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ON CERTAIN NONLINEAR DIFFERENTIAL MONOMIAL SHARING NON-ZERO POLYNOMIAL

ПРО НЕЛІНІЙНИЙ ДИФЕРЕНЦІАЛЬНИЙ ОДНОЧЛЕН ЗІ СПІЛЬНИМ НЕНУЛЬОВИМ МНОГОЧЛЕНОМ

With the idea of normal family we study the uniqueness of meromorphic functions f and g when $f^n(\mathcal{L}(f))^m - p$ and $g^n(\mathcal{L}(g))^m - p$ share two values, where $\mathcal{L}(f) = a_k f^{(k)} + a_{k-1} f^{(k-1)} + \ldots + a_1 f' + a_0 f$, $a_k(\neq 0), a_{k-1}, \ldots, a_1, a_0 \in \mathbb{C}$ and $p(z)(\not\equiv 0)$ is a polynomial. The obtained result significantly improves and generalizes the result in [A. Banerjee, S. Majumder, *On certain non-linear differential polynomial sharing a non-zero polynomial*, Bol. Soc. Mat. Mex. (2016), https://doi.org/10.1007/s40590-016-0156-0].

На базі ідеї про нормальні сім'ї функцій вивчається єдиність мероморфних функцій f і g у випадку, коли $f^n(\mathcal{L}(f))^m - p$ і $g^n(\mathcal{L}(g))^m - p$ мають спільні значення, де $\mathcal{L}(f) = a_k f^{(k)} + a_{k-1} f^{(k-1)} + \ldots + a_1 f' + a_0 f$, $a_k(\neq 0), a_{k-1}, \ldots, a_1, a_0 \in \mathbb{C}$, а $p(z)(\not\equiv 0)$ — поліном. Отриманий результат є істотним узагальненням результату з [A. Banerjee, S. Majumder, *On certain non-linear differential polynomial sharing a non-zero polynomial*, Bol. Soc. Mat. Mex. (2016), https://doi.org/10.1007/s40590-016-0156-0].

1. Introduction definitions and results. In this paper, by meromorphic functions we mean that meromorphic functions in the whole complex plane $\mathbb{C}$. We adopt the standard notations of value distribution theory (see [9]). We denote by $T(r)$ the maximum of $T(r, f)$ and $T(r, g)$. The notation $S(r)$ denotes any quantity satisfying $S(r) = o(T(r))$ as $r \longrightarrow \infty$, outside of a possible exceptional set of finite linear measure. A meromorphic function a is said to be a small function of f if $T(r, a) = S(r, f)$. We denote by $S(f)$ the set of all small functions of f. We use the symbol $\rho(f)$ to denote the order of f.

Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions. Let $a(z) \in S(f) \cap S(g)$. We say that $f(z)$ and $g(z)$ share $a(z)$ counting multiplicities (CM) if the zeros of $f(z) - a(z)$ and $g(z) - a(z)$ have the same locations and same multiplicities, and we say that $f(z)$ and $g(z)$ share $a(z)$ ignoring multiplicities (IM) if the zeros of $f(z) - a(z)$ and $g(z) - a(z)$ have the same locations but different multiplicities.

We say that a finite value z_0 is called a fixed point of f if $f(z_0) = z_0$. For the sake of simplicity, we use the notion $(m)^*$ defined by $(m)^* = m - 1$, if m is a positive integer; $(m)^* = [m]$, if m is positive rational, where $[m]$ denotes the greatest integer not exceeding m.

Let h be a meromorphic function in $\mathbb{C}$. Then h is called a normal function if there exists a positive real number M such that $h^{\#}(z) \leq M \ \forall z \in \mathbb{C}$, where

$$h^{\#}(z) = \frac{|h'(z)|}{1 + |h(z)|^2}$$

denotes the spherical derivative of h.

Let $\mathcal{F}$ be a family of meromorphic functions in a domain $D \subset \mathbb{C}$. We say that $\mathcal{F}$ is normal in D if every sequence $\{f_n\}_n \subseteq \mathcal{F}$ contains a subsequence which converges spherically and uniformly on the compact subsets of D (see [16]).

The following well-known theorem in value distribution theory was posed by Hayman and settled by several authors almost at the same time [3, 5].

Theorem A. *Let f be a transcendental meromorphic function and $n \in \mathbb{N}$. Then $f^n f' = 1$ has infinitely many solutions.*

To investigate the uniqueness result corresponding to Theorem A, both Fang and Hua [7], Yang and Hua [20] obtained the following result.

Theorem B. *Let f and g be two non-constant entire (meromorphic) functions, $n \in \mathbb{N}$ with $n \geq 6$ $(n \geq 11)$. If $f^n f'$ and $g^n g'$ share 1 CM, then either $f(z) = c_1 e^{cz}$ and $g(z) = c_2 e^{-cz}$, where $c, c_1, c_2 \in \mathbb{C} \setminus \{0\}$ satisfying $4(c_1 c_2)^{n+1} c^2 = -1$ or $f \equiv tg$, $t \in \mathbb{C} \setminus \{0\}$ such that $t^{n+1} = 1$.*

Considering the uniqueness question of entire or meromorphic functions having fixed points, Fang and Qiu [8] obtained the following theorem.

Theorem C. *Let f and g be two non-constant meromorphic (entire) functions, $n \in \mathbb{N}$ with $n \geq 11$ $(n \geq 6)$. If $f^n(z) f'(z) - z$ and $g^n(z) g'(z) - z$ share 0 CM, then either $f(z) = c_1 e^{cz^2}$ and $g(z) = c_2 e^{-cz^2}$, where $c, c_1, c_2 \in \mathbb{C} \setminus \{0\}$ satisfying $4(c_1 c_2)^{n+1} c^2 = -1$ or $f \equiv tg$, $t \in \mathbb{C} \setminus \{0\}$ such that $t^{n+1} = 1$.*

It is instinctive to ask what happens if the first derivative f' in Theorem A is replaced by the general derivative $f^{(k)}$. By considering this problem, Xu et al. [17] and Li [24], respectively, proved the following result.

Theorem D. *Let f be a transcendental meromorphic function and $k, n \in \mathbb{N}$ with $n \geq 2$. Then $f^n f^{(k)}$ takes every finite non-zero value infinitely many times or has infinitely many fixed points.*

Recently, Cao and Zhang [6] proved the following theorem.

Theorem E. *Let f, g be two non-constant meromorphic functions, whose zeros are of multiplicities at least $k+1$, $k \in \mathbb{N}$ with $1 \leq k \leq 5$ and let $n \in \mathbb{N}$ with $n \geq 10$. If $f^n f^{(k)}$ and $g^n g^{(k)}$ share 1 CM, $f^{(k)}$ and $g^{(k)}$ share 0 CM, f and g share ∞ IM, then one of the following two conclusions hold:*

(i) $f \equiv tg$, $t \in \mathbb{C} \setminus \{0\}$ *such that* $t^{n+1} = 1$;

(ii) $f(z) = c_1 e^{az}$ *and* $g(z) = c_2 e^{-az}$, *where* $a, c_1, c_2 \in \mathbb{C} \setminus \{0\}$ *such that* $(-1)^k (c_1 c_2)^{n+1} a^{2k} = 1$.

Regarding Theorem E, the following questions are inevitable.

***Question* 1.** Can the lower bound of n be further reduced in Theorem E?

***Question* 2.** Can the condition "Let f and g be two non-constant meromorphic functions, whose zeros are of multiplicities at least $k+1$, $k \in \mathbb{N}$" in Theorem E be further weakened?

***Question* 3.** Does Theorem E hold for $k \geq 6$?

We now explain the notation of weighted sharing as introduced in [11].

Definition 1 [11]. *Let $k \in \mathbb{N} \cup \{0\} \cup \{\infty\}$. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f and g share the value a with weight k. We write f and g share (a, k) to mean that f and g share the value a with weight k.*

Keeping in mind the above questions, Banerjee and Majumder [2] obtained the following result in 2016.

Theorem F. *Let f, g be two transcendental meromorphic functions, whose zeros are of multiplicities at least $k \in \mathbb{N}$ and $n \in \mathbb{N}$ such that $n > \left(\dfrac{k^2+4k+4}{k}\right)^*$. Let $p(z)(\not\equiv 0)$ be a polynomial such that either $\deg(p) \leq n-1$ or zeros of $p(z)$ be of multiplicities at most $n-1$. If $f^n f^{(k)} - p$ and $g^n g^{(k)} - p$ share $(0, k_1)$, where $k_1 = \left[\dfrac{k+2}{n-k}\right] + 3$ and f, g share ∞ IM and $f^{(k)}$, $g^{(k)}$ share 0 CM, then $f \equiv tg$, $t \in \mathbb{C} \setminus \{0\}$ such that $t^{n+1} = 1$.*

Throughout this paper, we always use $\mathcal{L}(f)$ to denote a differential polynomial as follows:

$$\mathcal{L}(f) = a_k f^{(k)} + a_{k-1} f^{(k-1)} + \ldots + a_1 f' + a_0 f, \quad a_k(\neq 0), \quad a_{k-1}, \ldots, a_1, a_0 \in \mathbb{C}. \tag{1.1}$$

Now we observe Theorem F. Then it is natural to ask the following questions which are the motive of the present paper.

