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ON THE DYNAMICS OF THE IMPULSIVE PREDATOR-PREY SYSTEMS WITH BEDDINGTON – DEANGELIS TYPE FUNCTIONAL RESPONSE

ПРО ДИНАМІКУ ІМПУЛЬСНИХ СИСТЕМ ТИПУ ХИЖАК–ЗДОБИЧ ІЗ ФУНКЦІОНАЛЬНОЮ ВІДПОВІДДЮ ТИПУ БЕДДІНГТОНА – ДЕАНГЕЛІСА

In this study, the two-dimensional predator-prey system with Beddington – DeAngelis type functional response with impulses is considered in a periodic environment. For this special case, necessary and sufficient conditions are found for the considered system when it has at least one w-periodic solution. This result is mainly based on the continuation theorem in the coincidence degree theory and to get the globally attractive w-periodic solution of the given system, an inequality is given as the necessary and sufficient condition by using the analytic structure of the system.

Вивчається двовимірна система типу хижак–здобич із функціональною відповіддю типу Беддінгтона – ДеАнгеліса та імпульсами у періодичному середовищі. Для цього спеціального випадку знайдено необхідні та достатні умови для того, щоб система мала принаймні один w-періодичний розв'язок. Цей результат базується головним чином на теоремі продовження з теорії степенів збігу, а для того, щоб знайти глобально притягуючий w-періодичний розв'язок розглядуваної системи, за допомогою аналітичної структури системи отримано нерівність, яка відіграє роль необхідної та достатньої умови.

1. Introduction. Population dynamics is an important branch of the mathematical ecology and biomathematics. Predator-prey systems is one of the research field of this subject and many studies have been done on the these type of dynamical systems. Studying on these systems is important because it helps us to understand the future of the considered species.

In this paper, we have investigate the impulsive predator-prey dynamic systems, since giving impulse to a system has many important examples in the real life. For instance, if you have used a pesticide against to a insect species, then there is an immediate decrease in the population or if there is an immigration from one territory to another for the same species, then there is also an immediate increase in the population. All of these can be expressed mathematically by using impulses and these type of equations are said to be impulsive differential equations. There are many studies on this type of differential and difference equations and especially, its theory has been investigated in [1, 17 – 19, 21, 25].

The other significant notion that is important for this study, is being in a periodic environment or not, since many things in real life has a periodic structure. Therefore, to consider the dynamical systems in a periodic environment becomes important. Global existence and the existence of the positive periodic solutions are significant aspects of the periodic predator-prey systems and the studies [7 – 11, 14, 15, 20, 24] investigated these problems on the nonautonomous predator-prey systems by using coincidence degree theory and the continuation theorem.

Another important notion for this work is functional responses. In this study, we have used the Beddington – DeAngelis type functional response because of some advantages of this one to the other functional responses like Holling type, ratio dependent, semi ratio dependent, mono type and etc. Beddington and DeAngelis uses Beddington – DeAngelis type functional response, according to their observation on the populations on the fishes in Adriatic. Also the advantages of this type of functional response can be found in their work [2, 6].

Especially, the singularity problem in Holling type and similar functional responses when predator or prey goes to extinction is solved by using Beddington – DeAngelis functional response. Because of that singularity problem, in [22], they can not use results of coincidence degree theory in their system directly. They have divided their system, applied continuation theorem to some part of it and obtained their goal with Brouwer's fixed point theorem. In addition to these, they need to use constant impulses in the variable that symbolize predator to obtain the globally stable w-periodic solution. In our system, with the advantage of Beddington – DeAngelis type functional response, we have used different impulses for both prey and predator which is more meaningful in the real life. We apply continuation theorem directly to our system to get the w-periodic solution and to show the global stability of that solution, we find a connection between the extiction of prey and predator and its consequences. Nevertheless, there exist some difficulties in the application of the continuation theorem to the whole system and these are solved by some analytic technics.

2. Preliminaries. The following informations are obtained from [3]. Let $L\colon \operatorname{Dom} L \subset X \to Y$ be a linear mapping, $C\colon X \to Y$ be a continuous mapping where X, Y be normed vector spaces. If $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < +\infty$ and $\operatorname{Im} L$ is closed in Y, then the mapping L will be called a Fredholm mapping of index zero. There exist continuous projections $U: X \to X$ and $V: Y \to Y$ when L is a Fredholm mapping of index zero such that $\operatorname{Im} U = \operatorname{Ker} L$, $\operatorname{Im} L = \operatorname{Ker} V = \operatorname{Im}(I - V)$, then it follows that $L|_{\operatorname{Dom} L \cap \operatorname{Ker} U} : (I - U)X \to \operatorname{Im} L$ is invertible. The inverse of that map is denoted as K_U. The mapping C will be called L-compact on Ω if $VC(\Omega)$ is bounded and $K_U(I - V)C\colon \Omega \to X$ is compact, where Ω is an open bounded subset of X. Since $\operatorname{Im} V$ is isomorphic to $\operatorname{Ker} L$, the isomorphism $J\colon \operatorname{Im} V \to \operatorname{Ker} L$ is exist and the above informations are important for the continuation theorem that we give below.

Definition 1 [5]. *The codimension (or quotient or factor dimension) of a subspace L of a vector space V is the dimension of the quotient space V/L; it is denoted by $\operatorname{codim}_V L$ or simply by $\operatorname{codim} L$ and is equal to the dimension of the orthogonal complement of L in V, and one has* $\dim L + \operatorname{codim} L = \dim V$.

Theorem 1 [12] (continuation theorem). *Suppose that L is a Fredholm mapping of index zero and C is L-compact on Ω. Assume that*:

(a) *For any y that satisfies $Ly = \lambda Cy$ is not on $\partial\Omega$, for each $\lambda \in (0, 1)$;*

(b) *$VCy \neq 0$ and the Brouwer degree $\deg\{JVC, \partial\Omega \cap \operatorname{Ker} L, 0\} \neq 0$ for each $y \in \partial\Omega \cap \operatorname{Ker} L$.*

Then $Ly = Cy$ has at least one solution lying in $\operatorname{Dom} L \cap \partial\Omega$.

