(z) > 0$ is a necessary and sufficient condition for $f \in \mathcal{H}$ to be locally univalent and sense-preserving in $\mathbb{D}$. Consequently, $f = h + \overline{g} \in \mathcal{H}$ is sense-preserving in $\mathbb{D}$ if and only if $|w(z)| < 1$, where $w(z) = \dfrac{g'(z)}{h'(z)}$ is the analytic dilatation of $f = h + \overline{g}$. For many basic results on univalent harmonic mappings, see [3, 7, 18]. Denote by $\mathcal{S}_H$ the class of all sense-preserving harmonic univalent mappings $f = h + \overline{g} \in \mathcal{H}$ and by $\mathcal{S}_H^0$ the class of functions $f \in \mathcal{S}_H$ such that $f_{\overline{z}}(0) = 0$. We denote by $\mathcal{K}_H^0$ and S_H^{*0} the subclasses of S_H^0 whose functions map $\mathbb{D}$ onto convex and starlike domains.

A domain $\Omega \subset \mathbb{C}$ is said to be convex in the direction γ, $0 \leq \gamma < \pi$, if every line parallel to the the line joining 0 to $e^{i\gamma}$ has a connected intersection with Ω. In particular, if $\gamma = 0$, we say that Ω is convex in horizontal direction (CHD).

Let $\{f_\beta\}$, where $f_\beta = h_\beta + \overline{g}_\beta$, be the collection of those harmonic mappings which are obtained by shearing of analytic vertical strip mappings

$$h_\beta(z) + g_\beta(z) = \frac{1}{2i\sin\beta}\log\left(\frac{1+ze^{i\beta}}{1+ze^{-i\beta}}\right), \quad 0<\beta<\pi, \tag{1.1}$$

with suitable dilatations (see [4]).

If

$$f(z) = h(z) + \overline{g(z)} = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b_n}\overline{z^n}$$

and

$$F(z) = H(z) + \overline{G(z)} = z + \sum_{n=2}^{\infty} A_n z^n + \sum_{n=1}^{\infty} \overline{B_n}\overline{z^n},$$

then the *convolution* $f * F$ is defined to be the function

$$(f*F)(z) = (h*H)(z) + \overline{(g+G)(z)} = z + \sum_{n=2}^{\infty} a_n A_n z^n + \sum_{n=1}^{\infty} \overline{b_n B_n}\overline{z^n}.$$

The properties of the harmonic convolutions are not as nice as that of the analytic functions. For example, harmonic convolution of two mappings from class $\mathcal{K}_H^0$, is not necessarily in $\mathcal{K}_H^0$. In view of this statement, a good number of papers appeared on this topic (see, for example, [5, 6, 9, 10, 14]). In particular, the following results were obtained by Dorff [5].

Theorem 1.1 [5]. *Let $f = h + \overline{g} \in \mathcal{K}_H^0$ with $h(z) + g(z) = z/(1-z)$. Then $f * f_\beta \in \mathcal{S}_H^0$ and is CHD, provided $f * f_\beta$ is locally univalent and sense-preserving, where f_β is given by* (1.1).

Theorem 1.2 [5]. *Let $f_1 = h_1 + \overline{g}_1 \in \mathcal{S}_H^0$ and $f_2 = h_2 + \overline{g}_2 \in \mathcal{S}_H^0$ with $h_i(z) + g_i(z) = z/(1-z)$ for $i = 1, 2$. If $f_1 * f_2$ is locally univalent and sense-preserving, then $f_1 * f_2 \in \mathcal{S}_H^0$ and is convex in the horizontal direction.*

In [9], Kumar et al. defined harmonic right half-plane mappings $F_a = H_a + \overline{G}_a$ such that $H_a(z) + G_a(z) = z/(1-z)$ with dilatations $w_a(z) = (a-z)/(1-az)$, $-1 < a < 1$. Clearly, for $a = 0$ the mapping F_a reduces to the standard right half-plane mapping.

Recently, Kumar et al. [10] studied the harmonic convolution of mapping f_β with the mapping F_a and posed the following conjecture.