***Question* 4.** Can one remove the condition "$\deg(p) \leq n-1$ or zeros of $p(z)$ be of multiplicities at most $n-1$" in Theorem F?

***Question* 5.** What happens when "$f^n(\mathcal{L}(f))^m - p$ and $g^n(\mathcal{L}(g))^m - p$" share the value 0 CM, where $p(z)(\not\equiv 0)$ is a polynomial in Theorem F?

***Question* 6.** Can the lower bound of n be further reduced in Theorem F?

2. Main result. In this paper, taking the possible answers of the above questions into background we obtain the following result which significantly improves and generalizes Theorem F.

Theorem 1. *Let f and g be two transcendental meromorphic functions having zeros of multiplicities at least $k \in \mathbb{N}$. Let $m, n \in \mathbb{N}$ such that $n \geq \dfrac{k^2+2mk+6}{k}$ and $p(z)(\not\equiv 0)$ be a polynomial. If $f^n(\mathcal{L}(f))^m - p$ and $g^n(\mathcal{L}(g))^m - p$ share $(0, k_1)$, where $k_1 = \left[\dfrac{3+(k-1)m}{n+m+(m-2)k-1}\right] + 3$ and f, g share ∞ IM and $\mathcal{L}(f)$, $\mathcal{L}(g)$ share 0 CM, then $f \equiv tg$, where $t \in \mathbb{C} \setminus \{0\}$ with $t^{n+m} = 1$.*

***Remark* 1.** It is easy to see that the condition "Let f and g be two transcendental meromorphic functions having zeros of multiplicities at least $k \in \mathbb{N}$" in Theorem 1 is sharp by the following example.

***Example* 1.** Let

$$f(z) = c_1 e^{az} \quad \text{and} \quad g(z) = c_2 e^{-az},$$

where $a, c_1, c_2 \in \mathbb{C} \setminus \{0\}$. Note that

$$\mathcal{L}(f(z)) = a_2 f''(z) + a_1 f'(z) + a_0 f(z) = c_1 \left(a_2 a^2 + a_1 a + a_0\right) e^{az}$$

and

$$\mathcal{L}(g(z)) = a_2 g''(z) + a_1 g'(z) + a_0 g(z) = c_2 \left(a_2 a^2 - a_1 a + a_0\right) e^{-az},$$

where $a_2(\neq 0), a_1, a_0 \in \mathbb{C}$ such that

$$c_1^{n+m} \left(a_2 a^2 + a_1 a + a_0\right)^m = c_2^{n+m} \left(a_2 a^2 - a_1 a + a_0\right)^m, \quad m, n \in \mathbb{N}.$$

Since f and g have no zeros, it follows that the condition "Let f and g be two transcendental meromorphic functions having zeros of multiplicities at least $k \in \mathbb{N}$" does not hold. Here we see that $f,\ g$ share ∞ CM and $\mathcal{L}(f),\ \mathcal{L}(g)$ share 0 CM. On the other hand, we see that

$$f^n(z)(\mathcal{L}(f(z)))^m - p(z) = c_1^{n+m}\left(a_2a^2 + a_1a + a_0\right)^m\left(e^{a(n+m)z} - 1\right)$$

and

$$g^n(z)(\mathcal{L}(g(z)))^m - p(z) = c_2^{n+m}\left(a_2a^2 - a_1a + a_0\right)^m\left(e^{-a(n+m)z} - 1\right),$$

where $p(z) = c_1^{n+m}\big(a_2a^2 + a_1a + a_0\big)^m$. Clearly $f^n(\mathcal{L}(f))^m - p$ and $g^n(\mathcal{L}(g))^m - p$ share $(0,\infty)$, but $f \not\equiv tg$, where $t \in \mathbb{C}\setminus\{0\}$ with $t^{n+m} = 1$.

We now explain some definitions and notations which are used in the paper.

Definition 2 [14]. *Let $p \in \mathbb{N}$ and $a \in \mathbb{C}\cup\{\infty\}$.*

(i) $N(r,a;f\mid\geq p)$ $(\overline{N}(r,a;f\mid\geq p))$ *denotes the counting function (reduced counting function) of those a-points of f whose multiplicities are not less than p.*

(ii) $N(r,a;f\mid\leq p)$ $(\overline{N}(r,a;f\mid\leq p))$ *denotes the counting function (reduced counting function) of those a-points of f whose multiplicities are not greater than p.*

Definition 3 [22]. *For $a \in \mathbb{C}\cup\{\infty\}$ and $p \in \mathbb{N}$ we denote by $N_p(r,a;f)$ the sum $\overline{N}(r,a;f) + {}$ ${}+\overline{N}(r,a;f\mid\geq 2) + \ldots + \overline{N}(r,a;f\mid\geq p)$. Clearly $N_1(r,a;f) = \overline{N}(r,a;f)$.*

Definition 4. *We denote by $\overline{N}(r,a;f\mid= k)$ the reduced counting function of those a-points of f whose multiplicities exactly $k \in \mathbb{N}$. Clearly $\overline{N}(r,a;f\mid= 1) = N(r,a;f\mid= 1)$.*

Definition 5 [1]. *Let f and g be two non-constant meromorphic functions such that f and g share 1 IM. Let z_0 be a 1-point of f with multiplicity p and a 1-point of g with multiplicity q. We denote by $\overline{N}_L(r,1;f)$, the counting function of those 1-points of f and g where $p > q$ and by $\overline{N}_E^{(l}(r,1;f)$, the counting function of those 1-points of f and g where $p = q \geq l$, each point in these counting functions is counted only once, where $l \in \mathbb{N}\setminus\{1\}$. In the same way we can define $\overline{N}_L(r,1;g)$ and $\overline{N}_E^{(l}(r,1;g)$.*

Definition 6 [11]. *Let $f,\ g$ share a value a IM. We denote by $\overline{N}_*(r,a;f,g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g. Clearly $\overline{N}_*(r,a;f,g) = \overline{N}_L(r,a;f) + \overline{N}_L(r,a;g)$.*

3. Lemmas. In this section, we present some lemmas which will be needed in the sequel. Now we define the following two auxiliary functions H and G, respectively:

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right) \tag{3.1}$$

and

$$V = \left(\frac{F'}{F-1} - \frac{F'}{F}\right) - \left(\frac{G'}{G-1} - \frac{G'}{G}\right) = \frac{F'}{F(F-1)} - \frac{G'}{G(G-1)}, \tag{3.2}$$

where F and G are two non-constant meromorphic functions.

Lemma 1 [23]. *Let f be a non-constant meromorphic function and $L(f)$ be a differential polynomial defined as follows:*

$$L(f) = f^{(k)} + a_{k-1}f^{(k-1)} + a_{k-2}f^{(k-2)} + \ldots + a_1f' + a_0f,$$

where $k \in \mathbb{N}$, $a_j \in S(f)$, $j = 0,1,\ldots,k-1$. If $L(f) \not\equiv 0$ and $p \in \mathbb{N}$, we have

$$N_p(r,0;L(f)) \leq k\overline{N}(r,\infty;f) + N_{p+k}(r,0;f) + S(r,f).$$

Lemma 2 [12]. *If $N(r,0;f^{(k)}\mid f\neq 0)$ denotes the counting function of those zeros of $f^{(k)}$ which are not the zeros of f, where a zero of $f^{(k)}$ is counted according to its multiplicity. Then*

$$N\left(r,0;f^{(k)}\mid f\neq 0\right)\leq k\overline{N}(r,\infty;f)+N(r,0;f\mid<k)+k\overline{N}(r,0;f\mid\geq k)+S(r,f).$$

Lemma 3 [19]. *Let f be a non-constant meromorphic function and $P(f)=a_0+a_1f+a_2f^2+\ldots$ $\ldots+a_nf^n$, where $a_0,a_1,a_2\ldots,a_n(\neq 0)\in\mathbb{C}$. Then $T(r,P(f))=nT(r,f)+O(1)$.*

Lemma 4 [13]. *Let f be a transcendental meromorphic function and $\alpha(\not\equiv 0,\infty)\in S(f)$, then $\psi=\alpha(f)^n(f^{(k)})^p\notin\mathbb{C}$, where $n\in\mathbb{N}\cup\{0\}$ and $p,k\in\mathbb{N}$.*

Lemma 5 [21]. *Let f_j, $j=1,2,3$, be a meromorphic and f_1 be non-constant. Suppose that $\sum_{j=1}^{3}f_j\equiv 1$ and $\sum_{j=1}^{3}N(r,0;f_j)+2\sum_{j=1}^{3}\overline{N}(r,\infty;f_j)<(\lambda+o(1))T_1(r)$ as $r\to+\infty$, $r\in I$, $\lambda<1$ and $T_1(r)=\max_{1\leq j\leq 3}T(r,f_j)$, where I is a set of infinite linear measure. Then either $f_2\equiv 1$ or $f_3\equiv 1$.*

Lemma 6 ([21], Theorem 1.24). *Let f be a non-constant meromorphic function and $k\in\mathbb{N}$. Suppose that $f^{(k)}\not\equiv 0$, then $N\big(r,0;f^{(k)}\big)\leq N(r,0;f)+k\overline{N}(r,\infty;f)+S(r,f)$.*