Definition 2 [26]. *A w-periodic semiflow $F(t)\colon X \to X$ (X is the initial value space) in the sense that $F(t)x$ is continuous in $(t, x) \in [0, +\infty) \times X$, $F(0) = I$ and $F(t + w) = F(t)F(w)$ for all $t > 0$ is generated by the solutions of a w-periodic system.*

Definition 3 [26]. *If there exists $\eta > 0$ such that, for any $x \in X_0$,*

$$\liminf_{t\to\infty} d(F(t)x, \partial X_0) \geq \eta,$$

then the periodic semiflow $F(t)$ is said to be uniformly persistent with respect to $(X_0, \partial X_0)$.

Definition 4 [13]. *Suppose that $F: \mathbb{R}^n \to \mathbb{R}^n$. If there exists a bounded set B such that, for each $x \in \mathbb{R}^n$, there is an integer $n_0 = n_0(x, B)$ such that $F^n(x) \in B$ for each $n \geq n_0$, then the map F is called point dissipative.*

Lemma 1 [26]. *Assume that* $S\colon X \to X$ *is continuous such that* $S(X_0) \subset X_0$. *Suppose that* S *is uniformly persistent with respect to* $(X_0, \partial X_0)$, *compact and point dissipative. Then, for* S *in* X_0 *relative to strongly bounded sets in* X_0, *there exists a global attractor* A_0 *and* S *has coexistence state* $x_0 \in A_0$.

Definition 5 [11]. *The system* (3) *is called permanent if there exist positive constants* r_1, r_2, R_1, *and* R_2 *such that solution* $\big(\tilde{x}(t), \tilde{y}(t)\big)$ *of system* (3) *satisfies*

$$r_1 \le \lim_{t\to\infty} \inf\ \tilde{x}(t) \le \lim_{t\to\infty} \sup\ \tilde{x}(t) \le R_1,$$

$$r_2 \le \lim_{t\to\infty} \inf\ \tilde{y}(t) \le lim_{t\to\infty} \sup\ \tilde{y}(t) \le R_2.$$

Lemma 2 [22]. *Consider the system*

$$\begin{aligned} &\tilde{x}'(t) = a(t)\tilde{x}(t) - b(t)\tilde{x}^2(t), \quad t \ne t_k, \\ &\tilde{x}(t_k^+) = (1+g_k)\tilde{x}(t_k). \end{aligned} \tag{1}$$

Then system (1) *admits a unique, positive,* w*-periodic solution if and only if*

$$\int_0^w a(t)dt + \ln\prod_{i=1}^q (1+g_i) > 0, \tag{2}$$

which, moreover, is globally asymptotically stable.

3. Main result. The equation that we investigate is

$$\begin{aligned} &\tilde{x}'(t) = a(t)\tilde{x}(t) - b(t)\tilde{x}^2(t) - c(t)\tilde{x}(t)E(t, \tilde{y}(t), \tilde{x}(t), \tilde{y}(t)), \quad t \ne t_k, \\ &\tilde{y}'(t) = -d(t)\tilde{y}(t) + f(t)\tilde{y}(t)E(t, \tilde{x}(t), \tilde{x}(t), \tilde{y}(t)), \quad t \ne t_k, \\ &\tilde{x}(t_k^+) = (1+g_k)\tilde{x}(t_k), \\ &\tilde{y}(t_k^+) = (r_k)\tilde{y}(t_k), \end{aligned} \tag{3}$$

where $E\big(t, u(t), x(t), y(t)\big) = \dfrac{u(t)}{\alpha(t) + \beta(t)x(t) + m(t)y(t)}$. In (3), if we take $\tilde{x}(t) = \exp(x(t))$ and $\tilde{y}(t) = \exp(y(t))$, then we obtain the following system:

$$\begin{aligned} &x'(t) = a(t) - b(t)\exp(x(t)) - c(t)E\big(t, \exp(y(t)), \exp(x(t)), \exp(y(t))\big), \quad t \ne t_k, \\ &y'(t) = -d(t) + f(t)E\big(t, \exp(x(t)), \exp(x(t)), \exp(y(t))\big), \quad t \ne t_k, \\ &\Delta x(t_k) = \ln(1+g_k), \\ &\Delta y(t_k) = \ln(r_k), \end{aligned} \tag{4}$$

where $t_{k+q} = t_k + w$, $a(t+w) = a(t)$, $b(t+w) = b(t)$, $c(t+w) = c(t)$, $d(t+w) = d(t)$, $f(t+w) = f(t)$, $\alpha(t+w) = \alpha(t), \beta(t+w) = \beta(t)$, $m(t+w) = m(t)$, k, g_k and r_k are the constants with $1 > g_k > -1$ and $r_k > 0$. Here, $x(t_{k+q}) = x(t_k) + w$, $\tilde{x}(t_{k+q}) = \tilde{x}(t_k) + w$, $y(t_{k+q}) = y(t_k) + w$, $\tilde{y}(t_{k+q}) = \tilde{y}(t_k) + w$. In equations (3) and (4) each coefficient function is from continuous functions class and all the coefficient fuctions are positive.

Definition 6. *In system* (4), *we say that* $x(t)$ $(y(t))$ (*prey* (*predator*)) *goes to extinction if and only if* $\exp(x(t))$ $(\exp(y(t)))$ *tends to* 0 *as* t *tends to infinity, for all solutions of* $x(t)$ $(y(t))$. *Equivalently, we also say that prey (predator) goes to extinction if and only if* $\tilde{x}(t)$ $(\tilde{y}(t))$ *tends to zero as* t *tends to infinity, for all solutions of system* (3).