Conjecture A. *Let $f_\beta = h_\beta + \overline{g}_\beta$ be the harmonic mappings given by* (1.1) *with dilatation $w(z) = e^{i\varphi}z^n$, $\varphi \in \mathbb{R}$. Then $F_a * f_\beta \in \mathcal{S}_H^0$ and is CHD for all $n \in \mathbb{N}$ provided $a \in [(n-2)/(n+ + 2), 1)$.*

They proved the above conjecture for $n = 1, 2, 3, 4$. Also, in [11], it has been verified only for $\beta = \pi/2$. In this paper (see Theorem 3.1), by using of a new method, we settle this conjecture in the affirmative for all $n \in \mathbb{N}$ and any $\beta \in (0, \pi)$. Namely, we prove that this conjectue is true for all for all $n \in \mathbb{N}$ provided $a \in [(n-2)/(n+2), 1)$ and any $\beta \in (0, \pi)$.

Note that in the most of papers, Cohn rule and Schur – Cohn algorithm play central role to prove the obtained results on the harmonic convolution (see [6, 9 – 11, 13, 14]). For example, the following results were proved by using of Cohn rule and Schur – Cohn algorithm.

Theorem B [9]. *If* $f_n = h+\overline{g}$ *is the right half-plane mapping given by* $h(z)+g(z) = z/(1-z)$ *with* $w(z) = e^{i\varphi}z^n$, $\varphi \in \mathbb{R}$, $n \in \mathbb{N}$, *then* $F_a * f_n \in \mathcal{S}_H^0$ *is CHD for* $a \in [(n-2)/(n+2), 1)$.

Theorem C [15]. *Let* $f = h + \overline{g} \in \mathcal{S}_H^0$ *with* $h(z) + g(z) = z/(1-z)$ *and* $w(z) = -z(z+a)/(1+az)$, *then* $F_0 * f \in \mathcal{S}_H^0$ *and is convex in the horizontal direction for* $a = 1$ *or* $-1 \le a \le 0$.

In this paper, we will use a new method to prove the above results which remarkably simplifies the calculation and shortens the proof of results compared with Cohn rule and Schur – Cohn algorithm and this is an advantage.

A more general class of harmonic univalent mappings, $L_c = H_c + \overline{G}_c$, $c > 0$, was defined by Muir [17]

$$L_c(z) = H_c(z) + \overline{G_c(z)} = \frac{1}{1+c}\left[\frac{z}{1-z} + \frac{cz}{(1-z)^2}\right] + \overline{\frac{1}{1+c}\left[\frac{z}{1-z} + \frac{cz}{(1-z)^2}\right]}. \tag{1.2}$$

Clearly, for $c = 1$, we obtain the standard right half-plane mapping.

In view of Lemmas 2.1 and 2.2 in [16], similar to the approach used in the proof of Theorem 3.1 (or Conjecture A), we get the following result which solves the problem 4.4 proposed in [16].

Theorem 1.3. *Let* $L_c = H_c + \overline{G}_c \in \mathcal{K}_H^0$ *be a mapping given by* (1.2). *If* $f_\beta = h_\beta + \overline{g}_\beta$ *is given by* (1.1) *with dilatation* $w(z) = e^{i\varphi}z^n$, $\varphi \in \mathbb{R}$, $n \in \mathbb{N}$, *then* $L_c * f_\beta \in \mathcal{S}_H^0$ *and is CHD for* $0 < c \le 2/n$.

2. Preliminaries. The following lemmas will be required in the proof of our main results.

Lemma 2.1. *For* $\eta > 1$ *and* $n \in \mathbb{N}$, *we have* $\eta^n > \eta^{n-1} + \eta^{n-2} + \ldots + \eta + 1$.

Proof. By mathematical induction the proof is easy, so we skip the details.

Lemma 2.2 (see [1] and also [8]). *Let* $p(z) = z^n + \sum_{j=0}^{n-1} a_j z^j$, *be a complex polynomial. Then all the zeros of* $p(z)$ *lie in the disk*

$$\{z : |z| < \eta\} \subset \{z : |z| < 1 + A\},$$

where

$$A = \max_{0 \le j \le n-1} |a_j|$$

and η *is the unique positive root of the real-coefficient polynomial*

$$Q(x) = x^n - |a_{n-1}|x^{n-1} - |a_{n-2}|x^{n-2} - \ldots - |a_1|x - |a_0|.$$

In view of Lemma 2.2, we obtain the result stated below which play a central role in proofs of our results in this paper.

Corollary **2.1.** *If* $|a_0| \le 1$, $|a_1| \le 1, \ldots, |a_{n-1}| \le 1$, *then all the zeros of complex polynomial* $p(z) = z^n + \sum_{j=0}^{n-1} a_j z^j$ *lie in the unit disk* $\{z : |z| < \eta \le 1\}$.