Lemma 7. *Let f, g be two non-constant meromorphic functions, whose zeros are of multiplicities at least k, where $k\in\mathbb{N}$ and $F=f^n(\mathcal{L}(f))^m/p$, $G=g^n(\mathcal{L}(g))^m/p$, where $p(z)(\not\equiv 0)$ is a polynomial and $m,n\in\mathbb{N}$ such that $n+m+(m-2)k>1$. Suppose $H\not\equiv 0$. If F, G share $(1,k_1)$ except for the zeros of p and f, g share $(\infty,0)$, where $0\leq k_1\leq\infty$, then*

$$\overline{N}(r,\infty;f)\leq\frac{k+1}{k(n+m+(m-2)k-1)}\left(T(r,f)+T(r,g)\right)+$$

$$+\frac{1}{n+m+(m-2)k-1}\,\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g).$$

Proof. First, we suppose ∞ is a Picard exceptional value of both f and g. Then the lemma follows immediately. Next we suppose ∞ is not a Picard exceptional value of both f and g. We claim that $V\not\equiv 0$. If possible suppose $V\equiv 0$. Then by integration we obtain $1-\dfrac{1}{F}\equiv A\left(1-\dfrac{1}{G}\right)$, $A\in\mathbb{C}\setminus\{0\}$. It is that if z_0 is a pole of f, then it is a pole of g. Hence from the definition of F and G we have $\dfrac{1}{F(z_0)}=0$ and $\dfrac{1}{G(z_0)}=0$. So, $A=1$ and hence $F\equiv G$. Since $H\not\equiv 0$, it follows that $F\not\equiv G$. Therefore we arrive at a contradiction. Hence $V\not\equiv 0$. Also $m(r,V)=S(r,f)+S(r,g)$.

Let z_0 be a pole of f with multiplicity q and a pole of g with multiplicity r such that $p(z_0)\neq 0$. Clearly z_0 is a pole of F with multiplicity $(n+m)q+mk$ and a pole of G with multiplicity $(n+m)r+mk$. Clearly

$$\frac{F'(z)}{F(z)(F(z)-1)}=O\left((z-z_0)^{(n+m)q+mk-1}\right)$$

and

$$\frac{G'(z)}{G(z)(G(z)-1)}=O\left((z-z_0)^{(n+m)r+mk-1}\right).$$

Consequently $V(z)=O\big((z-z_0)^{(n+m)t+mk-1}\big)$, where $t=\min\{q,r\}$. Since f and g share $(\infty,0)$, from the definition of V it is clear that z_0 is a zero of V with multiplicity at least $n+m+mk-1$. So from the definition of V and using Lemma 2 we have

$$(n+m+mk-1)\overline{N}(r,\infty;f) \leq$$

$$\leq N(r,0;V)+O(\log r) \leq T(r,V)+S(r,f)+S(r,g) \leq$$

$$\leq N(r,\infty;V)+S(r,f)+S(r,g) \leq$$

$$\leq \overline{N}(r,0;F)+\overline{N}(r,0;G)+\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g) \leq$$

$$\leq \overline{N}(r,0;f)+\overline{N}\left(r,0;f^{(k)}\mid f\neq 0\right)+\overline{N}(r,0;g)+\overline{N}\left(r,0;g^{(k)}\mid g\neq 0\right)+$$

$$+\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g) \leq$$

$$\leq \overline{N}(r,0;f)+k\,\overline{N}(r,\infty;f)+N_k(r,0;f)+\overline{N}(r,0;g)+k\,\overline{N}(r,\infty;g)+$$

$$+N_k(r,0;g)+\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g) \leq$$

$$\leq \frac{k+1}{k}\,N(r,0;f)+\frac{k+1}{k}\,N(r,0;g)+2k\overline{N}(r,\infty;f)+\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g) \leq$$

$$\leq \frac{k+1}{k}\,\left(T(r,f)+T(r,g)\right)+2k\,\overline{N}(r,\infty;f)+\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g).$$

Lemma 7 is proved.

Lemma 8. *Let f be a non-constant meromorphic function and let $F = f^n(\mathcal{L}(f))^m$, where $m,n,k\in\mathbb{N}$ satisfying $n>m$. Then*

$$(n-m)T(r,f) \leq T(r,F)-mN(r,\infty;f)-N\left(r,0;(\mathcal{L}(f))^m\right)+S(r,f).$$

Proof. Note that

$$N(r,\infty;F)=N\left(r,\infty;f^n\right)+N\left(r,\infty;(\mathcal{L}(f))^m\right)=$$

$$=N\left(r,\infty;f^n\right)+mN(r,\infty;f)+mk\overline{N}(r,\infty;f)+S(r,f),$$

i.e.,

$$N\left(r,\infty;f^n\right)=N(r,\infty,F)-mN(r,\infty;f)-mk\overline{N}(r,\infty,f)+S(r,f).$$

Also

$$m\left(r,f^n\right)=m\left(r,\frac{F}{(\mathcal{L}(f))^m}\right) \leq$$

$$\leq m(r,F)+m\left(r,\frac{1}{(\mathcal{L}(f))^m}\right)+S(r,f)=$$

$$=m(r,F)+T\left(r,(\mathcal{L}(f))^m\right)-N\left(r,0;(\mathcal{L}(f))^m\right)+S(r,f)=$$

$$=m(r,F)+N\left(r,\infty;(\mathcal{L}(f))^m\right)+m\left(r,(\mathcal{L}(f))^m\right)-N\left(r,0;(\mathcal{L}(f))^m\right)+S(r,f) \leq$$

$$\leq m(r,F)+mN(r,\infty;f)+mk\,\overline{N}(r,\infty;f)+m\left(r,\frac{(\mathcal{L}(f))^m}{f^m}\right)+$$

$$+m\left(r,f^m\right)-N\left(r,0;(\mathcal{L}(f))^m\right)+S(r,f)=$$

$$=m(r,F)+mT(r,f)+mk\,\overline{N}(r,\infty;f)-N\left(r,0;(\mathcal{L}(f))^m\right)+S(r,f).$$

Now

$$nT(r,f)=N\left(r,\infty;f^n\right)+m\left(r,f^n\right)\leq$$

$$\leq T(r,F)+mT(r,f)-mN(r,\infty;f)-N\left(r,0;(\mathcal{L}(f))^m\right)+S(r,f),$$

i.e.,

$$(n-m)T(r,f)\leq T(r,F)-mN(r,\infty;f)-N\left(r,0;(\mathcal{L}(f))^m\right)+S(r,f).$$

Lemma 8 is proved.

Lemma 9. *Let f be a transcendental meromorphic function and let $a(z)(\not\equiv 0,\infty)\in S(f)$. If $n>m+1$, then $f^n(\mathcal{L}(f))^m-a$ has infinitely many zeros, where $n,m,k\in\mathbb{N}$.*

Proof. Let $F=f^n(\mathcal{L}(f))^m$. Note that

$$T(r,F)=N(r,\infty;F)+m(r,F)\leq$$

$$\leq N\left(r,\infty;f^n\right)+N\left(r,\infty;(\mathcal{L}(f))^m\right)+m\left(r,f^{n+m}\right)+m\left(r,\left(\frac{\mathcal{L}(f)}{f}\right)^m\right)\leq$$

$$\leq nN(r,\infty;f)+mN(r,\infty;\mathcal{L}(f))+(n+m)m(r,f)+mm\left(r,\frac{\mathcal{L}(f)}{f}\right)\leq$$

$$\leq nN(r,\infty;f)+m(N(r,\infty;f)+k\,\overline{N}(r,\infty;f))+(n+m)m(r,f)+S(r,f)\leq$$

$$\leq (n+(k+1)m)N(r,\infty;f)+(n+m)m(r,f)+S(r,f)\leq$$

$$\leq (n+(k+1)m)T(r,f)+S(r,f). \tag{3.3}$$

Also by Lemma 8 we have

$$(n-m)T(r,f)\leq T(r,F)+S(r,f). \tag{3.4}$$

Since $n>m+1$, from (3.3) and (3.4) we conclude that $S(r,F)=S(r,f)$. Now we prove that $F-a$ has infinitely many zeros. If possible suppose $F-a$ has finitely many zeros. Then $N(r,a;F)=O(\log r)=S(r,f)=o(T(r,f))$. Now in view of Lemma 8, (3.3) and the second fundamental theorem for small functions (see [18]) we get

$$(n-m)T(r,f)\leq T(r,F)-mN(r,\infty;f)-N\left(r,0;(\mathcal{L}(f))^m\right)+S(r,f)\leq$$

$$\leq \overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,a;F)-m\,N(r,\infty;f)-N\left(r,0;(\mathcal{L}(f))^m\right)+$$

$$+(\varepsilon+o(1))T(r,F)+S(r,f)\leq$$

$$\leq\overline{N}(r,0;f)+\overline{N}\left(r,0;(\mathcal{L}(f))^m\right)+\overline{N}(r,\infty;f)-mN(r,\infty;f)-N\left(r,0;(\mathcal{L}(f))^m\right)+$$

$$+\varepsilon T(r,F)+o(T(r,F))+S(r,f)\leq$$

$$\leq N(r,0;f)+\varepsilon T(r,F)+S(r,F)+S(r,f)\leq$$

$$\leq T(r,f)+(n+(k+1)m)\varepsilon T(r,f)+\varepsilon S(r,f)+S(r,f)$$

for all $\varepsilon>0$. Therefore,

$$(n-m-1)T(r,f)\leq(n+(k+1)m)\varepsilon T(r,f)+S(r,f). \tag{3.5}$$

If we take $0<\varepsilon<\dfrac{n-m-1}{n+(k+1)m}$, then from (3.5) we arrive at a contradiction. Hence $F-a$ has infinitely many zeros.