Lemma 3. *Assume that*

$$\int_0^w d(t)dt - \ln \prod_{i=1}^q r_i > 0 \tag{5}$$

is satisfied. If $y(t)$ *does not go to extinction, then neither* $x(t)$ *does.*

Proof. The statement of the above lemma is the same with the statement: assume that (5) is satisfied. Then if $x(t)$ goes to extinction, then $y(t)$ also goes to extinction. By using the second equation in system (4) and taking the integral of that equation from 0 to t, we obtain

$$\exp(y(t)) = \exp(y(0)) \prod_{ti<t} (r_i) \exp\left(\int_0^t -d(s) + f(s)E(t, \exp(x(s)), \exp(x(s)), \exp(y(s)))ds \right). \tag{6}$$

If $x(t)$ goes to extinction, then $\exp(x(t))$ tends to 0 as t tends to infinity. Since all the coefficient functions are positive, $f(t)E\big(t, \exp(x(t)), \exp(x(t)), \exp(y(t))\big)$ also tends to 0 as t tends to infinity. For sufficiently large t the integral

$$\int_0^t -d(s) + \ln \prod_{i=1}^q r_i + f(s)E\big(t, \exp(x(s)), \exp(x(s)), \exp(y(s))\big)ds$$

becomes negative and the right-hand side of the equation (6) tends to 0 as t tends to infinity which means $\exp(y(t))$ tends to 0 as t tends to infinity. Thus, $y(t)$ goes to extinction. Hence, we are done.

3.1. Permanence and extinction of the solutions.

Lemma 4. *If inequalities* (2) *and* (5) *are satisfied*, *then for the given system* (3) $\liminf_{t\to\infty} \tilde{x}(t) \geq r_1$ *for some* $\tilde{r}_1 > 0$.

Proof. Assume that prey goes to extiction, then by Lemma 3 predator also goes to extinction. Then for sufficiently large $T > 0$ there exists $\epsilon_0 > 0$ such that, for each $t > T$,

$$\tilde{y}(t) < \epsilon_0.$$

If

$$\int_0^w a(t)dt + \ln \prod_{k=1}^q (1 + g_k) > 0$$

for sufficiently small ϵ_0, we have

$$\int_0^w a(t) - \frac{\epsilon_0 c(t)}{\alpha(t) + \epsilon_0 m(t)} - b(t)\tilde{x}(t)dt + \ln \prod_{k=1}^q (1 + g_k) > 0. \tag{7}$$

Additionally, for sufficiently small ϵ_0, the following inequality becomes also true:

$$\tilde{x}'(t) > \tilde{x}(t)\left(a(t) - \frac{\epsilon_0 c(t)}{\alpha(t) + \epsilon_0 m(t)} - b(t)\tilde{x}(t)\right).$$

Then consider the system

$$\begin{aligned} \bar{x}'(t) &= \bar{x}(t)\left(a(t) - \frac{\epsilon_0 c(t)}{\alpha(t) + \epsilon_0 m(t)} - b(t)\bar{x}(t)\right), \\ \bar{x}(t_k^+) &= (1 + g_k)\bar{x}(t_k). \end{aligned} \tag{8}$$

For system (8), since inequality (7) is true, we can apply Lemma 2. Then system (8) has globally attractive, w-periodic solution $\check{x}^*(t)$. By the comparison theorem for impulsive differential equations $\tilde{x}(t) > \check{x}^*(t)$. Therefore prey does not go to extiction which is a contradiction. Hence, $\liminf_{t\to\infty} \tilde{x}(t) \geq \tilde{r}_1$, for some $\tilde{r}_1 > 0$ is true.

Lemma 5. *Asume that inequality* (2) *is satisfied*. *Predator in system* (3) *goes to extinction if and only if*

$$\int\limits_0^w -d(t) + \frac{f(t)x^*(t)}{\alpha(t) + \beta(t)x^*(t)} dt + \ln\left(\prod_{k=1}^q (r_k)\right) \leq 0 \tag{9}$$

holds, *where* $x^*(t)$ *is the unique*, *positive*, *globally attractive*, w*-periodic solution of the system* (1).

Proof. By taking the contrapositive of the necessary part of the lemma, we have if (9) does not holds then predator does not go to extiction. From now on, it is a proof by contradiction. Therefore, assume that (9) does not holds and predator goes to extiction. If we can find a contradiction, then we are able to get the desired result for the first side of the lemma. Here suppose that system (3) satisfies the equation

$$\int\limits_0^w -d(t) + \frac{f(t)x^*(t)}{\alpha(t) + \beta(t)x^*(t)} dt + \ln\left(\prod_{k=1}^q (r_k)\right) > 0. \tag{10}$$

Then there exists $\tilde{\epsilon} > 0$ such that

$$\int\limits_0^w -d(t) + \frac{f(t)(x^*(t) - \tilde{\epsilon})}{\alpha(t) + \beta(t)(x^*(t) - \tilde{\epsilon}) + m(t)\tilde{\epsilon}} dt + \ln\left(\prod_{k=1}^q (r_k)\right) > 0. \tag{11}$$

Consider the system

$$\begin{aligned} \tilde{x}'(t) &= \tilde{x}(t)(a(t) - \frac{2\gamma c(t)}{\alpha(t) + 2\gamma m(t)} - b(t)\tilde{x}(t)), \\ \tilde{x}(t_k^+) &= (1 + g_k)\big(\tilde{x}(t_k)\big). \end{aligned} \tag{12}$$

where γ is a positive constant. It is obvious that for sufficiently small γ, $a(t) - \dfrac{2\gamma c(t)}{\alpha(t) + 2\gamma m(t)} > 0$. Thus, system (12) has a globally attractive, unique, w-periodic solution from Lemma 2 for sufficiently small γ. Assume that x_γ be the globally attractive solution of the system (12). Thus, $x_\gamma(t) \to x^*(t)$

as $\gamma \to 0$. Then there exists $\hat{\gamma}$ such that $x_{\hat{\gamma}}(t) \geq x^*(t) - \tilde{\epsilon}/2$ and $2\hat{\gamma} < \tilde{\epsilon}$. Since predator goes to extinction, then

$$\limsup_{t\to\infty} \tilde{y}(t) < \hat{\gamma}.$$

So, there exists T such that, for any $t > T$,

$$\tilde{y(t)} < 2\hat{\gamma} < \tilde{\epsilon}.$$

Since $\tilde{y(t)} < 2\hat{\gamma}$, then $\tilde{x}'(t) > \tilde{x}(t)(a(t) - \dfrac{2\gamma c(t)}{\alpha(t) + 2\gamma m(t)} - b(t)\tilde{x}(t))$. By the comparison theorem for impulsive differential equations $\tilde{x}(t) > x^*(t) - \tilde{\epsilon}$. Therefore, we get the system

$$\tilde{y}'(t) \geq \tilde{y}(t)\left(- d(t) + \frac{f(t)(x^*(t) - \tilde{\epsilon})}{\alpha(t) + \beta(t)(x^*(t) - \tilde{\epsilon}) + m(t)\tilde{\epsilon}}\right),$$
$$\tilde{y}(t_k^+) = (r_k)(\tilde{y}(t_k)).$$

