DOI: 10.37863/umzh.v73i3.152

UDC 517.9

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SOLUTIONS OF SYLVESTER EQUATION IN C^*-MODULAR OPERATORS
РОЗВ'ЯЗКИ РІВНЯННЯ СІЛЬВЕСТРА ДЛЯ C^*-МОДУЛЬНИХ ОПЕРАТОРІВ

We study the solvability of the Sylvester equation $AX + YB = C$ and the operator equation $AXD + FYB = C$ in the general setting of the adjointable operators between Hilbert C^*-modules. Based on the Moore – Penrose inverses of the associated operators, we propose necessary and sufficient conditions for the existence of solutions to these equations, and obtain the general expressions of the solutions in the solvable cases. We also provide an approach to the study of the positive solutions for a special case of Lyapunov equation.

Розглянуто розв'язність рівняння Сільвестра $AX + YB = C$ та операторного рівняння $AXD + FYB = C$ при загальних умовах суміжності операторів між гільбертовими C^*-модулями. На основі обернених Мура – Пенроуза для пов'язаних операторів отримано необхідні та достатні умови існування розв'язків цих рівнянь, а також загальні вирази для розв'язків у випадку, коли вони існують. Крім того, запропоновано підхід до вивчення додатних розв'язків у спеціальному випадку рівняння Ляпунова.

1. Introduction. In the year 2001, the Sylvester equation $CX - XA^T = B$ was studied for matrices by [12]. Thereafter, more general equation $AX - XF = BY$ was considered in [24]. The generalized Sylvester equation $AV + BW = EVJ + R$ with unknown matrices V and W, has many applications in linear systems theory [7, 19]. One special and important case is the Lyapunov matrix equation $AX + X^TC = B$, which has important applications in the control theory and robust fault detection [8]. In 2007, Piao et al. [18] studied this equation when A and C are square matrices with different dimensions. Mor et al. [16] obtained the explicit solution and eigenvalue bounds for the Lyapunov matrix equation $A^TP + PA = -BB^T$ and determined the number of positive eigenvalues of the positive semidefinite solution through the controllability matrix. The other special case of interest is the equation $A^*X + X^*A = B$. It was studied for matrices by Braden [3], and for the Hilbert space operators by Djordjević [6]. Fang and Yu [9] investigated the solvability of the operator equations $A^*X + X^*A = C$ and $A^*XB + B^*X^*A = C$ for adjointable operators on Hilbert C^*-modules whose ranges may not be closed. On the other hand, Mousavi et al. [17] studied the operator equations $AX + YB = C$ and $AXA^* + BYB^* = C$ in Hilbert C^*-modules.

In this paper, by using the matrix forms of adjointable operators between Hilbert C^*-modules, we first propose necessary and sufficient conditions for the existence of solutions to the Sylvester equation

$$AX + YB = C, \tag{1.1}$$

and then with the help of that we describe necessary and sufficient conditions for the existence of solution to the operator equation

$$AXD + FYB = C, \tag{1.2}$$

when A, B, D and F are bounded adjointable operators with closed range between Hilbert C^*-modules. Moreover, we obtain the explicit solutions to these operator equations in the solvable cases. The results obtained in this paper generalize earlier results due to Djordjević [6].

It is a very active topic to study positive solutions to matrix equations or positive solutions to operator equations. The positive solutions to the operator equations $AX = C$ and $XB = D$ were studied by Dajić et al. [5] for Hilbert space operators. In 2008, Xu [20] considered the Hermitian and positive solutions to these equations in Hilbert C^*-modules setting. Also, Hermitian positive semidefinite solution to the matrix equation $AXB = C$ was studied by Khatri and Mitra [11] and then by Zhang [23] in 2004. Cvetković – Ilić and Koliha [4] described the positive solution to the special case $AXA^* = B$ for elements of C^*-algebras. The other purpose of this work, is to provide an approach to the study of the positive solutions to the operator equation

$$AX + X^*A^* = B, \tag{1.3}$$

in the framework of Hilbert C^*-modules. The paper is organized as follows. In Section 2, we recall some knowledge about the Hilbert C^*-modules. For this purpose, we use [13 – 15, 21]. In Section 3, we study the general solutions to the operator equation (1.1). Based on the Moore – Penrose inverses of the associated operators, in Section 4 we give the necessary and sufficient conditions for the existence of a solution to the operator equation (1.2), and provide a formula for the general solution to this operator equation. Finally, in Section 5, we find the positive solutions of Eq. (1.3) for adjointable operators over Hilbert C^*-modules, where X is the unknown operator.

2. Preliminaries. Throughout this paper, $\mathcal{A}$ denotes a C^*-algebra. An inner-product $\mathcal{A}$-module is a linear space $\mathcal{X}$ which is a right $\mathcal{A}$-module, together with a map $(x, y) \mapsto \langle x, y\rangle :$ $\mathcal{X} \times \mathcal{X} \to \mathcal{A}$ such that for any $x, y, z \in \mathcal{X}$, $\alpha, \beta \in \mathbb{C}$ and $a \in \mathcal{A}$, the following conditions hold:

(i) $\langle x, \alpha y + \beta z\rangle = \alpha\langle x, y\rangle + \beta\langle x, z\rangle$;

(ii) $\langle x, ya\rangle = \langle x, y\rangle a$;

(iii) $\langle y, x\rangle = \langle x, y\rangle^*$;

(iv) $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0 \Longleftrightarrow x = 0$.

An inner-product $\mathcal{A}$-module $\mathcal{X}$ which is complete with respect to the induced norm $\|x\| = \sqrt{\|\langle x, x\rangle\|}$ for any $x \in \mathcal{X}$ is called a (right) Hilbert $\mathcal{A}$-module. A closed submodule $\mathcal{M}$ of a Hilbert $\mathcal{A}$-module $\mathcal{X}$ is said to be orthogonally complemented if $\mathcal{X} = \mathcal{M} \oplus \mathcal{M}^\perp$, where

$$\mathcal{M}^\perp = \{x \in \mathcal{X} : \langle x, y\rangle = 0 \text{ for any } y \in \mathcal{M}\}.$$

Now, suppose that $\mathcal{X}$ and $\mathcal{Y}$ are two Hilbert $\mathcal{A}$-modules, let $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ be the set of operators $T :$ $\mathcal{X} \to \mathcal{Y}$ for which there is an operator $T^* : \mathcal{Y} \to \mathcal{X}$ such that

$$\langle Tx, y\rangle = \langle x, T^*y\rangle \text{ for any } x \in \mathcal{X} \text{ and } y \in \mathcal{Y}.$$

It is known that any element $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ must be a bounded linear operator, which is also $\mathcal{A}$-linear in the sense that $T(xa) = (Tx)a$ for $x \in \mathcal{X}$ and $a \in \mathcal{A}$. We call $\mathcal{L}(\mathcal{X}, \mathcal{Y})$, the set of adjointable operators from $\mathcal{X}$ to $\mathcal{Y}$. For any $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, the null and the range space of T are denoted by $\ker(T)$ and $\operatorname{ran}(T)$, respectively. In the case $\mathcal{X} = \mathcal{Y}, \mathcal{L}(\mathcal{X}, \mathcal{X})$ which is abbreviated to $\mathcal{L}(\mathcal{X})$, is a C^*-algebra. Let $\mathcal{L}(\mathcal{X})_{sa}$ be the set of self-adjoint elements and $\mathcal{L}(\mathcal{X})_+$ be the set of positive elements in $\mathcal{L}(\mathcal{X})$, respectively. The identity operator on $\mathcal{X}$ is denoted by $1_\mathcal{X}$ or 1 if there is no ambiguity.