Lemma 9 is proved.

Lemma 10 [10]**.** *Let f and g be two non-constant meromorphic functions. Suppose that f and g share 0 and ∞ CM, $f^{(k)}$ and $g^{(k)}$ share 0 CM for $k=1,2,\ldots,6$. Then f and g satisfy one of the following cases:*

(i) $f\equiv tg$, *where* $t\in\mathbb{C}\setminus\{0\}$;

(ii) $f(z)=e^{az+b}$ *and* $g(z)=e^{cz+d}$, *where* $a(\neq 0)$, b, $c(\neq 0)$, $d\in\mathbb{C}$;

(iii) $f(z)=\dfrac{a}{1-be^{\alpha(z)}}$ *and* $g(z)=\dfrac{a}{e^{-\alpha(z)}-b}$, *where* $a,b\in\mathbb{C}\setminus\{0\}$ *and* α *is a non-constant entire function*;

(iv) $f(z)=a\big(1-be^{cz}\big)$ *and* $g(z)=d\big(e^{-cz}-b\big)$, *where* $a,b,c,d\in\mathbb{C}\setminus\{0\}$.

Lemma 11. *Let f and g be two transcendental meromorphic functions having zeros of multiplicities at least $k\in\mathbb{N}$, $m,n\in\mathbb{N}$. Let $\mathcal{L}(f)$, $\mathcal{L}(g)$ share 0 CM and f, g share ∞ IM. If $f^n(\mathcal{L}(f))^m\equiv g^n(\mathcal{L}(g))^m$. Then $f\equiv tg$, where $t\in\mathbb{C}\setminus\{0\}$ such that $t^{n+m}=1$.*

Proof. Suppose

$$f^n(\mathcal{L}(f))^m\equiv g^n(\mathcal{L}(g))^m, \tag{3.6}$$

i.e.,

$$\frac{f^n}{g^n}\equiv\frac{(\mathcal{L}(f))^m}{(\mathcal{L}(g))^m}. \tag{3.7}$$

Since f and g share ∞ IM, it follows from (3.6) that f and g share ∞ CM and so $\mathcal{L}(f)$ and $\mathcal{L}(g)$ share ∞ CM. Again since $\mathcal{L}(f)$ and $\mathcal{L}(g)$ share 0 CM, it follows that f and g share 0 CM also. Let $h_1=\dfrac{f}{g}$ and $h_2=\dfrac{\mathcal{L}(f)}{\mathcal{L}(g)}$. Then $h_1\neq 0,\infty$ and $h_2\neq 0,\infty$. From (3.7) we see that

$$h_1^n h_2^m\equiv 1. \tag{3.8}$$

First we suppose h_1 is a non-constant entire function. Clearly h_2 is also a non-constant entire function. Let $F_1=h_1^n$ and $G_1=h_2^m$. Also from (3.8) we get

$$F_1G_1 \equiv 1. \tag{3.9}$$

Clearly $F_1 \not\equiv d_0G_1$, where $d_0 \in \mathbb{C}\setminus\{0\}$, otherwise $F_1 \in \mathbb{C}$ and so h_1 will be a constant. Since $F_1 \neq 0,\infty$ and $G_1 \neq 0,\infty$ then there exist two non-constant entire functions α and β such that $F_1 = e^{\alpha}$ and $G_1 = e^{\beta}$. Now from (3.9) we see that $\alpha+\beta = C$, where $C \in \mathbb{C}$. Therefore $\alpha' = -\beta'$. Note that $F_1' = \alpha' e^{\alpha}$ and $G_1' = \beta' e^{\beta}$. This shows that F_1' and G_1' share 0 CM. Note that $F_1 \neq 0,\infty$, $G_1 \neq 0,\infty$ and $F_1 \not\equiv d_0G_1$, where $d_0 \in \mathbb{C}\setminus\{0\}$. Now in view of Lemma 10 we have to consider the cases $F_1(z) = c_1e^{az}$ and $G_1(z) = c_2e^{-az}$, where $a, c_1, c_2 \in \mathbb{C}\setminus\{0\}$ such that $c_1c_2 = 1$. Since

$$\left(\frac{f(z)}{g(z)}\right)^n = c_1e^{az} \quad \text{and} \quad \left(\frac{\mathcal{L}(f(z))}{\mathcal{L}(g(z))}\right)^m = c_2e^{-az},$$

it follows that

$$\frac{f(z)}{g(z)} = t_1e^{\frac{a}{n}z} = t_1e^{cz} \quad \text{and} \quad \frac{\mathcal{L}(f(z))}{\mathcal{L}(g(z))} = t_2e^{-\frac{a}{m}z} = t_2e^{dz}, \tag{3.10}$$

where $c, d, t_1, t_2 \in \mathbb{C}\setminus\{0\}$ such that $t_1^n = c_1,\ t_2^m = c_2,\ c = \dfrac{a}{n}$ and $d = -\dfrac{a}{m}$. Let

$$\Phi_1 = \frac{\mathcal{L}'(f)}{\mathcal{L}(f)} - \frac{\mathcal{L}'(g)}{\mathcal{L}(g)}. \tag{3.11}$$

From (3.10), we see that

$$\Phi_1(z) = d. \tag{3.12}$$

Again from (3.10) we see that $f^{(j)}(z) = t_1\sum_{i=0}^{j} C_i^j (e^{cz})^{(i)} g^{(j-i)}(z)$, i.e.,

$$f^{(j)}(z) = t_1e^{cz}\left(g^{(j)}(z) + jcg^{(j-1)}(z) + \frac{j(j-1)}{2}\,c^2g^{(j-2)}(z) + \ldots + c^jg(z)\right).$$

Therefore

$$\mathcal{L}(f(z)) = t_1e^{cz}\Bigg(a_kg^{(k)}(z) + (kca_k + a_{k-1})g^{(k-1)}(z) +$$

$$+\left(\frac{k(k-1)}{2}\,c^2a_k + (k-1)ca_{k-1} + a_{k-2}\right)g^{(k-2)}(z) + \ldots\Bigg) \tag{3.13}$$

and

$$\mathcal{L}'(f(z)) = t_1e^{cz}\Bigg(a_kg^{(k+1)}(z) + \left((k+1)ca_k + a_{k-1}\right)g^{(k)}(z) +$$

$$+\left(\frac{k(k+1)}{2}\,c^2a_k + kca_{k-1} + a_{k-2}\right)g^{(k-1)}(z) + \ldots\Bigg). \tag{3.14}$$

Now from (3.11), (3.13) and (3.14), we have

$$\Phi_1 = \frac{G_2 + (k+1)cg^{(k)}g^{(k)} - kcg^{(k-1)}g^{(k+1)}}{G_3 + g^{(k)}g^{(k)}}, \tag{3.15}$$

where

$$G_2(z) = \sum_{\substack{0\le i\le k+1\\ 0\le j\le k\\ 0\le i+j\le 2k-1}} A_{i,j}g^{(i)}(z)g^{(j)}(z) \quad \text{and} \quad G_3(z) = \sum_{\substack{0\le i,j\le k\\ 0\le i+j\le 2k-1}} B_{i,j}g^{(i)}(z)g^{(j)}(z),$$

$A_{i,j}, B_{i,j} \in \mathbb{C}$. Let z_p be a zero of $g(z)$ with multiplicity $p(\ge k)$. Then the Taylor expansion of g about z_p is

$$g(z) = b_p(z-z_p)^p + b_{p+1}(z-z_p)^{p+1} + b_{p+2}(z-z_p)^{p+2} + \ldots, \quad b_p \ne 0. \tag{3.16}$$

We now consider following two cases.

Case 1. Suppose $p = k$. Then

$$g^{(k)}(z) = k!b_k + (k+1)!b_{k+1}(z-z_k) + \ldots \tag{3.17}$$

and

$$g^{(k+1)}(z) = (k+1)!b_{k+1} + (k+2)!b_{k+2}(z-z_k) + \ldots. \tag{3.18}$$

Now from (3.15), (3.17) and (3.18), we have

$$\Phi_1(z_k) = c\,\frac{(k+1)(k!)^2b_k^2}{(k!)^2b_k^2} = c(k+1). \tag{3.19}$$

Therefore, we arrive at a contradiction from (3.12) and (3.19).