Here,

$$\tilde{y}(t) \geq \tilde{y}(0)\exp\left(\int\limits_0^t -d(s) + \frac{f(s)(x^*(s) - \tilde{\epsilon})}{\alpha(s) + \beta(s)(x^*(s) - \tilde{\epsilon}) + m(s)\tilde{\epsilon}}ds + \ln\left(\prod_{0<t_k<t}(r_k)\right)\right). \tag{13}$$

Since inequality (11) is satisfied, the right-hand side of inequality (13) is always positive for sufficiently large t and does not go to zero as t tends to infinity. Therefore $\tilde{y}(t)$ becomes always positive and does not go to zero. In other words, predator does not go to extinction. Hence, we have proved that if predator goes to extinction then inequality (9) holds.

For converse, to prove the result, we use contradiction again. Assume that inequality (9) holds and predator does not go to extinction. Then $\liminf_{t\to\infty} \tilde{y}(t) \geq \tilde{r}_2$. Since (2) is true, then by Lemma 6, $\limsup_{t\to\infty} \tilde{x}(t) \leq R_1$. Thus, we have

$$\tilde{x}'(t) \leq \tilde{x}(t)\left(a(t) - \frac{c(t)\tilde{r}_2}{\alpha(t) + \beta(t)R_1 + m(t)\tilde{r}_2} - b(t)\tilde{x}(t)\right),$$
$$\tilde{x}(t_k^+) = (1 + g_k)\big(\tilde{x}(t_k)\big).$$

By the comparison theorem for impulsive differential equations $\tilde{x}(t) \leq x^*(t)$, therefore, the following inequality is true:

$$\begin{aligned}
&\tilde{y}'(t) \leq \tilde{y}(t)\left(-d(t) + \frac{f(t)x^*(t)}{\alpha(t) + \beta(t)x^*(t) + m(t)\tilde{r}_2}\right),\\
&\tilde{y}(t_k^+) = (r_k)(\tilde{y}(t_k)).
\end{aligned} \tag{14}$$

Since in this system each coefficient functions are positive, then

$$\frac{f(t)x^*(t)}{\alpha(t) + \beta x^*(t) + m(t)\tilde{r}_2} \leq \frac{f(t)x^*(t)}{\alpha(t) + \beta(t)x^*(t)} - \mu$$

for some $\mu > 0$. Then

$$\tilde{y}(t) \leq y(0) \exp\left(\int\limits_0^t -d(t) + \frac{f(t)x^*(t)}{\alpha(t) + \beta(t)x^*(t)} - \mu dt + \ln\left(\prod_{0 \leq t_k \leq t} (r_k) \right) \right). \tag{15}$$

Since inequality (9) holds, for sufficiently large t, the inside of the exponential function in inequality (15) is negative. Thus, as t tends to infinity, $\tilde{y}(t)$ tends to zero, which means predator goes to extinction which is a contradiction.

Lemma 5 is proved.

Lemma 6. *If inequalities* (2) *and* (5) *are satisfied*, *then there exist positive constants* R_1 *and* R_2 *such that*

$$\lim_{t\to\infty} \sup x(t) \leq R_1, \qquad \lim_{t\to\infty} \sup y(t) \leq R_2. \tag{16}$$

Proof. First consider the system (3), then the following inequality is true:

$$\begin{aligned} &\tilde{x}'(t) \leq a(t)\tilde{x}(t) - b(t)\tilde{x}^2(t), \quad t \neq t_k, \\ &\tilde{x}(t_k^+) = (1 + g_k)\tilde{x}(t_k). \end{aligned} \tag{17}$$

Suppose that (2) holds and consider the equations

$$\begin{aligned} &\tilde{u}'(t) = a(t)\tilde{u}(t) - b(t)\tilde{u}^2(t), \quad t \neq t_k, \\ &\tilde{u}(t_k^+) = (1 + g_k)\tilde{u}(t_k). \end{aligned} \tag{18}$$

By Lemma 2, system (18) has unique, positive, globally attractive (or globally asypmtotically stable), w-periodic solution $\bar{u}(t)$. By using comparison theorem for impulsive differential equations from [1], we obtain that

$$\tilde{x}(t) \leq u(t).$$

The attractivity of $\bar{u}(t)$ implies that there exists $T > 0$ such that

$$u(t) \leq \bar{u}(t) + 1 \quad \text{for} \quad t > T.$$

Therefore, it is clear that $\tilde{x}(t)$ is bounded above with a positive constant R_1. Secondly, consider the system (3). The coefficient functions in system (3) is bounded, positive and w-periodic, then the following inequality is true:

$$\begin{aligned} &\tilde{y}'(t) \leq -d(t)\tilde{y}(t) + \frac{f(t)\tilde{x}(t)}{m(t)} \leq \frac{f^M R_1}{m^L} - d(t)\tilde{y}(t), \quad t \neq t_k, \\ &\tilde{y}(t_k^+) = (r_k)\tilde{y}(t_k). \end{aligned} \tag{19}$$

Then, we get

$$\tilde{y}(t) \leq \tilde{y}(0) \prod_{0<t_k<t} r_k \exp\left(\int\limits_0^t -d(s)ds \right) + \int\limits_0^t \prod_{s<t_k<t} r_k \exp\left(\int\limits_s^t -d(\sigma)d\sigma \right) \frac{f^M R_1}{m^L} ds.$$