Lemma 2.1 [13]. *Let $\mathcal{X}$ be a Hilbert $\mathcal{A}$-module and $T \in \mathcal{L}(\mathcal{X})$. Then $T \in \mathcal{L}(\mathcal{X})_+$ if and only if $\langle Tx, x\rangle \geq 0$ for all x in $\mathcal{X}$.*

Theorem 2.1 [13]. *Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert $\mathcal{A}$-modules and $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ has closed range. Then*:

(i) $\ker(T)$ *is orthogonally complemented in $\mathcal{X}$, with complement* $\operatorname{ran}(T^*)$;

(ii) $\operatorname{ran}(T)$ *is orthogonally complemented in $\mathcal{Y}$, with complement* $\ker(T^*)$;

(iii) *the map $T^* \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ has closed range.*

Boichuk and Samoilenko suggested some methods for the construction of the Moore – Penrose pseudoinverse for the original linear Fredholm operators in Banach and Hilbert spaces [2]. Boichuk and Pokutnyi studied perturbation theory of operator equations in the Fréchet and Hilbert spaces by using the notion of strong generalized inverse operators in [1]. Xu and Sheng [21] showed that an adjointable operator between two Hilbert $\mathcal{A}$-modules admits a bounded Moore – Penrose inverse if and only if it has closed range. The Moore – Penrose inverse $T^\dagger$ of T is the unique element in $\mathcal{L}(\mathcal{Y}, \mathcal{X})$ which satisfies the following conditions:

$$TT^\dagger T = T, \quad T^\dagger TT^\dagger = T^\dagger, \quad (TT^\dagger)^* = TT^\dagger, \quad (T^\dagger T)^* = T^\dagger T.$$

From these conditions we obtain that $(T^\dagger)^* = (T^*)^\dagger$, $TT^\dagger$ and $T^\dagger T$ are orthogonal projections, in the sense that they are self-adjoint idempotent operators. Furthermore, we have

$$\operatorname{ran}(T) = \operatorname{ran}(TT^\dagger), \qquad \operatorname{ran}(T^\dagger) = \operatorname{ran}(T^\dagger T) = \operatorname{ran}(T^*),$$

$$\ker(T) = \ker(T^\dagger T), \qquad \ker(T^\dagger) = \ker(TT^\dagger) = \ker(T^*).$$

It is well-known, that $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is regular if there exists $S \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ such that $TST = T$. Note that if T is regular, then $T^\dagger$ exists (see [10], Theorem 6). If in addition, $\mathcal{X} = \mathcal{Y}$ and $T \geq 0$, then $TT^\dagger = T^\dagger T$ and $T^\dagger \geq 0$.

***Remark* 2.1.** Let $\mathcal{X}$ and $\mathcal{Y}$ be Hilbert $\mathcal{A}$-modules, we use the notation $\mathcal{X} \oplus \mathcal{Y}$ to denote the direct sum of $\mathcal{X}$ and $\mathcal{Y}$, which is also a Hilbert $\mathcal{A}$-module whose $\mathcal{A}$-valued inner product is given by

$$\left\langle \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right\rangle = \langle x_1, x_2\rangle + \langle y_1, y_2\rangle$$

for $x_i \in \mathcal{X}$ and $y_i \in \mathcal{Y}$, $i = 1, 2$. To simplify the notation, we use $x \oplus y$ to denote $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{X} \oplus \mathcal{Y}$.

A matrix form of a bounded adjointable operator $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ can be induced by some natural decompositions of Hilbert C^*-modules. Indeed, if $\mathcal{M}$ and $\mathcal{N}$ are closed orthogonally complemented submodules of $\mathcal{X}$ and $\mathcal{Y}$, respectively, and $\mathcal{X} = \mathcal{M} \oplus \mathcal{M}^\perp$, $\mathcal{Y} = \mathcal{N} \oplus \mathcal{N}^\perp$, then T can be written as the following 2×2 matrix

$$T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix},$$

where $T_1 = P_\mathcal{N} T P_\mathcal{M} \in \mathcal{L}(\mathcal{M}, \mathcal{N})$, $T_2 = P_\mathcal{N} T(1 - P_\mathcal{M}) \in \mathcal{L}(\mathcal{M}^\perp, \mathcal{N})$, $T_3 = (1 - P_\mathcal{N})TP_\mathcal{M} \in$ $\in \mathcal{L}(\mathcal{M}, \mathcal{N}^\perp)$, $T_4 = (1 - P_\mathcal{N})T(1 - P_\mathcal{M}) \in \mathcal{L}(\mathcal{M}^\perp, \mathcal{N}^\perp)$ and $P_\mathcal{M}$ and $P_\mathcal{N}$ denote the projections corresponding to $\mathcal{M}$ and $\mathcal{N}$, respectively.

The proof of the following lemma can be found in [14].

Lemma 2.2 [14]**.** *Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert $\mathcal{A}$-modules and $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ has closed range. Then T has the following matrix decomposition with respect to the orthogonal decompositions of closed submodules $\mathcal{X} = \operatorname{ran}(T^*) \oplus \ker(T)$ and $\mathcal{Y} = \operatorname{ran}(T) \oplus \ker(T^*)$:*

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \operatorname{ran}(T^*) \\ \ker(T) \end{bmatrix} \to \begin{bmatrix} \operatorname{ran}(T) \\ \ker(T^*) \end{bmatrix},$$

where T_1 is invertible. Moreover,

$$T^\dagger = \begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \operatorname{ran}(T) \\ \ker(T^*) \end{bmatrix} \to \begin{bmatrix} \operatorname{ran}(T^*) \\ \ker(T) \end{bmatrix}.$$

***Remark* 2.2** [22]**.** Suppose that $\mathcal{X}$ is a Hilbert $\mathcal{A}$-module and $P \in \mathcal{L}(\mathcal{X})$ is a projection. Let $\mathcal{X}_1 = P\mathcal{X}$ and $\mathcal{X}_2 = (1-P)\mathcal{X}$ and let $T : \mathcal{X}_1 \oplus \mathcal{X}_2 \to \mathcal{X}$, $T(\xi \oplus \eta) = \xi + \eta$ for $\xi \in \mathcal{X}_1$ and $\eta \in \mathcal{X}_2$. Then T is invertible with $T^{-1}(x) = P(x) \oplus (1-P)(x)$ for $x \in \mathcal{X}$, and it is easy to verify that $T^* = T^{-1}$ so that T is a unitary. Furthermore, for any self-adjoint element T of $\mathcal{L}(\mathcal{X})$, T can be decomposed as

$$T = T_{11} + T_{12} + T_{12}^* + T_{22},$$

where $T_{11} = PTP$, $T_{12} = PT(1-P)$ and $T_{22} = (1-P)T(1-P)$. By Lemma 2.1 we know that

$$\begin{aligned} T \geq 0 &\Longleftrightarrow \langle (T_{11} + T_{12} + T_{12}^* + T_{22})(\xi + \eta), \xi + \eta \rangle \geq 0 \quad \forall \xi \in \mathcal{X}_1, \quad \eta \in \mathcal{X}_2 \\ &\Longleftrightarrow \langle T_{11}\xi, \xi \rangle + \langle T_{12}\eta, \xi \rangle + \langle T_{12}^*\xi, \eta \rangle + \langle T_{22}\eta, \eta \rangle \geq 0 \quad \forall \xi \in \mathcal{X}_1, \quad \eta \in \mathcal{X}_2 \\ &\Longleftrightarrow \left\langle \begin{pmatrix} T_{11} & T_{12} \\ T_{12}^* & T_{22} \end{pmatrix} (\xi \oplus \eta), (\xi \oplus \eta) \right\rangle \geq 0 \quad \forall \xi \in \mathcal{X}_1, \quad \eta \in \mathcal{X}_2 \\ &\Longleftrightarrow \begin{pmatrix} T_{11} & T_{12} \\ T_{12}^* & T_{22} \end{pmatrix} \geq 0. \end{aligned}$$