Case 2. Suppose $p \ge k+1$. Then

$$g^{(k-1)}(z) = p(p-1)\ldots(p-k+2)b_p(z-z_p)^{(p-k+1)} + \ldots,$$

$$g^{(k)}(z) = p(p-1)\ldots(p-k+1)b_p(z-z_p)^{(p-k)} + \ldots,$$

and

$$g^{(k+1)}(z) = p(p-1)\ldots(p-k)b_p(z-z_p)^{(p-k-1)} + \ldots.$$

Therefore

$$g^{(k)}(z)g^{(k)}(z) = Kb_p^2(z-z_p)^{2p-2k} + \ldots, \tag{3.20}$$

$$g^{(k-1)}(z)g^{(k+1)}(z) = \frac{p-k}{p-k+1}\,Kb_p^2(z-z_p)^{2p-2k} + \ldots, \tag{3.21}$$

where $K = [p(p-1)\ldots(p-k+1)]^2$. Also

$$G_2(z) = O\left((z-z_p)^{2p-i-j}\right) \quad \text{and} \quad G_3(z) = O\left((z-z_p)^{2p-i-j}\right),$$

where $2p-2k+1 \le 2p-i-j \le 2p$. Now from (3.15), (3.20) and (3.21), we have

$$\Phi_1(z_p) = \frac{(k+1)cKb_p^2 - kc\,\dfrac{p-k}{p-k+1}\,Kb_p^2}{Kb_p^2} = c\,\frac{p+1}{p-k+1}. \tag{3.22}$$

Therefore we arrive at a contradiction from (3.12) and (3.22). Thus in either cases one can easily say that g has no zeros. Since f and g share 0 CM, it follows that f and g have no zeros. But this is impossible because zeros of f and g are of multiplicities at least $k \in \mathbb{N}$. Hence $h_1 \in \mathbb{C} \setminus \{0\}$. Then from (3.6) we get $h_1^{n+m} = 1$. Therefore, we have $f \equiv tg$, where $t \in \mathbb{C} \setminus \{0\}$ such that $t^{n+m} = 1$.

Lemma 11 is proved.

Lemma 12 [4]. *Let f be a meromorphic function on $\mathbb{C}$ with finitely many poles. If f has bounded spherical derivative on $\mathbb{C}$, then f is of order at most* 1.

Lemma 13 (Zalcman's [15, 23]). *Let F be a family of meromorphic functions in the unit disc Δ and α be a real number satisfying $-1 < \alpha < 1$. Then if F is not normal at a point $z_0 \in \Delta$ there exist for each α with $-1 < \alpha < 1$,*

(i) *points $z_n \in \Delta,\ z_n \to z_0$,*

(ii) *positive numbers $\rho_n,\ \rho_n \to 0^+$,*

(iii) *functions $f_n \in F$,*

such that $\rho_n^{-\alpha} f_n(z_n + \rho_n\zeta) \to g(\zeta)$ spherically uniformly on compact subset of $\mathbb{C}$, where g is a non-constant meromorphic function. The function g may be taken to satisfy the normalisation $g^{\#}(\zeta) \le g^{\#}(0) = 1,\ \zeta \in \mathbb{C}$.

Lemma 14. *Let f, g be two transcendental meromorphic functions having zeros of multiplicities at least $k \in \mathbb{N}$, and let $f^n(\mathcal{L}(f))^m - p$, $g^n(\mathcal{L}(g))^m - p$ share 0 CM and f, g share ∞ IM, where $p(z)(\not\equiv 0)$ is a polynomial and $m, n \in \mathbb{N}$. Then*

$$f^n(\mathcal{L}(f))^m g^n(\mathcal{L}(g))^m \not\equiv p^2.$$

Proof. Suppose

$$f^n(\mathcal{L}(f))^m g^n(\mathcal{L}(g))^m \equiv p^2. \tag{3.23}$$

Since f and g share ∞ IM, from (3.23) one can easily say that f and g are transcendental entire functions. We consider the following cases.

Case 1. Let $\deg(p) \in \mathbb{N}$. Now from (3.23) it follows that $N(r,0;f) = O(\log r)$ and $N(r,0;g) = O(\log r)$. Let

$$F = \frac{f^n(\mathcal{L}(f))^m}{p} \quad \text{and} \quad G = \frac{g^n(\mathcal{L}(g))^m}{p}. \tag{3.24}$$

From (3.23) we get

$$FG \equiv 1. \tag{3.25}$$

If $F \equiv d_1 G$, $d_1 \in \mathbb{C} \setminus \{0\}$, then $F \in \mathbb{C} \setminus \{0\}$, which is impossible by Lemma 4. Hence $F \not\equiv d_1 G$. Let

$$\Phi = \frac{f^n(\mathcal{L}(f))^m - p}{g^n(\mathcal{L}(g))^m - p}. \tag{3.26}$$

Since f and g are transcendental entire functions, it follows that $f^n(\mathcal{L}(f))^m - p \neq \infty$ and $g^n(\mathcal{L}(g))^m - p \neq \infty$. Also since $f^n(\mathcal{L}(f))^m - p$ and $g^n(\mathcal{L}(g))^m - p$ share 0 CM, we deduce from (3.26) that

$$\Phi \equiv e^{\beta}, \tag{3.27}$$

where β is an entire function. Let $f_1 = F$, $f_2 = -e^{\beta}G$ and $f_3 = e^{\beta}$. Here f_1 is transcendental. Now from (3.27), we have $f_1 + f_2 + f_3 \equiv 1$. Hence, by Lemma 6, we get

$$\sum_{j=1}^{3} N(r,0;f_j) + 2\sum_{j=1}^{3} \overline{N}(r,\infty;f_j) \leq$$

$$\leq N(r,0;F) + N(r,0;e^{\beta}G) + O(\log r) \leq (\lambda + o(1))T_1(r)$$

as $r \to +\infty$, $r \in I$, $\lambda < 1$. So, by Lemma 5, we get either $e^{\beta}G \equiv -1$ or $e^{\beta} \equiv 1$. But here the only possibility is that $e^{\beta}G \equiv -1$, i.e., $g^n(\mathcal{L}(g))^m \equiv -e^{-\beta}p$ and so from (3.23) we obtain $F \equiv e^{\gamma_1}G$, i.e., $f^n(\mathcal{L}(f))^m \equiv e^{\gamma_1}g^n(\mathcal{L}(g))^m$, where γ_1 is a non-constant entire function. Then, from (3.23), we get

$$f^n(\mathcal{L}(f))^m \equiv d_2 e^{\frac{1}{2}\gamma_1}p \quad \text{and} \quad g^n(\mathcal{L}(g))^m \equiv d_2 e^{-\frac{1}{2}\gamma_1}p, \tag{3.28}$$

where $d_2 = \pm 1$. This shows that $f^n(\mathcal{L}(f))^m$ and $g^n(\mathcal{L}(f))^m$ share 0 CM. Clearly, from (3.28), we see F and G are entire functions having no zeros.

Let z_p be a zero of $f(z)$ of multiplicity $p(\geq k)$ and z_q be a zero of $g(z)$ of multiplicity $q(\geq k)$. Clearly z_p will be a zero of $f^n(\mathcal{L}(f))^m$ of multiplicity $(n+1)p - k$ and z_q will be a zero of $g^n(\mathcal{L}(g))^m$ of multiplicity $(n+1)q - k$. Since $f^n(\mathcal{L}(f))^m$ and $g^n(\mathcal{L}(g))^m$ share 0 CM, it follows that $z_p = z_q$ and $p = q$. Consequently $f(z)$ and $g(z)$ share 0 CM. Since $N(r,0;f) = O(\log r)$ and $N(r,0;g) = O(\log r)$, so we can take

$$f(z) = h(z)e^{\alpha(z)} \quad \text{and} \quad g(z) = h(z)e^{\beta(z)}, \tag{3.29}$$

where $h(z)$ is a non-constant polynomial and α, β are two non-constant entire functions.

We deduce from (3.29) that

$$f^n(\mathcal{L}(f))^m \equiv P_1\left(h, h', \ldots, h^{(k)}, \alpha', \alpha'', \ldots, \alpha^{(k)}\right) e^{(n+m)\alpha}, \tag{3.30}$$

where

$$P_1\left(h, h', \ldots, h^{(k)}, \alpha', \alpha'', \ldots, \alpha^{(k)}\right) = h^n \left(\sum_{i=0}^{k} a_i P_{1i}\left(h, h', \ldots, h^{(i)}, \alpha', \alpha'', \ldots, \alpha^{(i)}\right)\right)^m,$$

$P_{1i}\big(h, h', \ldots, h^{(i)}, \alpha', \alpha'', \ldots, \alpha^{(i)}\big)$ is a differential polynomial in $h, h', \ldots, h^{(i)}, \alpha', \alpha'', \ldots, \alpha^{(i)}$, $i = 1, \ldots, k$, $P_{10} = a_0 h$ and

$$g^n(\mathcal{L}(g))^m \equiv P_2\left(h, h', \ldots, h^{(k)}, \beta', \beta'', \ldots, \beta^{(k)}\right) e^{(n+m)\beta}, \tag{3.31}$$

where

$$P_2\left(h,h',\ldots,h^{(k)},\beta',\beta'',\ldots,\beta^{(k)}\right)=h^n\left(\sum_{i=0}^{k}a_iP_{2i}\left(h,h',\ldots,h^{(i)},\beta',\beta'',\ldots,\beta^{(i)}\right)\right)^m,$$

$P_{2i}\big(h,h',\ldots,h^{(i)},\beta',\beta'',\ldots,\beta^{(i)}\big)$ is a differential polynomial in $h,h',\ldots,h^{(i)},\beta',\beta'',\ldots,\beta^{(i)}$, $i=1,\ldots,k$, $P_{20}=a_0h$. Let $\mathcal{F}=\{F_\omega\}$ and $\mathcal{G}=\{G_\omega\}$, where $F_\omega(z)=F(z+\omega)$ and $G_\omega(z)=$ $=G(z+\omega)$, $z\in\mathbb{C}$. Clearly $\mathcal{F}$ and $\mathcal{G}$ are two families of entire functions defined on $\mathbb{C}$. We now consider following two subcases.