We can rewrite the last inequality as

$$\tilde{y}(t) \leq \tilde{y}(0) \exp\left(\int_0^t -d(s)ds + \ln\left(\prod_{0<t_k<t} r_k \right) \right) +$$

$$+\frac{f^M R_1}{m^L} \int_0^t \exp\left(\int_s^t -d(\sigma)d\sigma + \ln\left(\prod_{s<t_k<t} r_k \right) \right) ds. \tag{20}$$

For sufficently large t, inside of the exponential function for the first term and the second term of inequality (20) is negative. So if we take $\dfrac{f^M R_1}{m^L} = M$, then

$$\tilde{y}(t) \leq \tilde{y}(0) e^{1+Dw-ct} + M e^{1+Dw} \int_0^t e^{c(s-t)} ds \leq$$

$$\leq \tilde{y}(0) \left(e^{1+Dw-ct} + \frac{M e^{1+Dw}}{c} (1 - e^{-ct}) \right) \leq$$

$$\leq \tilde{y}(0) \left(e^{1+Dw} + \frac{M e^{1+Dw}}{c} \right).$$

Here, $D = \max \{ |d(t)| : t \in [0, w] \}$ and

$$c = \min \left\{ \left(\int_0^w d(s)ds - \ln \left(\prod_{k=1}^q r_k \right) \right) \Big/ w, \ 1/w \right\}.$$

Thus, we have a positive constant R_2 such that $\tilde{y}(t)$ is bounded above with a positive constant R_2.

Lemma 7. *If* (2) *and* (10) *for* $x^*(t)$ *is the unique, positive, globally attractive,* w*-periodic solution of the system* (1) *is satisfied, then* $\lim_{t\to\infty} \tilde{y}(t) > \tilde{r}_2$ *for some positive* $\tilde{r}_2$.

Proof. This result is the immediate consequence of Lemma 5.

Lemma 8. *Assume that inequalities* (2) *and* (5) *are satisfied. Then system* (3) *is permanent if and only if inequality* (10) *is satisfied. Therefore, from Theorem* 2, *this system has at least one* w*-periodic solution.*

Proof. This is the immediate consequence of the Lemmas 3, 6, and 5.

Lemma 9. *In system* (3), *assume that* (2) *is satisfied. If at least one solution of* $\tilde{y}(t)$, *does not tend to* 0 *as* t *tends to infinity, then for all solutions of* $\tilde{y}(t)$, *does not tend to* 0 *as* t *tends to infinity.*

Proof. This is a proof by contradiction. Let us assume that there exist two solutions for system (3), $(\tilde{x}(t), \tilde{y}(t))$ and $(\hat{x}(t), \hat{y}(t))$ such that $\tilde{y}(t)$, does not tend to 0 as t tends to infinity and $\hat{y}(t)$ tends to 0 as t tends to infinity. Since $\hat{y}(t)$ tends to 0 as t tends to infinity, then $\hat{x}(t)$ tends to x^* as t tends to infinity. According to Definition 6 predator does not go to extinction, and as a consequence of Lemma 5,

$$\int_0^w -d(t) + \frac{f(t)x^*(t)}{\alpha(t) + \beta(t)x^*(t)} dt + \ln\left(\prod_{k=1}^q (r_k) \right) > 0$$

is satisfied. Then, by using Lemma 7, we have $\hat{y}(t) > \hat{r}$, for some positive $\hat{r}$ which is a contradiction. Hence, proof is completed.

3.2. w-Periodicity of the solutions.

Theorem 2. *Assume that all the coefficient functions in system* (4) *are bounded, positive, w-periodic, from $C(\mathbb{R},\mathbb{R}^2)$ and inequalities* (2) *and* (5) *are satisfied. Then there exists at least one w-periodic solution if and only if $y(t)$ does not go to extinction.*

Proof. $X := \{(p,z)^\intercal \in PC(\mathbb{R},\mathbb{R}^2) : p(t+w) = p(t), z(t+w) = z(t)\}$ with the norm

$$\|(p,z)^\intercal\| = \sup_{t\in[0,w]} \big(|p(t)|, |z(t)|\big)$$

and

$$Y := \Big\{\big[(p,z)^\intercal, (d_1,f_1)^\intercal, \ldots, (d_q,f_q)^\intercal\big] \in PC(\mathbb{R},\mathbb{R}^2) \times (\mathbb{R}^2)^q,\ p(t+w) = p(t), z(t+w) = z(t)\Big\}$$

with the norm

$$\Big\|\big[(p,z)^\intercal, (d_1,f_1)^\intercal, \ldots, (d_q,f_q)^\intercal\big]\Big\| = \sup_{t\in[0,w]} \Big(\|(p,z)^\intercal\|, \big\|(d_1,f_1)^\intercal\big\|, \ldots, \big\|(d_q,f_q)^\intercal\big\|\Big).$$

Let us define the mappings L and C by $L\colon \operatorname{Dom} L \subset X \to Y$ such that

$$L\big((p,z)^\intercal\big) = \big((p',z')^\intercal, (\Delta p(t_1), \Delta z(t_1))^\intercal, \ldots, (\Delta p(t_q), \Delta z(t_q))^\intercal\big)$$

and $C\colon X \to Y$ such that

$$C\big((p,z)^\intercal\big) =$$

$$= \left(\begin{bmatrix} a(t) - b(t)\exp(p(t)) - c(t)E\big(t, z(t), p(t), z(t)\big) \\ -d(t) + f(t)E\big(t, p(t), p(t), z(t)\big) \end{bmatrix}, \begin{bmatrix} \ln(1+g_1) \\ \ln(p_1) \end{bmatrix}, \ldots, \begin{bmatrix} \ln(1+g_q) \\ \ln(p_q) \end{bmatrix}\right).$$

Then $\operatorname{Ker} L = \{(p,z)^\intercal$ such that $(p,z)^\intercal = (c_1,c_2)^\intercal\}$, c_1 and c_2 are constants,

$$\operatorname{Im} L = \left\{ \big[(p,z)^\intercal, (d_1,f_1)^\intercal, \ldots, (d_q,f_q)^\intercal\big] : \begin{bmatrix} \int_0^w p(s)ds + \sum_{i=1}^q d_i \\ \int_0^w z(s)ds + \sum_{i=1}^q f_i \end{bmatrix} = (0,0)^\intercal \right\}.$$