Lemma 2.3 [21]**.** *Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be Hilbert $\mathcal{A}$-modules and $T = \begin{pmatrix} T_{11} & T_{12} \\ T_{12}^* & T_{22} \end{pmatrix}$ be a self-adjoint element of $\mathcal{L}(\mathcal{X}_1 \oplus \mathcal{X}_2)$ with $T_{ij} \in \mathcal{L}(\mathcal{X}_j \oplus \mathcal{X}_i)$, $i, j = 1, 2$. Suppose that $\operatorname{ran}(T_{11})$ is closed. Then $T \geq 0$ if and only if the following three conditions are satisfied:*

$$T_{11} \geq 0, \qquad T_{12} = T_{11}T_{11}^\dagger T_{12}, \qquad T_{22} - T_{12}^* T_{11}^\dagger T_{12} \geq 0.$$

Lemma 2.4 [15]**.** *Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be Hilbert $\mathcal{A}$-modules. Also, let $T \in \mathcal{L}(\mathcal{Y}, \mathcal{Z})$ and $S \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ have closed ranges, and $A \in \mathcal{L}(\mathcal{X}, \mathcal{Z})$. Then the equation*

$$TXS = A, \quad X \in \mathcal{L}(\mathcal{Y}), \tag{2.1}$$

has a solution if and only if $TT^\dagger AS^\dagger S = A$. In this case, any solution of Eq. (2.1) *has the form*

$$X = T^\dagger AS^\dagger.$$

Theorem 2.2 [15]. *Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be Hilbert $\mathcal{A}$-modules, $S \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and $T \in \mathcal{L}(\mathcal{Z}, \mathcal{Y})$ be invertible operators and $A \in \mathcal{L}(\mathcal{Y})$. Then the following statements are equivalent*:

(a) *there exists a solution $X \in \mathcal{L}(\mathcal{X}, \mathcal{Z})$ to the operator equation $TXS^* + SX^*T^* = A$*;

(b) $A = A^*$.

If (a) *or* (b) *is satisfied, then any solution to*

$$TXS^* + SX^*T^* = A, \quad X \in \mathcal{L}(\mathcal{X}, \mathcal{Z}),$$

has the form

$$X = \frac{1}{2}T^{-1}A(S^*)^{-1} - T^{-1}Z(S^*)^{-1},$$

where $Z \in \mathcal{L}(\mathcal{Y})$ satisfies $Z^* = -Z$.

3. Operator equation $AX + YB = C$. In this section, we study the solvability of operator equation (1.1) in the general context of the Hilbert C^*-modules and present a general solution of it.

Theorem 3.1. *Let $\mathcal{X}, \mathcal{Y}$ be Hilbert $\mathcal{A}$-modules, $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and $B \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ be invertible and $C \in \mathcal{L}(\mathcal{Y})$. Then the Eq.* (1.1) *has a solution $(X, Y) \in \mathcal{L}(\mathcal{Y}, \mathcal{X}) \times \mathcal{L}(\mathcal{X}, \mathcal{Y})$. In the case any solution of the Eq.* (1.1) *is represented by*

$$X = \frac{1}{2}A^{-1}C + \frac{1}{2}WB, \tag{3.1}$$

$$Y = \frac{1}{2}CB^{-1} - \frac{1}{2}AW, \tag{3.2}$$

where $W \in \mathcal{L}(\mathcal{X})$ is arbitrary.

Proof. It is easy to see that operators X and Y of the forms (3.1) and (3.2) are a solution of Eq. (1.1). On the other hand, let X be any solution of Eq. (1.1) then $X = A^{-1}C - A^{-1}YB$ and $A^{-1}Y = A^{-1}CB^{-1} - XB^{-1}$. We have

$$X = \frac{1}{2}A^{-1}C + \left(\frac{1}{2}A^{-1}CB^{-1} - A^{-1}Y\right)B =$$

$$= \frac{1}{2}A^{-1}C + \left(\frac{1}{2}[A^{-1}Y + XB^{-1}] - A^{-1}Y\right)B =$$

$$= \frac{1}{2}A^{-1}C + \left(\frac{1}{2}(XB^{-1} - A^{-1}Y)\right)B = \frac{1}{2}A^{-1}C + \frac{1}{2}WB,$$

where $W = XB^{-1} - A^{-1}Y$. Also, $Y = CB^{-1} - AXB^{-1}$ and $XB^{-1} = A^{-1}CB^{-1} - A^{-1}Y$ and so we can write

$$Y = \frac{1}{2}CB^{-1} + A\left(\frac{1}{2}A^{-1}CB^{-1} - XB^{-1}\right) =$$

$$= \frac{1}{2}CB^{-1} + A\left(\frac{1}{2}[XB^{-1} + A^{-1}Y] - XB^{-1}\right) =$$

$$= \frac{1}{2}CB^{-1} + A\left(\frac{1}{2}(A^{-1}Y - XB^{-1})\right) = \frac{1}{2}CB^{-1} - \frac{1}{2}AW.$$

Theorem 3.1 is proved.

Now, we solve Eq. (1.1) in the case when A and B have closed ranges.

Theorem 3.2. *Let $\mathcal{X}$ and $\mathcal{Y}$ be Hilbert $\mathcal{A}$-modules and $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and $B \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ have closed ranges and* $\mathrm{ran}(A) = \mathrm{ran}(B^*)$ *and* $\mathrm{ran}(A^*) = \mathrm{ran}(B)$ *and $C \in \mathcal{L}(\mathcal{Y})$. Then the following statements are equivalent*:

(a) *there exists a solution $(X, Y) \in \mathcal{L}(\mathcal{Y}, \mathcal{X}) \times \mathcal{L}(\mathcal{X}, \mathcal{Y})$ of Eq.* (1.1);

(b) $(1 - AA^\dagger)C(1 - B^\dagger B) = 0$.

If (a) *or* (b) *is satisfied, then any solution (X, Y) of equation* (1.1) *has the form*

$$X = \frac{1}{2}A^\dagger C + \frac{1}{2}A^\dagger C(1 - B^\dagger B) + \frac{1}{2}WB + (1 - A^\dagger A)Z,$$

$$Y = \frac{1}{2}AA^\dagger CB^\dagger + (1 - AA^\dagger)CB^\dagger - \frac{1}{2}AWBB^\dagger + V(1 - BB^\dagger),$$

where $Z \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$, $V \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ are arbitrary operators and $W \in \mathcal{L}(\mathcal{X})$ so that $(1 - AA^\dagger)WBB^\dagger = 0$.