Subcase 1.1. Suppose that one of the families $\mathcal{F}$ and $\mathcal{G}$, say $\mathcal{F}$, is normal on $\mathbb{C}$. Then by Marty's theorem $F^{\#}(\omega)=F^{\#}_\omega(0)\le M$ for some $M>0$ and for all $\omega\in\mathbb{C}$. Hence, by Lemma 12, we have F is of order at most 1. Now from (3.25), we obtain

$$\rho\left(f^n(\mathcal{L}(f))^m\right)=\rho(F)=\rho(G)=\rho\left(g^n(\mathcal{L}(g))^m\right)\le 1. \tag{3.32}$$

Consequently we get

$$f^n(z)(\mathcal{L}(f(z)))^m=d_3pe^{az}\quad\text{and}\quad g^n(z)(\mathcal{L}(g(z)))^m=d_4pe^{bz}, \tag{3.33}$$

where $a,b,d_3,d_4\in\mathbb{C}\setminus\{0\}$. From (3.23) we see that $a+b=0$. We claim that $(n+m)\alpha(z)-az\in\mathbb{C}$ and $(n+m)\beta(z)-bz\in\mathbb{C}$. If possible suppose $(n+m)\alpha(z)-az\notin\mathbb{C}$ and $(n+m)\beta(z)-bz\notin\mathbb{C}$. Let $\alpha_1(z)=(n+m)\alpha(z)-az$ and $\beta_1(z)=(n+m)\beta(z)-bz$. Note that

$$T\left(r,\alpha'\right)=m\left(r,\alpha'\right)\le m\left(r,(n+m)\alpha'\right)+O(1)=m\left(r,\alpha_1'+a\right)+O(1)\le$$

$$\le m\left(r,\alpha_1'\right)+O(1)=m\left(\frac{(e^{\alpha_1})'}{e^{\alpha_1}}\right)+O(1)=S\left(r,e^{\alpha_1}\right).$$

Clearly $\alpha^{(i)}\in S(\alpha_1)$ for $i\in\mathbb{N}$. Therefore $P_1\in S(\alpha_1)$ and so $\dfrac{p}{P_1}\in S(\alpha_1)$. Similarly we have $\dfrac{p}{P_2}\in S(\beta_1)$. Now from (3.30), (3.31) and (3.33), we conclude that $e^{\alpha_1}\in S\big(e^{\alpha_1}\big)$ and $e^{\beta_1}\in S\big(e^{\beta_1}\big)$, which is a contradiction. Hence $\alpha_1,\beta_1\in\mathbb{C}$ and so both α and β are polynomials of degree 1. Finally, we take

$$f(z)=d_5h(z)e^{az}\quad\text{and}\quad g(z)=d_6h(z)e^{-az}, \tag{3.34}$$

where $d_5,d_6\in\mathbb{C}\setminus\{0\}$. Now from (3.34), we get

$$f^n(z)(\mathcal{L}(f(z)))^m=d_5^{n+m}h^n(z)\left(a_0h(z)+\sum_{j=1}^{k}a_j\left(\sum_{i=0}^{j}C_i^j\,a^{j-i}\,h^{(i)}(z)\right)\right)^m e^{(n+m)az},$$

where we define $h^{(0)}(z)=h(z)$. Similarly we obtain

$$g^n(z)(\mathcal{L}(g(z)))^m=$$

$$=d_6^{n+m}h^n(z)\left(a_0h(z)+\sum_{j=1}^{k}a_j\left(\sum_{i=0}^{j}C_i^j\,(-1)^{j-i}\,a^{j-i}\,h^{(i)}(z)\right)\right)^m e^{-(n+m)az}.$$

Since $f^n(\mathcal{L}(f))^m$ and $g^n(\mathcal{L}(g))^m$ share 0 CM, it follows that

$$a_0h(z)+\sum_{j=1}^{k}a_j\left(\sum_{i=0}^{j}C_i^j a^{j-i}\,h^{(i)}(z)\right)\equiv$$

$$\equiv d_7\left(a_0h(z)+\sum_{j=1}^{k}a_j\left(\sum_{i=0}^{j}C_i^j(-1)^{j-i}a^{j-i}\,h^{(i)}(z)\right)\right), \tag{3.35}$$

where $d_7\in\mathbb{C}\setminus\{0\}$. But the relation (3.35) does not hold.

Subcase 1.2. Suppose that one of the families $\mathcal{F}$ and $\mathcal{G}$, say $\mathcal{F}$ is not normal on $\mathbb{C}$. Now by Marty's theorem there exists a sequence of meromorphic functions $\{F(z+\omega_j)\}\subset\mathcal{F}$, where $z\in\{z:|z|<1\}$ and $\{\omega_j\}\subset\mathbb{C}$ is some sequence such that $F^{\#}(\omega_j)\to\infty$, as $|\omega_j|\to\infty$. Then by Lemma 13 there exist:

(i) points $z_j,\ |z_j|<1$,

(ii) positive numbers $\rho_j,\ \rho_j\to 0^+$,

(iii) a subsequence $\{F(\omega_j+z_j+\rho_j\zeta)\}$ of $\{F(\omega_j+z)\}$

such that

$$\hat{h}_j(\zeta)=\rho_j^{-\frac{1}{2}}F(\omega_j+z_j+\rho_j\zeta)\to\hat{h}(\zeta) \tag{3.36}$$

spherically uniformly on compact subset of $\mathbb{C}$, where $\hat{h}(\zeta)$ is non-constant holomorphic function such that $\hat{h}^{\#}(\zeta)\le\hat{h}^{\#}(0)=1$. Now from Lemma 12 we see that $\rho(\hat{h})\le 1$. By Hurwitz's theorem we can see that $\hat{h}(\zeta)\ne 0$. In the proof of Zalcman's lemma (see [15, 23]) we see that

$$\rho_j=\frac{1}{F^{\#}(b_j)} \tag{3.37}$$

and

$$F^{\#}(b_j)\ge F^{\#}(\omega_j), \tag{3.38}$$

where $b_j=\omega_j+z_j$. Let

$$\check{h}_j(\zeta)=\rho_j^{\frac{1}{2}}G(\omega_j+z_j+\rho_j\zeta). \tag{3.39}$$

(3.25) yields $F(\omega_j+z_j+\rho_j\zeta)G(\omega_j+z_j+\rho_j\zeta)\equiv 1$ and so, from (3.36) and (3.39), we get

$$\hat{h}_j(\zeta)\check{h}_j(\zeta)\equiv 1. \tag{3.40}$$

Now, from (3.36) and (3.40), we can deduce that

$$\check{h}_j(\zeta)\to\check{h}(\zeta) \tag{3.41}$$

spherically uniformly on compact subset of $\mathbb{C}$, where $\check{h}(\zeta)$ is some non-constant holomorphic function in the complex plane. By Hurwitz's theorem we can see that $\check{h}(\zeta)\ne 0$. From (3.36), (3.40) and (3.41), we get

$$\hat{h}(\zeta)\check{h}(\zeta)\equiv 1. \tag{3.42}$$

Now, from (3.42) and $\rho(\hat{h}) \leq 1$, we see that

$$\rho(\hat{h}) = \rho(\check{h}) \leq 1. \tag{3.43}$$

Noting that $\hat{h}$ and $\check{h}$ are transcendental entire functions having no zeros, we observe from (3.43) that

$$\hat{h}(z) = d_8 e^{cz} \quad \text{and} \quad \check{h}(z) = d_9 e^{-cz}, \tag{3.44}$$

where $c, d_8, d_9 \in \mathbb{C} \setminus \{0\}$ such that $d_8 d_9 = 1$. Also from (3.44), we have

$$\frac{\hat{h}_j'(\zeta)}{\hat{h}_j(\zeta)} = \rho_j \frac{F'(w_j + z_j + \rho_j \zeta)}{F(w_j + z_j + \rho_j \zeta)} \to \frac{\hat{h}'(\zeta)}{\hat{h}(\zeta)} = c, \tag{3.45}$$

spherically uniformly on compact subset of $\mathbb{C}$. Now from (3.37) and (3.45), we obtain

$$\left|\frac{\hat{h}_j'(0)}{\hat{h}_j(0)}\right| = \rho_j \left|\frac{F'(\omega_j + z_j)}{F(\omega_j + z_j)}\right| = \frac{1 + |F(\omega_j + z_j)|^2}{|F'(\omega_j + z_j)|} \frac{|F'(\omega_j + z_j)|}{|F(\omega_j + z_j)|} =$$

$$= \frac{1 + |F(\omega_j + z_j)|^2}{|F(\omega_j + z_j)|} \to \left|\frac{\hat{h}'(0)}{\hat{h}(0)}\right| = |c|, \tag{3.46}$$

which implies that

$$\lim_{j \to \infty} F(\omega_j + z_j) \neq 0, \infty. \tag{3.47}$$

From (3.36) and (3.47) we see that

$$\hat{h}_j(0) = \rho_j^{-\frac{1}{2}} F(\omega_j + z_j) \to \infty. \tag{3.48}$$

Again from (3.36) and (3.44), we have

$$\hat{h}_j(0) \to \hat{h}(0) = c_1. \tag{3.49}$$

Now from (3.48) and (3.49) we arrive at a contradiction.