$\operatorname{Im} L$ is closed in Y and $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L = 2$. We can show this as follows. It is obvious that summation of any element from $\operatorname{Im} L$ and $\operatorname{Ker} L$ is in Y. Without loss of generality, take $p \in Y$ and $\int_\kappa^{w+\kappa} p(t)dt + \sum_{i=1}^q d_i = I \neq 0$. Let us define a new function

$$g = p - \frac{I}{w}.$$

Then $\dfrac{I}{w}$ is constant because, for all κ, $\int_\kappa^{w+\kappa} p(t)dt$ is always same by the definition of periodic time scales and the impulses are constant and there are same number of impulses in the interval $[\kappa, w+\kappa]$ for all κ. If we take the integral of g from κ to $w+\kappa$, we get

$$\int\limits_{\kappa}^{w+\kappa} g(t)dt + \sum_{i=1}^{q} d_i = \int\limits_{\kappa}^{w+\kappa} p(t)dt + \sum_{i=1}^{q} d_i - I = 0.$$

Then $p \in Y$ can be written as the summation of $g \in \operatorname{Im} L$ and $\dfrac{I}{w} \in \operatorname{Ker} L,$ since $\dfrac{I}{w}$ is constant. Similar steps are used for $z.$ $(p,z)^{\intercal} \in Y$ can be written as the summation of an element from $\operatorname{Im} L$ and an element from $\operatorname{Ker} L.$ Also, it is easy to show that any element in Y is uniquely expressed as the summation of an element $\operatorname{Ker} L$ and an element from $\operatorname{Im} L.$ So $\operatorname{codim} \operatorname{Im} L$ is also 2, we get the desired result. Therefore, L is a Fredholm mapping of index zero.

There exist continuous projectors $U : X \to X$ and $V : Y \to Y$ such that

$$U\big((p,z)^{\intercal}\big) = \frac{1}{w} \begin{bmatrix} \int\limits_0^w p(s)ds \\ \int\limits_0^w z(s)ds \end{bmatrix}$$

and

$$V\big((p,z)^{\intercal}, (d_1, f_1)^{\intercal}, \ldots, (d_q, f_q)^{\intercal}\big) = \frac{1}{w} \left(\begin{bmatrix} \int\limits_0^w p(s)ds + \sum\limits_{i=1}^{q} d_i \\ \int\limits_0^w z(s)ds + \sum\limits_{i=1}^{q} f_i \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right).$$

The generalized inverse $K_U = \operatorname{Im} L \to \operatorname{Dom} L \cap \operatorname{Ker} U$ is given,

$$K_U\big((p,z)^{\intercal}, (d_1, f_1)^{\intercal}, \ldots, (d_q, f_q)^{\intercal}\big) =$$

$$= \begin{bmatrix} \int\limits_0^t p(s)ds + \sum\limits_{t>t_i} d_i - \dfrac{1}{w} \int\limits_0^w \int\limits_0^t p(s)dsdt - \sum\limits_{i=1}^{q} d_i + \dfrac{1}{w} \sum\limits_{i=1}^{q} d_i t_i \\ \int\limits_0^t z(s)ds + \sum\limits_{t>t_i} f_i - \dfrac{1}{w} \int\limits_0^w \int\limits_0^t z(s)dsdt - \sum\limits_{i=1}^{q} f_i + \dfrac{1}{w} \sum\limits_{i=1}^{q} f_i t_i \end{bmatrix},$$

$$VC\big((p,z)^{\intercal}\big) =$$

$$= \frac{1}{w} \left(\begin{bmatrix} \int\limits_0^w a(s) - b(s)\exp(p(s)) - c(s)E(s, z(s), p(s), z(s))ds + \ln \prod\limits_{i=1}^{q} (1 + g_i) \\ \int\limits_0^w -d(s) + f(s)E(s, p(s), p(s), z(s))ds + \ln \prod\limits_{i=1}^{q} (r_i) \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right).$$

Let

$$a(t) - b(t)\exp(p(t)) - c(t)E\big(t, z(t), p(t), z(t)\big) = C_1(t),$$
$$-d(t) + f(t)E\big(t, p(t), p(t), z(t)\big) = C_2(t),$$
$$\frac{1}{w}\int_0^w a(s) - b(s)\exp(p(s)) - c(s)E(s, z(s), p(s), z(s))ds = \bar{C}_1,$$

and

$$\frac{1}{w}\int_0^w -d(s) + f(s)E(s, p(s), p(s), z(s))ds = \bar{C}_2,$$

$$K_U(I-V)C\big((p,z)^\intercal\big) = K_U\left(\begin{bmatrix} C_1(t)-\bar{C}_1 \\ C_2(t)-\bar{C}_2\end{bmatrix}, \begin{bmatrix}\ln(1+g_1)\\ \ln(p_1)\end{bmatrix}, \ldots, \begin{bmatrix}\ln(1+g_q)\\ \ln(p_q)\end{bmatrix}\right) =$$

$$= \begin{bmatrix} \int_0^t C_1(s)-\bar{C}_1 ds + \ln\prod_{t>t_i}(1+g_i) - \frac{1}{w}\int_0^w\int_0^t C_1(s)- \\ -\bar{C}_1 dsdt - \ln\prod_{i=1}^q(1+g_i) + \frac{1}{w}\sum_{i=1}^q \ln(1+g_i)t_i \\ \int_0^t C_2(s)-\bar{C}_2 ds + \ln\prod_{t>t_i} r_i - \frac{1}{w}\int_0^w\int_0^t C_2(s)- \\ -\bar{C}_2 dsdt - \ln\prod_{i=1}^q r_i + \frac{1}{w}\sum_{i=1}^q \ln(r_i)t_i \end{bmatrix}.$$

Clearly, VC and $K_U(I-V)C$ are continuous. Since X and Y are Banach spaces, then by using Arzela – Ascoli theorem we can find $\overline{K_U(I-V)C(\bar{\Omega})}$ is compact for any open bounded set $\Omega \subset X$. Additionally, $VC(\bar{\Omega})$ is bounded. Thus, C is L-compact on $\bar{\Omega}$ with any open bounded set $\Omega \subset X$.