Proof. In contrary, let $\eta > 1$, where η is the unique positive root of the real-coefficient polynomial

$$Q(x) = x^n - |a_{n-1}|x^{n-1} - |a_{n-2}|x^{n-2} - \ldots - |a_1|x - |a_0|.$$

Namely, $\eta > 1$ such that $Q(\eta) = 0$. So,

$$\eta^n = |a_{n-1}|\eta^{n-1} + |a_{n-2}|\eta^{n-2} + \ldots + |a_1|\eta + |a_0| \le$$

$$\le \eta^{n-1} + \eta^{n-2} + \ldots + \eta + 1.$$

This contradicts Lemma 2.1. Then $\eta \le 1$ and in view of Lemma 2.2 we get the desired result.

Lemma 2.3 (see [10], Lemma 2.1). *If $f_\beta = h_\beta + \overline{g}_\beta$ is the mapping given by with dilatation $w = g'_\beta/h'_\beta$, then $\widetilde{w}$, the dilatation of $F_a * f_\beta$, is given by*

$$\widetilde{w}(z) = \frac{2w(1+w)(a+az\cos\beta+z\cos\beta+z^2) - zw'(1-a)(1+2z\cos\beta+z^2)}{2(1+z\cos\beta+az\cos\beta+az^2)(1+w) - zw'(1-a)(1+2z\cos\beta+z^2)}. \tag{2.1}$$

Lemma 2.4 (see [9], Eq. (4)). *If $f = h + \overline{g} \in \mathcal{S}_H^0$ is right half-plane mapping, where $h(z) + g(z) = z/(1-z)$ with dilatation $w = g'/h'(h'(z) \neq 0, z \in \mathbb{D})$, then $\widetilde{w}_2$, the dilatation of $F_a * f$, is given by*

$$\widetilde{w}_1(z) = \frac{2(a-z)w(1+w) + (a-1)w'z(1-z)}{2(1-az)(1+w) + (a-1)w'z(1-z)}. \tag{2.2}$$

Lemma 2.5 (see [6], Eq. (6)). *If $f = h + \overline{g} \in \mathcal{S}_H^0$ with $h(z) + g(z) = z/(1-z)$ and dilatation $w = g'/h'$, then the dilatation $F_0 * f$ is given by*

$$\widetilde{w}_2(z) = -z\frac{w^2 + \left[w - \frac{1}{2}w'z\right] + \frac{1}{2}w'}{1 + \left[w - \frac{1}{2}w'z\right] + \frac{1}{2}w'z^2}. \tag{2.3}$$

3. Main results. In the following result, we prove Conjecture A.

Theorem 3.1. *If $f_\beta = h_\beta + \overline{g}_\beta$ is the harmonic mapping obtained from the relation* (1.1) *with dilatation $w(z) = e^{i\varphi}z^n$, $\varphi \in \mathbb{R}$, $n \in \mathbb{N}$, then $F_a * f_\beta \in \mathcal{S}_H^0$ and is CHD for $a \in \left[\frac{n-2}{n+2}, 1\right)$.*

Proof. By Theorem 1.1, it suffices to show that $F_a * f_\beta$ is locally univalent and sense-preserving or equivalently the dilatation $\widetilde{w}$ of $F_a * f_\beta$ satisfies $|\widetilde{w}(z)| < 1$ for all $z \in \mathbb{D}$. Setting $w(z) = e^{i\varphi}z^n$ in (2.1), we obtain

$$\widetilde{w}(z) = z^n e^{2i\varphi}\frac{p(z)}{p^*(z)},$$

where

$$p(z) = z^{n+2} + (a+1)\cos\beta z^{n+1} + az^n + \frac{1}{2}(2+an-n)e^{-i\theta}z^2 +$$

$$+[(a(1+n)+1-n)\cos\beta]e^{-i\theta}z + \frac{1}{2}(2a+an-n)e^{-i\theta}$$

and

$$p^*(z) = \frac{1}{2}(2a+an-n)e^{i\theta}z^{n+2} + [(a(1+n)+1-n)\cos\beta]e^{i\theta}z^{n+1} +$$

$$+\frac{1}{2}(2+an-n)e^{-i\theta}z^n + az^2 + (a+1)\cos\beta z + 1$$

such that $p^*(z) = z^{n+2}\overline{p(1/\overline{z})}$.