Proof. (a) $\Rightarrow$ (b). Suppose that (X, Y) is a solution of Eq. (1.1). Thus we have

$$(1 - AA^\dagger)C(1 - B^\dagger B) = (1 - AA^\dagger)(AX + YB)(1 - B^\dagger B) =$$

$$= (1 - AA^\dagger)AX(1 - B^\dagger B) + (1 - AA^\dagger)YB(1 - B^\dagger B) = 0.$$

(b) $\Rightarrow$ (a). Suppose $(1 - AA^\dagger)C(1 - B^\dagger B) = 0$. Since $\mathrm{ran}(\mathrm{A})$ and $\mathrm{ran}(B)$ are closed, then applying Theorem 2.1 we get $\mathcal{X} = \mathrm{ran}(A^*) \oplus \ker(A)$ and $\mathcal{Y} = \mathrm{ran}(A) \oplus \ker(A^*)$. By our assumptions and using Lemma 2.2, operators A and B have the matrix forms

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathrm{ran}(A^*) \\ \ker(A) \end{bmatrix} \to \begin{bmatrix} \mathrm{ran}(A) \\ \ker(A^*) \end{bmatrix}$$

and

$$B = \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathrm{ran}(A) \\ \ker(A^*) \end{bmatrix} \to \begin{bmatrix} \mathrm{ran}(A^*) \\ \ker(A) \end{bmatrix},$$

where A_1 and B_1 are invertible. Now, condition (b) implies that C has the form

$$C = \begin{bmatrix} C_1 & C_2 \\ C_3 & 0 \end{bmatrix} : \begin{bmatrix} \mathrm{ran}(A) \\ \ker(A^*) \end{bmatrix} \to \begin{bmatrix} \mathrm{ran}(A) \\ \ker(A^*) \end{bmatrix}.$$

Let us assume that the operators X and Y have the following matrix forms:

$$X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} : \begin{bmatrix} \mathrm{ran}(A) \\ \ker(A^*) \end{bmatrix} \to \begin{bmatrix} \mathrm{ran}(A^*) \\ \ker(A) \end{bmatrix}$$

and

$$Y = \begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix} : \begin{bmatrix} \mathrm{ran}(A^*) \\ \ker(A) \end{bmatrix} \to \begin{bmatrix} \mathrm{ran}(A) \\ \ker(A^*) \end{bmatrix}.$$

Then from $AX + YB = C$ it follows that

$$\begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} + \begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix} \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} A_1X_1 + Y_1B_1 & A_1X_2 \\ Y_3B_1 & 0 \end{bmatrix} = \begin{bmatrix} C_1 & C_2 \\ C_3 & 0 \end{bmatrix}.$$

That is,

$$A_1X_1 + Y_1B_1 = C_1, \tag{3.3}$$

$$A_1X_2 = C_2, \tag{3.4}$$

$$Y_3B_1 = C_3. \tag{3.5}$$

According Theorem 3.1, Eq. (3.3) has a solution (X_1, Y_1) of the form $X_1 = \frac{1}{2}A_1^{-1}C_1 + \frac{1}{2}W_1B_1$, $Y_1 = \frac{1}{2}C_1B_1^{-1} - \frac{1}{2}A_1W_1$, where $W_1 \in \mathcal{L}(\mathrm{ran(A^*)})$ is arbitrary. In view of (3.4) and (3.5) we deduce $X_2 = A_1^{-1}C_2$ and $Y_3 = C_3B_1^{-1}$. Hence,

$$X = \begin{bmatrix} \frac{1}{2}A_1^{-1}C_1 + \frac{1}{2}W_1B_1 & A_1^{-1}C_2 \\ X_3 & X_4 \end{bmatrix}$$

and

$$Y = \begin{bmatrix} \frac{1}{2}C_1B_1^{-1} - \frac{1}{2}A_1W_1 & Y_2 \\ C_3B_1^{-1} & Y_4 \end{bmatrix},$$

where X_3, X_4, Y_2 and Y_4 can be taken arbitrary. Let

$$Z = \begin{bmatrix} Z_1 & Z_2 \\ X_3 & X_4 \end{bmatrix} : \begin{bmatrix} \mathrm{ran}(A) \\ \ker(A^*) \end{bmatrix} \to \begin{bmatrix} \mathrm{ran}(A^*) \\ \ker(A) \end{bmatrix}$$

and

$$V = \begin{bmatrix} V_1 & Y_2 \\ V_3 & Y_4 \end{bmatrix} : \begin{bmatrix} \mathrm{ran}(A^*) \\ \ker(A) \end{bmatrix} \to \begin{bmatrix} \mathrm{ran}(A) \\ \ker(A^*) \end{bmatrix}.$$

From the condition $(1 - A^\dagger A)WBB^\dagger = 0$ we derive that W has the following matrix form:

$$W = \begin{bmatrix} W_1 & W_2 \\ 0 & W_4 \end{bmatrix} : \begin{bmatrix} \mathrm{ran}(A^*) \\ \ker(A) \end{bmatrix} \to \begin{bmatrix} \mathrm{ran}(A^*) \\ \ker(A) \end{bmatrix}.$$

Thus, we have

$$\frac{1}{2}A^\dagger C = \begin{bmatrix} \frac{1}{2}A_1^{-1}C_1 & \frac{1}{2}A_1^{-1}C_2 \\ 0 & 0 \end{bmatrix}, \qquad \frac{1}{2}A^\dagger C(1 - B^\dagger B) = \begin{bmatrix} 0 & \frac{1}{2}A_1^{-1}C_2 \\ 0 & 0 \end{bmatrix},$$

$$\frac{1}{2}WB = \begin{bmatrix} \frac{1}{2}W_1B_1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad (1 - A^\dagger A)Z = \begin{bmatrix} 0 & 0 \\ X_3 & X_4 \end{bmatrix}.$$

Consequently,

$$X = \frac{1}{2}A^\dagger C + \frac{1}{2}A^\dagger C(1 - B^\dagger B) + \frac{1}{2}WB + (1 - A^\dagger A)Z.$$

On the other hand,

$$\frac{1}{2}AA^\dagger CB^\dagger = \begin{bmatrix} \frac{1}{2}C_1B_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}, \qquad -\frac{1}{2}AWBB^\dagger = \begin{bmatrix} -\frac{1}{2}A_1W_1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$(1 - AA^\dagger)CB^\dagger = \begin{bmatrix} 0 & 0 \\ C_3B_1^{-1} & 0 \end{bmatrix}, \qquad V(1 - BB^\dagger) = \begin{bmatrix} 0 & Y_2 \\ 0 & Y_4 \end{bmatrix}.$$

Then

$$Y = \frac{1}{2}AA^\dagger CB^\dagger + (1 - AA^\dagger)CB^\dagger - \frac{1}{2}AWBB^\dagger + V(1 - BB^\dagger).$$

Theorem 3.2 is proved.

4. Operator equation $AXD + FYB = C$. In this section, we study the general solutions to Eq. (1.2) below in the general context of the Hilbert C^*-modules. The necessary and sufficient conditions for the existence of a solution are given and the set of solutions are completely described.