Now, continuation theorem will be used. To be able to use this theorem, we should investigate the following system:

$$\begin{aligned} &x'(t) = \lambda\big[a(t) - b(t)\exp(x(t)) - c(t)E\big(t, \exp(y(t)), \exp(x(t)), \exp(y(t))\big)\big], \quad t \neq t_k, \\ &y'(t) = \lambda\big[-d(t) + f(t)E\big(t, \exp(x(t)), \exp(x(t)), \exp(y(t))\big)\big], \quad t \neq t_k, \\ &\Delta x(t_k) = \lambda\ln(1+g_k), \\ &\Delta y(t_k) = \lambda\ln(r_k), \end{aligned} \tag{21}$$

$$\begin{aligned} &\int_0^w a(t)dt + \ln\prod_{i=1}^q(1+g_i) = \int_0^w b(t)\exp(x(t)) + c(t)E\big(t, \exp(y(t)), \exp(x(t)), \exp(y(t))\big)dt, \\ &\int_0^w d(t)dt - \ln\prod_{i=1}^q(r_i) = \int_0^w f(t)E\big(t, \exp(x(t)), \exp(x(t)), \exp(y(t))\big)dt. \end{aligned} \tag{22}$$

By using (21) and (22), we obtain

$$\int\limits_0^w |x'(t)|dt \le$$

$$\le \lambda \left[\int\limits_0^w |a(t)|dt + \int\limits_0^w b(t)\exp(x(t)) + c(t)E\big(t, \exp(y(t)), \exp(x(t)), \exp(y(t))\big)dt \right] \le$$

$$\le \lambda \left[\int\limits_0^w |a(t)|dt + \int\limits_0^w a(t)dt + \ln \prod_{i=1}^q (1+g_i) \right] \le M_1, \tag{23}$$

where $M_1 := 2\int_0^w a(t)dt + \ln \prod_{i=1}^q (1+g_i)$, and

$$\int\limits_0^w |y'(t)|dt \le \lambda \left[\int\limits_0^w |d(t)|dt + \int\limits_0^w f(t)E\big(t, \exp(x(t)), \exp(x(t)), \exp(y(t))\big)dt \right] \le$$

$$\le \lambda \left[\int\limits_0^w |d(t)|dt + \int\limits_0^w d(t)dt - \ln \prod_{i=1}^q r_i \right] \le M_2, \tag{24}$$

where $M_2 := 2\int_0^w d(t)dt - \ln \prod_{i=1}^q r_i$.

Since $(x,y)^\intercal \in X$ and there are q impulses which are constant, then we can say that there exist $\eta_i,\ \xi_i,\ i = 1,2$, such that

$$\begin{aligned} x(\xi_1) &= \min\Big\{ \inf_{t\in[0,t_1]} x(t), \inf_{t\in(t_1,t_2]} x(t), \ldots, \inf_{t\in(t_q,w]} x(t) \Big\}, \\ x(\eta_1) &= \max\Big\{ \sup_{t\in[0,t_1]} x(t), \sup_{t\in(t_1,t_2]} x(t), \ldots, \sup_{t\in(t_q,w]} x(t) \Big\}, \end{aligned} \tag{25}$$

$$\begin{aligned} y(\xi_2) &= \min\Big\{ \inf_{t\in[0,t_1]} y(t), \inf_{t\in(t_1,t_2]} y(t), \ldots, \inf_{t\in(t_q,w]} y(t) \Big\}, \\ y(\eta_2) &= \max\Big\{ \sup_{t\in[0,t_1]} y(t), \sup_{t\in(t_1,t_2]} y(t), \ldots, \sup_{t\in(t_q,w]} y(t) \Big\}. \end{aligned} \tag{26}$$

By the first equation of (22) and (23), we get $x(\xi_1) < l_1$, where

$$l_1 :\ = \ln\left(\frac{\int_0^w a(t)dt + \ln \prod_{i=1}^q (1+g_i)}{\int_0^w b(t)dt} \right).$$

Since $x(\xi_1)$ is the infimum of $x(t)$ for $t \in [0,w]$, then there exists $t_1 \in [0,w]$ such that $x(\xi_1) \le x(t_1) < l_1$. By using the first inequality in Lemma 1, we have

$$x(t) \le x(t_1) + \int\limits_0^w |x'(t)|dt \le x(t_1) + \left(2\int\limits_0^w a(t)dt + \ln \prod_{i=1}^q (1+g_i) \right) < H_1 := l_1 + M_1. \tag{27}$$

From the second equation of (22), we get $x(\eta_1) \geq l_2$, where

$$l_2 : \ = \ln\left(\frac{\displaystyle\int_0^w d(t)dt - \ln\prod\nolimits_{i=1}^q r_i}{\displaystyle\int_0^w \big(f(t)/\alpha(t)\big)dt}\right).$$

Since $x(\eta_1)$ is the supremum of $x(t)$ for $t \in [0, w]$, then there exists $t_2 \in [0, w]$ such that $x(\eta_1) \geq$ $\geq x(t_2) > l_2$. By using second inequality in Lemma 1, we obtain

$$x(t) \geq x(t_2) - \int\limits_0^w |x'(t)|dt \geq x(t_2) - \left(2\int\limits_0^w a(t)dt + \ln\prod_{i=1}^q (1 + g_i)\right) >$$

$$> H_2 := l_2 - M_1. \tag{28}$$

By (27) and (28) $\max_{t\in[0,w]} |x(t)| \leq B_1 := \max\{|H_1|, |H_2|\}$. By using

$$f(t)E\big(t, \exp(x(t)), \exp(x(t)), \exp(y(t))\big) =$$

$$= f(t)E\big(t, \exp(y(t)), \exp(x(t)), \exp(y(t))\big) \exp(x(t) - y(t)),$$

we get

$$\int\limits_0^w d(t)dt - \ln\prod_{i=1}^q r_i < \int\limits_0^w (f(t)/m(t))[\exp(x(t) - y(t))]dt \leq$$

$$\leq \big[\exp(x(\eta_1) - y(\xi_2))\big] \int\limits_0^w (f(t)/m(t))dt.$$

We have the following inequality, since (27) is true, for each $t \in [0, w]$:

$$y(\xi_2) < H_1 - \ln\left(\frac{\displaystyle\int_0^w d(t)dt - \ln\prod\nolimits_{i=1}^q r_i}{\displaystyle\int_0^w (f(t)/m(t))dt}\right) := l_3.$$