Clearly, if $z_0, z_0 \neq 0$, is a zero of p then $\frac{1}{z_0}$ is a zero of p^*. Hence, if $\alpha_1, \alpha_2, \ldots, \alpha_{n+2}$ are the zeros of p (not necessarily distinct), then we can write

$$\widetilde{w}(z) = z^n e^{2i\varphi} \frac{(z-\alpha_1)}{(1-\overline{\alpha}_1 z)} \frac{(z-\alpha_2)}{(1-\overline{\alpha}_2 z)} \cdots \frac{(z-\alpha_{n+2})}{(1-\overline{\alpha}_{n+2} z)}$$

for $|\alpha_i| \le 1$, since $\dfrac{z-\alpha_i}{1-\overline{\alpha}_i z}$ maps the closed unit disk onto itself. Therefore, to prove that $|\widetilde{w}(z)| < 1$ in $\mathbb{D}$, we will show that all the zeros of polynomial p, i.e., $\alpha_1, \alpha_2, \ldots, \alpha_{n+2}$ lie inside or on the unit circle $|z| = 1$ for $a \in [(n-2)/(n+2), 1)$. We obviously have $a_0 = \dfrac{1}{2}(2a + an - n)e^{-i\theta}$, $a_1 = [(a(1+n) + 1 - n)\cos\beta]e^{-i\theta}$, $a_2 = \dfrac{1}{2}(2 + an - n)e^{-i\theta}$, $a_3 = a_4 = \ldots = a_{n-1} = 0$, $a_n = a$, $a_{n+1} = (a+1)\cos\beta$ and $a_{n+2} = 1$.

For $a \in \left[\dfrac{n-2}{n+2}, 1\right)$, we can easily see that

$$|a_0| \le 1, \ |a_2| \le 1, \ldots, |a_n| \le 1.$$

Also, if $a \in \left(\dfrac{n-1}{n+1}, 1\right)$ we observe that $|a_1| < |a_0| \le 1$ and if $a \in \left[\dfrac{n-2}{n+2}, \dfrac{n-1}{n+1}\right)$, we have $|a_1| < |a_2| \le 1$ $\left(\text{for } a = \dfrac{n-1}{n+1}, \text{ separately, it is clear that } |a_1| < 1\right)$.

So, Corollary 2.1 implies that all the zeros of polynomial

$$q(z) = p(z) - a_{n+1} z^{n+1} =$$

$$= z^{n+2} + az^n + \frac{1}{2}(2 + an - n)e^{-i\theta} z^2 + [(a(1+n) + 1 - n)\cos\beta]e^{-i\theta} z + \frac{1}{2}(2a + an - n)e^{-i\theta}$$

lie inside the unit circle $|z| = 1$ (namely in the unit disk $\mathbb{D}$).

On the other hand, since $p(z) = q(z) + a_{n+1} z^{n+1}$, then

$$\frac{p(z)}{q(z)} = 1 + \frac{a_{n+1} z^{n+1}}{q(z)}$$

and this approaches 1 as z goes to infinity. So, there is a sufficiently large number R with

$$\left| \frac{p(z)}{q(z)} - 1 \right| < 1$$

for $|z| = R$, that is, $|p(z) - q(z)| < |q(z)|$ for $|z| = R$. Now, the application of Rouché's theorem (see [2]) allows us to conclude that all the zeros of polynomial p lie in the unit disk $\mathbb{D}$ for $a \in$ $\in [(n-2)/(n+2), 1)$ and this completes the proof.

Similar to the method used in he proof of the above theorem, by using of Corollary 2.1, Theorem 1.2 and the relation (2.2), we get Theorem B. So, we skip the proof.

Theorem 3.2. *Let* $f = h + \overline{g} \in \mathcal{S}_H^0$ *with* $h(z) + g(z) = z/(1-z)$ *and* $w(z) = -z(z+a)/(1+az)$. *Then* $F_0 * f \in \mathcal{S}_H^0$ *and is convex in the horizontal direction for* $a = 1$ *or* $-1 \le a \le 0$.