Case 2. Let $p \in \mathbb{C} \setminus \{0\}$. Then from (3.23) we get $f^n(\mathcal{L}(f))^m g^n(\mathcal{L}(g))^m \equiv b^2$, where f and g are transcendental entire functions. Clearly f and g have no zeros. But this is impossible because zeros of f and g are of multiplicities at least $k \in \mathbb{N}$.

Lemma 14 is proved.

Lemma 15. *Let f, g be two transcendental meromorphic functions having zeros of multiplicities at least $k \in \mathbb{N}$ and let $F = \dfrac{f^n(\mathcal{L}(f))^m}{p}$, $G = \dfrac{g^n(\mathcal{L}(g))^m}{p}$, where $p(z)(\not\equiv 0)$ is a polynomial and $m, n \in \mathbb{N}$ such that $n > \dfrac{mk + k^2 + k + 2}{k}$. Suppose $f^n(\mathcal{L}(f))^m - p$, $g^n(\mathcal{L}(g))^m - p$ share $(0, k_1)$ where $k_1 \in \mathbb{N} \cup \{0\} \cup \{\infty\}$ and f, g share $(\infty, 0)$. If $H \equiv 0$, then either $f^n(\mathcal{L}(f))^m g^n(\mathcal{L}(g))^m \equiv p^2$, where $f^n(\mathcal{L}(f))^m - p$, $g^n(\mathcal{L}(g))^m - p$ share 0 CM or $f^n(\mathcal{L}(f))^m \equiv g^n(\mathcal{L}(g))^m$.*

Proof. Since $H \equiv 0$, by integration, we get $\dfrac{F'}{(F-1)^2} = d_{10}\,\dfrac{G'}{(G-1)^2}$, where $d_{10} \in \mathbb{C} \setminus \{0\}$, i.e.,

$$\frac{\left(\dfrac{F_1 - p}{p}\right)'}{\left(\dfrac{F_1 - p}{p}\right)^2} = d_{10}\,\frac{\left(\dfrac{G_1 - p}{p}\right)'}{\left(\dfrac{G_1 - p}{p}\right)^2},$$

where $F_1 = f^n(\mathcal{L}(f))^m$ and $G_1 = f^n(\mathcal{L}(g))^m$. This shows that $\dfrac{F_1 - p}{p}$ and $\dfrac{G_1 - p}{p}$ share 0 CM. Since $F_1 - p$ and $G_1 - p$ share $(0, k_1)$, it follows that $F_1 - p$ and $G_1 - p$ share 0 CM. Finally by integration we get

$$\frac{1}{F-1} \equiv \frac{d_{12}G + d_{11} - d_{12}}{G-1}, \tag{3.50}$$

where $d_{11}(\neq 0), d_{12} \in \mathbb{C}$. We now consider the following cases.

Case 1. Let $d_{12} \neq 0$ and $d_{11} \neq d_{12}$. If $d_{12} = -1$, then from (3.50) we have

$$F \equiv \frac{-d_{11}}{G - d_{11} - 1}.$$

Therefore $\overline{N}(r, a+1; G) = \overline{N}(r, \infty; F) = \overline{N}(r, \infty; f) + \overline{N}(r, 0; p)$. Now in view of Lemma 8 and the second fundamental theorem we get

$$(n-m)T(r,g) \leq T(r,G) - mN(r,\infty;g) - N\left(r, 0; (\mathcal{L}(g))^m\right) + S(r,g) \leq$$

$$\leq \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r,a+1;G) - mN(r,\infty;g) - N\left(r,0;(\mathcal{L}(g))^m\right) + S(r,g) \leq$$

$$\leq \overline{N}(r,0;g) + \overline{N}\left(r,0;(\mathcal{L}(g))^m\right) + \overline{N}(r,\infty;f) - N\left(r,0;(\mathcal{L}(g))^m\right) + S(r,g) \leq$$

$$\leq \overline{N}(r,0;g) + \overline{N}(r,\infty;g) + S(r,g) \leq$$

$$\leq \frac{1}{k}\,N(r,0;g) + N(r,\infty;g) + S(r,g) \leq \frac{k+1}{k}\,T(r,g) + S(r,g),$$

which is contradiction since $n > \dfrac{mk+k+1}{k}$.

If $d_{12} \neq -1$, from (3.50) we obtain

$$F - \left(1 + \frac{1}{d_{12}}\right) \equiv \frac{-d_{11}}{d_{12}^2\left(G + \dfrac{d_{11} - d_{12}}{d_{12}}\right)}.$$

So, $\overline{N}\left(r, \dfrac{d_{12} - d_{11}}{d_{12}}; G\right) = \overline{N}(r,\infty;F) = \overline{N}(r,\infty;f) + \overline{N}(r,0;p)$. By using Lemma 8 and the same argument as used in the case when $d_{12} = -1$, we can get a contradiction.

Case 2. Let $d_{12} \neq 0$ and $d_{11} = d_{12}$. If $d_{12} = -1$, then from (3.50) we have $FG \equiv 1$, i.e., $f^n(\mathcal{L}(f))^m g^n(\mathcal{L}(g))^m \equiv p^2$, where $f^n(\mathcal{L}(f))^m - p$ and $g^n(\mathcal{L}(g))^m - p$ share 0 CM.

If $d_{12} \neq -1,$ from (3.50) we have

$$\frac{1}{F} \equiv \frac{d_{12}G}{(1+d_{12})G-1}.$$

Therefore $\overline{N}\left(r, \dfrac{1}{1+d_{12}}; G\right) = \overline{N}(r,0;F).$ So, in view of Lemmas 1 and 8 and the second fundamental theorem we get

$$(n-m)T(r,g) \leq T(r,G) - mN(r,\infty;g) - N\left(r,0;(\mathcal{L}(g))^m\right) + S(r,g) \leq$$

$$\leq \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}\left(r, \frac{1}{1+d_{12}}; G\right) - m\,N(r,\infty;g) -$$

$$-N\left(r,0;(\mathcal{L}(g))^m\right) + S(r,g) \leq \overline{N}(r,0;g) +$$

$$+\overline{N}\left(r,0;(\mathcal{L}(g))^m\right) + \overline{N}(r,0;F) - N\left(r,0;(\mathcal{L}(g))^m\right) + S(r,g) \leq$$

$$\leq \overline{N}(r,0;g) + \overline{N}(r,0;f) + \overline{N}(r,0;\mathcal{L}(f)) + S(r,g) \leq$$

$$\leq \overline{N}(r,0;g) + \overline{N}(r,0;f) + N_{k+1}(r,0;f) + k\overline{N}(r,\infty;f) + S(r,g) \leq$$

$$\leq \frac{1}{k}\,T(r,g) + \frac{1}{k}\,T(r,f) + T(r,f) + k\,T(r,f) + S(r,f) + S(r,g).$$

We suppose that there exists a set I with infinite measure such that $T(r,f) \leq T(r,g)$ for $r \in I$ and so for $r \in I$ we have $(n-m)\; T(r,g) \leq \dfrac{k^2+k+2}{k}\; T(r,g) + S(r,g),$ which is a contradiction since $n > \dfrac{mk+k^2+k+2}{k}.$

Case 3. Let $d_{12} = 0.$ From (3.50) we obtain

$$F \equiv \frac{G+d_{11}-1}{d_{11}}.$$

If $d_{11} \neq 1$ then we obtain $\overline{N}(r, 1-d_{11}; G) = \overline{N}(r,0;F).$ We can similarly deduce a contradiction as in Case 2. Therefore $d_{11} = 1$ and so we obtain $F \equiv G,$ i.e., $f^n(\mathcal{L}(f))^m \equiv g^n(\mathcal{L}(g))^m.$

Lemma 15 is proved.

Lemma 16 [1]. *Let f and g be non-constant meromorphic functions sharing $(1,k_1),$ where $2 \leq k_1 \leq \infty.$ Then*

$$\overline{N}(r,1;f\mid=2) + 2\,\overline{N}(r,1;f\mid=3) + \ldots + (k_1-1)\,\overline{N}(r,1;f\mid=k_1) + k_1\,\overline{N}_L(r,1;f) +$$

$$+(k_1+1)\,\overline{N}_L(r,1;g) + k_1\,\overline{N}_E^{(k_1+1}(r,1;g) \leq N(r,1;g) - \overline{N}(r,1;g).$$

4. Proof of Theorem 1. Let $F = \dfrac{f^n(\mathcal{L}(f))^m}{p}$ and $G = \dfrac{g^n(\mathcal{L}(g))^m}{p}.$ Clearly $F,$ G share $(1,k_1)$ except for the zeros of p and $f,$ g share $(\infty,0).$

Case 1. Let $H \not\equiv 0$.