Theorem 4.1. *Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{W}, \mathcal{V}, \mathcal{K}$ be Hilbert $\mathcal{A}$-modules, $A \in \mathcal{L}(\mathcal{Z}, \mathcal{K})$, $B \in \mathcal{L}(\mathcal{X}, \mathcal{W})$, $D \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, $F \in \mathcal{L}(\mathcal{V}, \mathcal{K})$ be invertible and $C \in \mathcal{L}(\mathcal{X}, \mathcal{K})$. Then the operator equation* (1.2) *has a solution $(X, Y) \in \mathcal{L}(\mathcal{Y}, \mathcal{Z}) \times \mathcal{L}(\mathcal{W}, \mathcal{V})$. In the case any solution of the Eq.* (1.2) *is represented by*

$$X = \frac{1}{2}A^{-1}CD^{-1} - A^{-1}KD^{-1} \tag{4.1}$$

and

$$Y = \frac{1}{2}F^{-1}CB^{-1} + F^{-1}KB^{-1}, \tag{4.2}$$

where $K \in \mathcal{L}(\mathcal{X}, \mathcal{K})$ is arbitrary.

Proof. We show that if the Eq. (1.2) has a solution $(X, Y) \in \mathcal{L}(\mathcal{Y}, \mathcal{Z}) \times \mathcal{L}(\mathcal{W}, \mathcal{V})$, then it must be of the forms (4.1) and (4.2). Let $T = \begin{bmatrix} A & 0 \\ 0 & B^* \end{bmatrix} : \mathcal{Z} \oplus \mathcal{W} \to \mathcal{K} \oplus \mathcal{X}$, $\hat{Y} = \begin{bmatrix} 0 & X \\ Y^* & 0 \end{bmatrix}$: $\mathcal{V} \oplus \mathcal{Y} \to \mathcal{Z} \oplus \mathcal{W}$, $S = \begin{bmatrix} F & 0 \\ 0 & D^* \end{bmatrix} : \mathcal{V} \oplus \mathcal{Y} \to \mathcal{K} \oplus \mathcal{X}$ and $N = \begin{bmatrix} 0 & C \\ C^* & 0 \end{bmatrix} : \mathcal{X} \oplus \mathcal{X} \to \mathcal{K} \oplus \mathcal{K}$. By $T\hat{Y}S^* + S(\hat{Y})^*T^* = N$, we have

$$T\hat{Y}S^* + S(\hat{Y})^*T^* = \begin{bmatrix} A & 0 \\ 0 & B^* \end{bmatrix} \begin{bmatrix} 0 & X \\ Y^* & 0 \end{bmatrix} \begin{bmatrix} F^* & 0 \\ 0 & D \end{bmatrix} +$$

$$+ \begin{bmatrix} F & 0 \\ 0 & D^* \end{bmatrix} \begin{bmatrix} 0 & Y \\ X^* & 0 \end{bmatrix} \begin{bmatrix} A^* & 0 \\ 0 & B \end{bmatrix} = \begin{bmatrix} 0 & C \\ C^* & 0 \end{bmatrix},$$

or, equivalently,

$$\begin{bmatrix} 0 & AXD + FYB \\ B^*Y^*F^* + D^*X^*A^* & 0 \end{bmatrix} = \begin{bmatrix} 0 & C \\ C^* & 0 \end{bmatrix}.$$

Therefore $AXD + FYB = C$. Obviously, by our assumptions T and S are invertible and from Theorem 2.2, it follows that $\hat{Y}$ has the following representation:

$$\hat{Y} = \frac{1}{2}T^{-1}N(S^*)^{-1} - T^{-1}Z(S^*)^{-1}, \tag{4.3}$$

where $Z \in \mathcal{L}(\mathcal{K} \oplus \mathcal{X})$ satisfies $Z^* = -Z$. Let $Z = \begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix}$. From (4.3), we have

$$\begin{bmatrix} 0 & X \\ Y^* & 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} A^{-1} & 0 \\ 0 & (B^*)^{-1} \end{bmatrix}\begin{bmatrix} 0 & C \\ C^* & 0 \end{bmatrix}\begin{bmatrix} (F^*)^{-1} & 0 \\ 0 & D^{-1} \end{bmatrix} -$$

$$-\begin{bmatrix} A^{-1} & 0 \\ 0 & (B^*)^{-1} \end{bmatrix}\begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix}\begin{bmatrix} (F^*)^{-1} & 0 \\ 0 & D^{-1} \end{bmatrix}.$$

It enforces that

$$X = \frac{1}{2}A^{-1}CD^{-1} - A^{-1}Z_2D^{-1},$$

$$Y^* = \frac{1}{2}(B^*)^{-1}C^*(F^*)^{-1} - (B^*)^{-1}Z_3(F^*)^{-1},$$

$$A^{-1}Z_1(F^*)^{-1} = (B^*)^{-1}Z_4D^{-1} = 0.$$

From invertibility of the operators A, B, D and F it follows that $Z_1 = 0$ and $Z_4 = 0$. On the other hand, from $Z^* = -Z$, we get $Z_3^* = -Z_2$. For $K = Z_2$, we have

$$X = \frac{1}{2}A^{-1}CD^{-1} - A^{-1}KD^{-1}$$

and

$$Y = \frac{1}{2}F^{-1}CB^{-1} + F^{-1}KB^{-1}.$$

Theorem 4.1 is proved.

Now, we are ready to state our main result of this section.

Theorem 4.2. *Suppose that* $\mathcal{X}$ *and* $\mathcal{Y}$ *are Hilbert* $\mathcal{A}$*-modules,* $A, F \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, $B, D \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ *and* A, B, D, F *have closed ranges and* $C \in \mathcal{L}(\mathcal{Y})$ *such that* $\operatorname{ran}(F) = \operatorname{ran}(D^*)$, $\operatorname{ran}(F^*) = \operatorname{ran}(D)$, *and*

$$FF^\dagger A(1 - DD^\dagger) = 0, \qquad (1 - FF^\dagger)ADD^\dagger = 0,$$

$$F^\dagger FB(1 - D^\dagger D) = 0, \qquad (1 - F^\dagger F)BD^\dagger D = 0,$$

$$D^\dagger DB^\dagger BA = D^\dagger DA, \qquad D^\dagger DAA^\dagger B^* = D^\dagger DB^*,$$

$$F^\dagger FBB^\dagger A^* = F^\dagger FA^*, \qquad F^\dagger FA^\dagger AB = F^\dagger FB,$$

$$F^\dagger CB^\dagger B(1 - D^\dagger D) = F^\dagger C(1 - D^\dagger D),$$

$$(1 - FF^\dagger)AA^\dagger CD^\dagger = (1 - FF^\dagger)CD^\dagger.$$

Then the following statements are equivalent:

(a) *there exists a solution* $(X, Y) \in \mathcal{L}(\mathcal{X}) \times \mathcal{L}(\mathcal{X})$ *to Eq.* (1.2);

(b) $(1 - AA^\dagger)C(1 - B^\dagger B) = 0$ *and* $(1 - FF^\dagger)C(1 - D^\dagger D) = 0$.

If (a) *or* (b) *is satisfied*, *then any solution to Eq.* (1.2) *has the form*

$$X = \frac{1}{2}F^\dagger FA^\dagger CD^\dagger + (1 - F^\dagger F)A^\dagger CD^\dagger + \frac{1}{2}F^\dagger FA^\dagger C(1 - B^\dagger B)D^\dagger +$$

$$+\frac{1}{2}F^\dagger FWBD^\dagger + F^\dagger F(1 - A^\dagger A)ZD^\dagger + U(1 - DD^\dagger)$$

and

$$Y = \frac{1}{2}F^\dagger AA^\dagger CB^\dagger DD^\dagger + F^\dagger(1 - AA^\dagger)CB^\dagger DD^\dagger + F^\dagger CB^\dagger(1 - DD^\dagger) -$$

$$-\frac{1}{2}F^\dagger AWBB^\dagger DD^\dagger + F^\dagger V(1 - BB^\dagger)DD^\dagger + (1 - F^\dagger F)T,$$

where $Z \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ *and* $T, U, V, W \in \mathcal{L}(\mathcal{X})$ *are arbitrary operators*, *so that* $F^\dagger F(1 - -AA^\dagger)WBB^\dagger DD^\dagger = 0$.