Proof. By Theorem 1.2, we need only to prove that $F_0 * f$ is locally univalent and sense-preserving. Setting $\widetilde{w}_2(z) = -z(z+a)/(1+az)$ in (2.3) and simplifying, we obtain

$$\widetilde{w}_2(z) = z \frac{z^3 + \dfrac{2+3a}{2} z^2 + (1+a)z + a/2}{1 + \dfrac{2+3a}{2} z + (1+a)z^2 + (a/2)z^3} = z \frac{\varphi(z)}{\varphi^*(z)},$$

where

$$\varphi(z) = z^3 + \frac{2+3a}{2} z^2 + (1+a)z + a/2.$$

To prove that $|\widetilde{w}_2(z)| < 1$ in $\mathbb{D}$, it suffices to show that all the zeros of φ lie inside $|z| = 1$ or on $|z| = 1$. If $a = 1$, then $\varphi(z) = z^3 + \dfrac{2+3a}{2} z^2 + (1+a)z + a/2 = \dfrac{1}{2}(1+z)^2(1+2z)$ has all its zeros in $\overline{\mathbb{D}}$. The repeated application of Corollary 2.1 (as in the proof of Theorem 3.1) shows that this is in fact true, also for $-1 \leq a \leq 0$. We skip the details. This completes the proof.

The new method used in the proof of above theorems can be applied to prove many problems in convolutions of univalent harmonic mappings. For example, we can derive the following theorem. We skip the details for similarity.

Theorem 3.3 ([15], Theorem 1.2)**.** *Let* $L_c = H_c + \overline{G}_c \in \mathcal{K}_H^0$ *be a mapping given by* (1.2). *If* $F_a = H_a + \overline{G}_a$ *is the right half-plane mapping, then* $L_c * F_a$ *is univalent and convex in the horizontal direction for* $0 < c \leq 2(1+a)/(1-a)$.

References

1. A. Cauchy, *Exercises de mathematique*, Oeuvres (2), **9** (1829).
2. J. Conway, *Functions of one complex variable*, Second Ed., Vol. I, Springer-Verlag (1978).
3. M. Dorff, *Anamorphosis, mapping problems, and harmonic univalent functions*, Explorat. Complex Anal., 197 – 269 (2012).
4. M. Dorff, *Harmonic univalent mappings onto asymmetric vertical strips*, Comput. Methods and Funct. Theory, 171 – 175 (1997).
5. M. Dorff, *Convolution of planar harmonic convex mappings*, Complex Var. Theory and Appl., **45**, № 3, 263 – 271 (2001).
6. M. Dorff, M. Nowak, M. Wołoszkiewicz, *Convolution of harmonic convex mappings*, Complex Var. Elliptic Equat., **57**, № 5, 489 – 503 (2012).
7. P. Duren, *Harmonic mappings in the plane*, Cambridge Tracts Math., **156**, Cambridge Univ. Press, Cambridge (2004).
8. R. B. Gardner, N. K. Govil, *Eneström – Kakeya theorem and some of its generalizations*, Current Topics in Pure and Comput. Complex Anal., 171 – 199 (2014).
9. R. Kumar, M. Dorff, S. Gupta, S. Singh, *Convolution properties of some harmonic mappings in the right half-plane*, Bull. Malays. Math. Sci. Soc., **39**, № 1, 439 – 455 (2016).
10. R. Kumar, S. Gupta, S. Singh, M. Dorff, *An application of Cohn's rule to convolutions of univalent harmonic mappings*, Rocky Mountain J. Math., **46**, № 2, 559 – 570 (2016).
11. R. Kumar, S. Gupta, S. Singh, M. Dorff, *On harmonic convolutions involving a vertical strip mapping*, Bull. Korean Math. Soc., **52**, № 1, 105 – 123 (2015).
12. H. Lewy, *On the nonvanishing of the Jacobian in certain one-to-one mappings*, Bull. Amer. Math. Soc., **42**, 689 – 692 (1936).
13. L. Li, S. Ponnusamy, *Convolutions of slanted half-plane harmonic mappings*, Analysis (Munich), **33**, № 2, 159 – 176 (2013).
14. L. Li, S. Ponnusamy, *Solution to an open problem on convolution of harmonic mappings*, Complex Var. Elliptic Equat., **58**, № 12, 1647 – 1653 (2013).
15. Y. Li, Z. Liu, *Convolution of harmonic right half-plane mappings*, Open Math., **14**, 789 – 800 (2016).
16. Z. Liu, Y. Jiang, Y. Sun, *Convolutions of harmonic half-plane mappings with harmonic vertical strip mappings*, Filomat, **31**, № 7, 1843 – 1856 (2017).
17. S. Muir, *Weak subordination for convex univalent harmonic functions*, J. Math. Anal. and Appl., **348**, 689 – 692 (2008).
18. S. Ponnusamy, A. Rasila, *Planar harmonic and quasiregular mappings*, Topics in Modern Function Theory, Chapter in CMFT, RMS-Lect. Notes Ser., № 19, 267 – 333 (2013).

Received 18.05.18