From (3.1) it can be easily calculated that the possible poles of H occur at (i) multiple zeros of F and G, (ii) those 1 points of F and G whose multiplicities are different, (iii) those poles of F and G whose multiplicities are different, (iv) zeros of F' which are not the zeros of $F(F-1)$, (v) zeros of G' which are not the zeros of $G(G-1)$. Since H has only simple poles we get

$$N(r,\infty;H) \leq \overline{N}_*(r,\infty;f,g) + \overline{N}_*(r,1;F,G) + \overline{N}(r,0;F \mid\geq 2) + \overline{N}(r,0;G \mid\geq 2)+$$

$$+\overline{N}_0(r,0;F') + \overline{N}_0(r,0;G'), \tag{4.1}$$

where $\overline{N}_0(r,0;F')$ is the reduced counting function of those zeros of F' which are not the zeros of $F(F-1)$ and $\overline{N}_0(r,0;G')$ is similarly defined. Now from Nevanlinna's fundamental estimate of the logarithmic derivative we obtain $m(r,H) = S(r,F) + S(r,G)$. Since

$$T(r,F) \leq [n+(k+1)m]\,T(r,f) + S(r,f), \quad T(r,G) \leq [n+(k+1)m]\,T(r,g) + S(r,g),$$

it follows that

$$m(r,H) = S(r,f) + S(r,g).$$

Let z_0 be a simple zero of $F-1$ but $p(z_0) \neq 0$. Clearly z_0 is a simple zero of $G-1$. Then an elementary calculation gives that $H(z) = O(z-z_0)$, which proves that z_0 is a zero of H. By the first fundamental theorem of Nevanlinna we get

$$N(r,1;F \mid= 1) \leq N(r,0;H) \leq T(r,H) + O(1) =$$

$$= N(r,\infty;H) + m(r,H) + O(1) \leq N(r,\infty;H) + S(r,f) + S(r,g). \tag{4.2}$$

By using (4.1) and (4.2), we obtain

$$\overline{N}(r,1;F) \leq N(r,1;F \mid= 1) + \overline{N}(r,1;F \mid\geq 2) \leq$$

$$\leq \overline{N}_*(r,\infty;f,g) + \overline{N}(r,0;F \mid\geq 2) + \overline{N}(r,0;G \mid\geq 2) + \overline{N}_*(r,1;F,G)+$$

$$+\overline{N}(r,1;F \mid\geq 2) + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G') + S(r,f) + S(r,g) \leq$$

$$\leq \overline{N}(r,\infty;f) + \overline{N}(r,0;F \mid\geq 2) + \overline{N}(r,0;G \mid\geq 2) + \overline{N}_*(r,1;F,G)+$$

$$+\overline{N}(r,1;F \mid\geq 2) + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G') + S(r,f) + S(r,g). \tag{4.3}$$

Now in view of Lemmas 2 and 16 we have

$$\overline{N}_0(r,0;G') + \overline{N}(r,1;F \mid\geq 2) + \overline{N}_*(r,1;F,G) \leq$$

$$\leq \overline{N}_0(r,0;G') + \overline{N}(r,1;F \mid= 2) + \overline{N}(r,1;F \mid= 3) + \ldots + \overline{N}(r,1;F \mid= k_1)+$$

$$+\overline{N}_E^{(k_1+1}(r,1;F) + \overline{N}_L(r,1;F) + \overline{N}_L(r,1;G) + \overline{N}_*(r,1;F,G) \leq$$

$$\leq \overline{N}_0(r,0;G') - \overline{N}(r,1;F \mid= 3) - \ldots - (k_1-2)\overline{N}(r,1;F \mid= k_1)-$$

$$-(k_1-1)\overline{N}_L(r,1;F)-k_1\overline{N}_L(r,1;G)-(k_1-1)\overline{N}_E^{(k_1+1}(r,1;F)+$$

$$+N(r,1;G)-\overline{N}(r,1;G)+\overline{N}_*(r,1;F,G)\leq$$

$$\leq\overline{N}_0(r,0;G')+N(r,1;G)-\overline{N}(r,1;G)-(k_1-2)\overline{N}_L(r,1;F)-(k_1-1)\overline{N}_L(r,1;G)\leq$$

$$\leq N(r,0;G'\mid G\neq 0)-(k_1-2)\overline{N}_L(r,1;F)-(k_1-1)\overline{N}_L(r,1;G)\leq$$

$$\leq\overline{N}(r,0;G)+\overline{N}(r,\infty;g)-(k_1-2)\overline{N}_L(r,1;F)-(k_1-1)\overline{N}_L(r,1;G)=$$

$$=\overline{N}(r,0;G)+\overline{N}(r,\infty;g)-(k_1-2)\overline{N}_*(r,1;F,G)-\overline{N}_L(r,1;G). \tag{4.4}$$

Hence, by using (4.3), (4.4) and Lemma 1, we get from second fundamental theorem that

$$T(r,F)\leq\overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,1;F)-N_0(r,0;F')\leq$$

$$\leq 2\,\overline{N}(r,\infty,f)+N_2(r,0;F)+\overline{N}(r,0;G\mid\geq 2)+\overline{N}(r,1;F\mid\geq 2)+$$

$$+\overline{N}_*(r,1;F,G)+\overline{N}_0(r,0;G')+S(r,f)+S(r,g)\leq$$

$$\leq 3\,\overline{N}(r,\infty;f)+N_2(r,0;F)+N_2(r,0;G)-(k_1-2)\,\overline{N}_*(r,1;F,G)+$$

$$+S(r,f)+S(r,g)\leq 3\,\overline{N}(r,\infty;f)+2\,\overline{N}(r,0;f)+N_2\left(r,0;(\mathcal{L}(f))^m\right)+$$

$$+2\overline{N}(r,0;g)+mN_2(r,0;\mathcal{L}(g))-(k_1-2)\,\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)\leq$$

$$\leq 3\overline{N}(r,\infty;f)+2\overline{N}(r,0;f)+N\left(r,0;(\mathcal{L}(f))^m\right)+2\overline{N}(r,0;g)+$$

$$+m\;N_{k+2}(r,0;g)+mk\,\overline{N}(r,\infty;g)-(k_1-2)\,\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)\leq$$

$$\leq(3+mk)\,\overline{N}(r,\infty;f)+2\overline{N}(r,0;f)+2\overline{N}(r,0;g)+mN(r,0;g)+$$

$$+N\left(r,0;(\mathcal{L}(f))^m\right)-(k_1-2)\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g). \tag{4.5}$$

Now, by using Lemmas 7 and 8, we get from (4.5)

$$(n-m)T(r,f)\leq T(r,F)-mN(r,\infty;f)-N\left(r,0;(\mathcal{L}(f))^m\right)+S(r,f)\leq$$

$$\leq(3+(k-1)m)\,\overline{N}(r,\infty;f)+2\,\overline{N}(r,0;f)+2\,\overline{N}(r,0;g)+m\,N(r,0;g)-$$

$$-(k_1-2)\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)\leq$$

$$\leq\frac{(k+1)(3+(k-1)m)}{k(n+m+(m-2)k-1)}\,(T(r,f)+T(r,g))+$$

$$+\frac{2}{k}\,(T(r,f)+T(r,g))+\frac{3+(k-1)m}{n+m+(m-2)k-1}\,\overline{N}_*(r,1;F,G)+$$

$$+mT(r,g)-(k_1-2)\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)\le$$

$$\le\left(\frac{(mk+4)n+m^2k^2+\left(m^2+3m-2\right)k+2(m+1)}{k(n+m+(m-2)k-1)}\right)T(r)+S(r). \tag{4.6}$$

In a similar way we can obtain

$$(n-m)\,T(r,g)\le\left(\frac{(mk+4)n+m^2k^2+\left(m^2+3m-2\right)k+2(m+1)}{k(n+m+(m-2)k-1)}\right)T(r)+S(r). \tag{4.7}$$

Combining (4.6) and (4.7) we see that

$$(n-m)T(r)\le\left(\frac{(mk+4)n+m^2k^2+\left(m^2+3m-2\right)k+2(m+1)}{k(n+m+(m-2)k-1)}\right)T(r)+S(r),$$

i.e.,

$$(k(n-K_1)(n-K_2))\,T(r)\le S(r), \tag{4.8}$$

where

$$K_1=\frac{(2-m)k^2+(m+1)k+4+\sqrt{L_1}}{2k},\quad K_2=\frac{(2-m)k^2+(m+1)k+4-\sqrt{L_1}}{2k},$$

$$L_1=\left[(2-m)k^2+(m+1)k+4\right]^2+8k\left\{\left(m^2-m\right)k^2+\left(m^2+m-1\right)k+(m+1)\right\}=$$

$$=m^2k^4+9m^2k^2+2mk^2+6m^2k^3-6mk^3+$$

$$+4k^4(1-m)+16k(m+1)+9k^2+4k^3+16<$$

$$<m^2k^4+9m^2k^2+6m^2k^3+10mk^2-2mk^3+16(3m-1)k+$$

$$+k^2+64+8k^2(1-m)+4k^3(1-m)+32k(1-m)\le\left[mk^2+(3m-1)k+8\right]^2.$$

Therefore,

$$K_1<\frac{(2-m)k^2+(m+1)k+4+mk^2+(3m-1)k+8}{2k}=\frac{k^2+2mk+6}{k}.$$

Since $n\ge\dfrac{k^2+2mk+6}{k}$, (4.8) leads to a contradiction.

Case 2. Let $H\equiv 0$. Then theorem follows from Lemmas 15, 11 and 14.

Theorem 1 is proved.

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