Proof. (a) $\Rightarrow$ (b). Suppose that $(X, Y) \in \mathcal{L}(\mathcal{X}) \times \mathcal{L}(\mathcal{X})$ is a solution of Eq. (1.2). Then

$$(1 - AA^\dagger)C(1 - B^\dagger B) = (1 - AA^\dagger)(AXD + FYB)(1 - B^\dagger B) =$$

$$= (1 - AA^\dagger)AXD(1 - B^\dagger B) + (1 - AA^\dagger)FYB(1 - B^\dagger B) = 0$$

and

$$(1 - FF^\dagger)C(1 - D^\dagger D) = (1 - FF^\dagger)(AXD + FYB)(1 - D^\dagger D) =$$

$$= (1 - FF^\dagger)AXD(1 - D^\dagger D) + (1 - FF^\dagger)FYB(1 - D^\dagger D) = 0.$$

(b) $\Rightarrow$ (a). Suppose that $(X, Y) \in \mathcal{L}(\mathcal{X}) \times \mathcal{L}(\mathcal{X})$ is a solution of Eq. (1.2). Since $\operatorname{ran}(\mathrm{D})$ and $\operatorname{ran}(F)$ are closed, we have $\mathcal{X} = \operatorname{ran}(D^*) \oplus \ker(D)$ and $\mathcal{Y} = \operatorname{ran}(F) \oplus \ker(F^*)$. Applying our hypotheses A, B, D and F have the following matrix forms:

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} : \begin{bmatrix} \operatorname{ran}(D) \\ \ker(D^*) \end{bmatrix} \to \begin{bmatrix} \operatorname{ran}(F) \\ \ker(F^*) \end{bmatrix},$$

$$B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} : \begin{bmatrix} \operatorname{ran}(D^*) \\ \ker(D) \end{bmatrix} \to \begin{bmatrix} \operatorname{ran}(F^*) \\ \ker(F) \end{bmatrix},$$

$$D = \begin{bmatrix} D_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \operatorname{ran}(D^*) \\ \ker(D) \end{bmatrix} \to \begin{bmatrix} \operatorname{ran}(D) \\ \ker(D^*) \end{bmatrix}$$

and

$$F = \begin{bmatrix} F_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \operatorname{ran}(F^*) \\ \ker(F) \end{bmatrix} \to \begin{bmatrix} \operatorname{ran}(F) \\ \ker(F^*) \end{bmatrix}.$$

Hence D_1 and F_1 are invertible by Lemma 2.2. Also, we can assume that unknown operators X and Y have the matrix forms

$$X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} : \begin{bmatrix} \operatorname{ran}(D) \\ \ker(D^*) \end{bmatrix} \to \begin{bmatrix} \operatorname{ran}(D) \\ \ker(D^*) \end{bmatrix}$$

and

$$Y = \begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix} : \begin{bmatrix} \operatorname{ran}(F^*) \\ \ker(F) \end{bmatrix} \to \begin{bmatrix} \operatorname{ran}(F^*) \\ \ker(F) \end{bmatrix}.$$

From condition $(1 - FF^\dagger)C(1 - D^\dagger D) = 0$ it obtains C has the form

$$C = \begin{bmatrix} C_1 & C_2 \\ C_3 & 0 \end{bmatrix} : \begin{bmatrix} \operatorname{ran}(D^*) \\ \ker(D) \end{bmatrix} \to \begin{bmatrix} \operatorname{ran}(F) \\ \ker(F^*) \end{bmatrix}.$$

Conditions $FF^\dagger A(1 - DD^\dagger) = 0$ and $(1 - FF^\dagger)ADD^\dagger = 0$ imply that A has the matrix form

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_4 \end{bmatrix} : \begin{bmatrix} \operatorname{ran}(D) \\ \ker(D^*) \end{bmatrix} \to \begin{bmatrix} \operatorname{ran}(F) \\ \ker(F^*) \end{bmatrix},$$

and conditions $F^\dagger FB(1 - D^\dagger D) = 0$ and $(1 - F^\dagger F)BD^\dagger D = 0$ imply that B have the matrix form

$$B = \begin{bmatrix} B_1 & 0 \\ 0 & B_4 \end{bmatrix} : \begin{bmatrix} \operatorname{ran}(D^*) \\ \ker(D) \end{bmatrix} \to \begin{bmatrix} \operatorname{ran}(F^*) \\ \ker(F) \end{bmatrix}.$$

By replacing these matrix forms in the operator equation $AXD + FYB = C$ we get

$$\begin{bmatrix} A_1 & 0 \\ 0 & A_4 \end{bmatrix} \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} \begin{bmatrix} D_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} F_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix} \begin{bmatrix} B_1 & 0 \\ 0 & B_4 \end{bmatrix} =$$

$$= \begin{bmatrix} A_1X_1D_1 + F_1Y_1B_1 & F_1Y_2B_4 \\ A_4X_3D_1 & 0 \end{bmatrix} = \begin{bmatrix} C_1 & C_2 \\ C_3 & 0 \end{bmatrix}.$$

That is,

$$A_1X_1D_1 + F_1Y_1B_1 = C_1, \tag{4.4}$$

$$F_1Y_2B_4 = C_2, \tag{4.5}$$

$$A_4X_3D_1 = C_3. \tag{4.6}$$

Taking $X_0 = X_1D_1$ and $Y_0 = F_1Y_1$, then equation (4.4) becomes

$$A_1X_0 + Y_0B_1 = C_1. \tag{4.7}$$

Due to the fact that $D^\dagger DB^\dagger BA = D^\dagger DA$ and the above matrix forms we have

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} B_1^\dagger B_1 & 0 \\ 0 & B_4^\dagger B_4 \end{bmatrix} \begin{bmatrix} A_1 & 0 \\ 0 & A_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_1 & 0 \\ 0 & A_4 \end{bmatrix},$$

which gives us the identity $B_1^\dagger B_1A_1 = A_1$. Hence, $\operatorname{ran}(A_1) \subseteq \operatorname{ran}(B_1^*)$. Similarly, applying condition $D^\dagger DAA^\dagger B^* = D^\dagger DB^*$ we find

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_1A_1^\dagger & 0 \\ 0 & A_4A_4^\dagger \end{bmatrix} \begin{bmatrix} B_1^* & 0 \\ 0 & B_4^* \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} B_1^* & 0 \\ 0 & B_4^* \end{bmatrix}.$$

It in turn gives us $A_1A_1^\dagger B_1^* = B_1^*$ and so $\operatorname{ran}(B_1^*) \subseteq \operatorname{ran}(\mathrm{A}_1)$. Therefore,$\operatorname{ran}(A_1) = \operatorname{ran}(B_1^*)$. Taking into account conditions $F^\dagger FBB^\dagger A^* = F^\dagger FA^*$ and $F^\dagger FA^\dagger AB = F^\dagger FB$ and by using a similar method we yield $\operatorname{ran}(A_1^*) = \operatorname{ran}(B_1)$, too. Also, by multiplication $FF^\dagger$ on the left and $D^\dagger D$ on the right to the identity $(1 - AA^\dagger)C(1 - B^\dagger B) = 0$, we obtain $(1 - A_1A_1^\dagger)C_1(1 - B_1^\dagger B_1) = 0$. Now, all conditions of Theorem 3.2 hold and so Eq. (4.7) is solvable and any solution of it can be represented in the following forms:

$$\begin{aligned} X_0 &= \frac{1}{2}A_1^\dagger C_1 + \frac{1}{2}A_1^\dagger C_1(1 - B_1^\dagger B_1) + \frac{1}{2}W_1B_1 + (1 - A_1^\dagger A_1)Z_1, \\ Y_0 &= \frac{1}{2}A_1A_1^\dagger C_1B_1^\dagger + (1 - A_1A_1^\dagger)C_1B_1^\dagger - \frac{1}{2}A_1W_1B_1B1^\dagger + V_1(1 - B_1B_1^\dagger), \end{aligned} \tag{4.8}$$

where $Z_1 \in \mathcal{L}(\operatorname{ran}(F), \operatorname{ran}(D)), V_1 \in \mathcal{L}(\operatorname{ran}(D), \operatorname{ran}(F))$ and $W_1 \in \mathcal{L}(\operatorname{ran}(D))$ that W_1 satisfies in $(1 - A_1A_1^\dagger)W_1B_1B_1^\dagger = 0$. This last identity implies that $F^\dagger F(1 - AA^\dagger)WBB^\dagger DD^\dagger = 0$. Since D_1 and F_1 are invertible, then from (4.8) we deduce

$$X_1 = \frac{1}{2}A_1^\dagger C_1D_1^{-1} + \frac{1}{2}A_1^\dagger C_1(1 - B_1^\dagger B_1)D_1^{-1} + \frac{1}{2}W_1B_1D_1^{-1} + (1 - A_1^\dagger A_1)Z_1D_1^{-1},$$

$$Y_1 = \frac{1}{2}F_1^{-1}A_1A_1^\dagger C_1B_1^\dagger + F_1^{-1}(1 - A_1A_1^\dagger)C_1B_1^\dagger - \frac{1}{2}F_1^{-1}A_1W_1B_1B1^\dagger + F_1^{-1}V_1(1 - B_1B_1^\dagger).$$

The assumptions $F^\dagger CB^\dagger B(1 - D^\dagger D) = F^\dagger C(1 - D^\dagger D)$ and $(1 - FF^\dagger)AA^\dagger CD^\dagger = (1 - FF^\dagger)CD^\dagger$ give us $F_1^{-1}C_2B_4^\dagger B_4 = F_1^{-1}C_2$ and $A_4(A_4)^\dagger C_3D_1^{-1} = C_3D_1^{-1}$, respectively. With the aid of these facts and by using Lemma 2.4 it yields that the operator equations (4.5) and (4.6) are solvable, and further we have $Y_2 = F_1^{-1}C_2(B_4)^\dagger$ and $X_3 = (A_4)^\dagger C_3D_1^{-1}$. Hence,

$$X = \begin{bmatrix} \frac{1}{2}A_1^\dagger C_1D_1^{-1} + \frac{1}{2}A_1^\dagger C_1(1 - B_1^\dagger B_1)D_1^{-1} + \frac{1}{2}W_1B_1D_1^{-1} + (1 - A_1^\dagger A_1)Z_1D_1^{-1} & X_2 \\ (A_4)^\dagger C_3D_1^{-1} & X_4 \end{bmatrix}$$

and

$$Y =$$

$$= \begin{bmatrix} \frac{1}{2}F_1^{-1}A_1A_1^{\dagger}C_1B_1^{\dagger} + F_1^{-1}(1 - A_1A_1^{\dagger})C_1B_1^{\dagger} - \frac{1}{2}F_1^{-1}A_1W_1B_1B_1^{\dagger} + F_1^{-1}V_1(1 - B_1B_1^{\dagger}) & F_1^{-1}C_2B_4^{\dagger} \\ Y_3 & Y_4 \end{bmatrix},$$

where X_2, X_4, Y_3 and Y_4 can be taken arbitrary. On the other hand, from the assumptions $\operatorname{ran}(F) = \operatorname{ran}(D^*)$ and $\operatorname{ran}(F^*) = \operatorname{ran}(D)$ we observe that the operators W, Z, U, V and T have the following matrix forms:

$$W = \begin{bmatrix} W_1 & W_2 \\ W_3 & W_4 \end{bmatrix} : \begin{bmatrix} \operatorname{ran}(D) \\ \ker(D^*) \end{bmatrix} \to \begin{bmatrix} \operatorname{ran}(D) \\ \ker(D^*) \end{bmatrix},$$

$$Z = \begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix} : \begin{bmatrix} \operatorname{ran}(F) \\ \ker(F^*) \end{bmatrix} \to \begin{bmatrix} \operatorname{ran}(D) \\ \ker(D^*) \end{bmatrix},$$

$$U = \begin{bmatrix} U_1 & X_2 \\ U_3 & X_4 \end{bmatrix} : \begin{bmatrix} \operatorname{ran}(D) \\ \ker(D^*) \end{bmatrix} \to \begin{bmatrix} \operatorname{ran}(D) \\ \ker(D^*) \end{bmatrix},$$

$$V = \begin{bmatrix} V_1 & V_2 \\ V_3 & V_4 \end{bmatrix} : \begin{bmatrix} \operatorname{ran}(D) \\ \ker(D^*) \end{bmatrix} \to \begin{bmatrix} \operatorname{ran}(F) \\ \ker(F^*) \end{bmatrix}$$

and

$$T = \begin{bmatrix} T_1 & T_2 \\ Y_3 & Y_4 \end{bmatrix} : \begin{bmatrix} \operatorname{ran}(D) \\ \ker(D^*) \end{bmatrix} \to \begin{bmatrix} \operatorname{ran}(D) \\ \ker(D^*) \end{bmatrix}.$$

Hence, we obtain

$$\frac{1}{2}F^{\dagger}FA^{\dagger}CD^{\dagger} = \begin{bmatrix} \frac{1}{2}A_1^{\dagger}C_1D_1^{-1} & 0 \\ 0 & 0 \end{bmatrix},$$

$$(1 - F^{\dagger}F)A^{\dagger}CD^{\dagger} = \begin{bmatrix} 0 & 0 \\ A_4^{\dagger}C_3D_1^{-1} & 0 \end{bmatrix},$$

$$\frac{1}{2}F^{\dagger}FA^{\dagger}C(1 - B^{\dagger}B)D^{\dagger} = \begin{bmatrix} \frac{1}{2}A_1^{\dagger}C_1(1 - B_1^{\dagger}B_1)D_1^{-1} & 0 \\ 0 & 0 \end{bmatrix},$$

$$\frac{1}{2}F^{\dagger}FWBD^{\dagger} = \begin{bmatrix} \frac{1}{2}W_1B_1D_1^{-1} & 0 \\ 0 & 0 \end{bmatrix},$$

$$F^{\dagger}F(1 - A^{\dagger}A)ZD^{\dagger} = \begin{bmatrix} (1 - A_1^{\dagger}A_1)Z_1D_1^{-1} & 0 \\ 0 & 0 \end{bmatrix},$$

$$U(1-DD^\dagger)=\begin{bmatrix}0 & X_2\\ 0 & X_4\end{bmatrix}.$$

From these we get

$$X=\frac{1}{2}F^\dagger FA^\dagger CD^\dagger+(1-F^\dagger F)A^\dagger CD^\dagger+\frac{1}{2}F^\dagger FA^\dagger C(1-B^\dagger B)D^\dagger+\frac{1}{2}F^\dagger FWBD^\dagger+$$
$$+F^\dagger F(1-A^\dagger A)ZD^\dagger+U(1-DD^\dagger).$$

Also, we have

$$\frac{1}{2}F^\dagger AA^\dagger CB^\dagger DD^\dagger=\begin{bmatrix}\frac{1}{2}F_1^{-1}A_1A_1^\dagger C_1B_1^\dagger & 0\\ 0 & 0\end{bmatrix},$$

$$F^\dagger(1-AA^\dagger)CB^\dagger DD^\dagger=\begin{bmatrix}F_1^{-1}(1-A_1A_1^\dagger)C_1B_1^\dagger & 0\\ 0 & 0\end{bmatrix},$$

$$F^\dagger CB^\dagger(1-DD^\dagger)=\begin{bmatrix}0 & F_1^{-1}C_2B_4^\dagger\\ 0 & 0\end{bmatrix},$$

$$-\frac{1}{2}F^\dagger AWBB^\dagger DD^\dagger=\begin{bmatrix}-\frac{1}{2}F_1^{-1}A_1W_1B_1B_1^\dagger & 0\\ 0 & 0\end{bmatrix},$$

$$F^\dagger V(1-BB^\dagger)DD^\dagger=\begin{bmatrix}F_1^{-1}V_1(1-B_1B_1^\dagger) & 0\\ 0 & 0\end{bmatrix},$$

$$(1-F^\dagger F)T=\begin{bmatrix}0 & 0\\ Y_3 & Y_4\end{bmatrix}.$$

Accordingly, we derive

$$Y=\frac{1}{2}F^\dagger AA^\dagger CB^\dagger DD^\dagger+F^\dagger(1-AA^\dagger)CB^\dagger DD^\dagger+F^\dagger CB^\dagger(1-DD^\dagger)-\frac{1}{2}F^\dagger AWBB^\dagger DD^\dagger+$$
$$+F^\dagger V(1-BB^\dagger)DD^\dagger+(1-F^\dagger F)T.$$

Theorem 4.2 is proved.

5. Positive solutions of operator equation $AX+X^*A^*=B$. In this section, we provide an approach to the study of the positive solutions to the operator equation (1.3) for adjointable operators between Hilbert C^*-modules.

Theorem 5.1. *Let $\mathcal{X}$ be a Hilbert $\mathcal{A}$-module, and $A,B,C\in\mathcal{L}(\mathcal{X})$. Let A, B and BA^* have closed ranges and BA^* be a positive operator,* $\operatorname{ran}(B)=\operatorname{ran}(BA^*)$ *and* $(1-A^\dagger A)B^\dagger=0$. *Then the Eq.* (1.3) *has a positive solution $X\in\mathcal{L}(\mathcal{X})_+$ if and only if B is self-adjoint and* $(1-AA^\dagger)B(1-$ $-AA^\dagger)=0$. *In this case the general positive solution has the form*

$$X = \frac{1}{2}X_0 + X_0V(1 - A^\dagger A) + (1 - A^\dagger A)V^*X_0 + 2(1 - A^\dagger A)V^*X_0V(1 - A^\dagger A)+$$
$$+(1 - A^\dagger A)W(1 - A^\dagger A), \tag{5.1}$$

where $X_0 = B^*(BA^*)^\dagger B$ *is a particular positive solution,* $V \in \mathcal{L}(\mathcal{X})_{sa}$ *and* $W \in \mathcal{L}(\mathcal{X})_+$ *are arbitrary.*

Proof. Suppose that Eq. (1.3) has a solution $X \in \mathcal{L}(\mathcal{X})_+$. Obviously, $B = B^*$. Also, we have

$$(1 - AA^\dagger)B(1 - AA^\dagger) = (1 - AA^\dagger)(AX + X^*A^*)(1 - AA^\dagger) =$$
$$= (1 - AA^\dagger)AX(1 - AA^\dagger) + (1 - AA^\dagger)X^*A^*(1 - AA^\dagger) = 0.$$

Conversely, assume that B is self-adjoint and $(1 - AA^\dagger)B(1 - AA^\dagger) = 0$. We prove that the general positive solution of Eq. (1.3) can be expressed as (5.1). For this purpose, take $P_1 = B^\dagger B,\ P_2 = 1 - B^\dagger B$ and $X_1 = P_1XP_1$, where X is as in (5.1). From the assumption $(1 - A^\dagger A)B^\dagger = 0$, we have

$$X_1 = P_1XP_1 = B^\dagger BXB^\dagger B = \frac{1}{2}B^*(BA^*)^\dagger B,$$

and hence $BA^* \in \mathcal{L}(\mathcal{X})_+$ implies that $X_1 \in \mathcal{L}(\mathcal{X})_+$. Moreover, since $\mathrm{ran}(B) = \mathrm{ran}(BA^*)$ we get

$$X_1(2B^\dagger(BA^*)(B^*)^\dagger)X_1 = \frac{1}{2}B^*(BA^*)^\dagger B(B^\dagger(BA^*)(B^*)^\dagger)B^*(BA^*)^\dagger B =$$
$$= \frac{1}{2}B^*(BA^*)^\dagger BB^\dagger(BA^*)BB^\dagger(BA^*)^\dagger B = \frac{1}{2}B^*(BA^*)^\dagger B,$$

thus, X_1 is regular and $X_1^\dagger = B^\dagger(BA^*)(B^*)^\dagger$. According to Remark 2.2 we can write

$$X = X_1 + X_2 + X_3 + X_4 = P_1XP_1 + P_1XP_2 + P_2XP_1 + P_2XP_2,$$

where

$$X_2 = P_1XP_2 = B^\dagger BX(1 - B^\dagger B) = B^*(BA^*)^\dagger BV(1 - A^\dagger A),$$
$$X_3 = P_2XP_1 = (1 - B^\dagger B)XB^\dagger B = (1 - A^\dagger A)V^*B^*(BA^*)^\dagger B,$$
$$X_4 = P_2XP_2 = (1 - B^\dagger B)X(1 - B^\dagger B) = 2(1 - A^\dagger A)V^*B^*(BA^*)^\dagger BV(1 - A^\dagger A)+$$
$$+(1 - A^\dagger A)W(1 - A^\dagger A).$$

Evidently, $X_3 = X_2^*$ and $X_1X_1^\dagger X_2 = X_2$. Furthermore, we have

$$X_4 - X_3X_1^\dagger X_2 = 2(1 - A^\dagger A)V^*B^*(BA^*)^\dagger BV(1 - A^\dagger A) + (1 - A^\dagger A)W(1 - A^\dagger A)-$$
$$-((1 - A^\dagger A)V^*B^*(BA^*)^\dagger B)(B^\dagger(BA^*)(B^*)^\dagger)(2B^*(BA^*)^\dagger BV(1 - A^\dagger A)) =$$
$$= (1 - A^\dagger A)W(1 - A^\dagger A) \in \mathcal{L}(\mathcal{X})_+.$$

Applying Lemma 2.3, we derive that X is positive and any arbitrary positive solution to Eq. (1.3) can be expressed as (5.1).

Theorem 5.1 is proved.

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Received 12.06.18,
after revision — 24.12.18