Since $y(\xi_2)$ is the infimum of $y(t)$ for $t \in [0, w]$, then there exists $t_3 \in [0, w]$ such that $y(\xi_2) \leq$ $\leq y(t_3) < l_3$. By using first equation of Lemma 1, we obtain

$$y(t) \leq y(t_3) + \int\limits_0^w |y'(t)|dt \leq y(t_3) + \left(2\int\limits_0^w d(t)dt - \ln\prod_{i=1}^q r_i\right) <$$

$$< H_3 := l_3 + M_2. \tag{29}$$

Here, all the coefficient functions in $f(t)E\big(t, \exp(y(t)), \exp(x(t)), \exp(y(t))\big)$ are positive and $y(t)$ does not go to extinction and by Lemma 9, since systems (3) and (4) are equivalent, for all solutions of $y(t)$ as t tends to infinity $\exp(y(t))$ does not tend to 0, then we obtain

$$\frac{f(t)}{m(t)} > f(t)E\big(t, \exp(y(t)), \exp(x(t)), \exp(y(t))\big) > 0.$$

Then there exists $k \in \mathbb{N}$ such that

$$f(t)E\big(t, \exp(y(t)), \exp(x(t)), \exp(y(t))\big) > \frac{1}{k}\frac{f(t)}{m(t)} > 0,$$

$$\int_0^w d(t)dt - \ln\prod_{i=1}^{q} r_i = \int_0^w f(t)E\big(t, \exp(y(t)), \exp(x(t)), \exp(y(t))\big)\big[\exp(x(t) - y(t))\big]dt \geq$$

$$\geq \big[\exp(x(\xi_1) - y(\eta_2))\big]\int_0^w f(t)E\big(t, \exp(y(t)), \exp(x(t)), \exp(y(t))\big)dt >$$

$$> \big[\exp(x(\xi_1) - y(\eta_2))\big]\frac{1}{k}\int_0^w \frac{f(t)}{m(t)}dt.$$

Then, we get

$$y(\eta_2)) > x(\xi_1) - \ln\left(\frac{\int_0^w d(t)dt - \ln\prod_{i=1}^{q} r_i}{1/k\dfrac{f(t)}{m(t)}}\right).$$

By (28), we have

$$y(\eta_2)) > H_2 - \ln\left(\frac{\int_0^w d(t)dt - \ln\prod_{i=1}^{q} r_i}{1/k\dfrac{f(t)}{m(t)}}\right) := l_4.$$

Since $y(\eta_2)$ is the supremum of $y(t)$ for $t \in [0, w]$, then there exists $t_4 \in [0, w]$ such that $y(\eta_2) \geq \geq y(t_4) > l_4$. If we use second inequality of Lemma 1, we get

$$y(t) \geq y(t_4) - \int_0^w |x'(t)|dt \geq$$

$$\geq y(t_4) - \left(2\int_0^w d(t)dt - \ln\prod_{i=1}^{q} r_i\right) > H_4 := l_4 - M_2. \tag{30}$$

By (29) and (30) we obtain $\max_{t\in[0,w]}|y(t)| \leq B_2 :\, = \max\big\{|H_3|, |H_4|\big\}$. Obviously, B_1 and B_2 are both independent of λ. Let $M = B_1 + B_2 + 1$. Then $\max_{t\in[0,w]}\big\|(x, y)^{\intercal}\big\| < M$. Let $\Omega = \big\{\big\|(x, y)^{\intercal}\big\| \in X : \big\|(x, y)^{\intercal}\big\| < M\big\}$ and Ω verifies the requirement (a) in Theorem 1. If $\big\|(x, y)^{\intercal}\big\| \in \operatorname{Ker} L \cap \partial\Omega$, $\big\|(x, y)^{\intercal}\big\|$ is a constant with $\big\|(x, y)^{\intercal}\big\| = M$, then

$$VC\big((x, y)^{\intercal}\big) =$$

$$= \left(\begin{bmatrix} \int\limits_0^w a(s) - b(s)\exp(x) - c(s)E(s, y(s), x(s), y(s))ds + \ln \prod\limits_{i=1}^{q}(1+g_i) \\ \int\limits_0^w -d(s) + f(s)E(s, x(s), x(s), y(s))ds + \ln \prod\limits_{i=1}^{q}(r_i) \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right) \neq$$

$$\neq \left((0,0)^\intercal, \ldots, (0,0)^\intercal\right),$$

$$JVC\big((x,y)^\intercal\big) = \begin{bmatrix} \int\limits_0^w a(s) - b(s)\exp(x(s)) - c(s)E(s, y(s), x(s), y(s))ds + \ln \prod\limits_{i=1}^{q}(1+g_i) \\ \int\limits_0^w -d(s) + f(s)E(s, x(s), x(s), y(s))ds + \ln \prod\limits_{i=1}^{q}(r_i) \end{bmatrix},$$

where $J\colon \operatorname{Im} V \to \operatorname{Ker} L$ such that $J\big((x,y)^\intercal, (0,0)^\intercal, \ldots, (0,0)^\intercal\big) = (x,y)^\intercal$.

Define the homotopy $H_\nu = \nu(JVC) + (1-\nu)G$, where

$$G\big((x,y)^\intercal\big) = \begin{bmatrix} \int\limits_0^w a(s) - b(s)\exp(x)ds + \ln \prod\limits_{i=1}^{q}(1+g_i) \\ \int\limits_0^w d(s) - f(s)E(s, x, x, y)ds + \ln \prod\limits_{i=1}^{q}(r_i) \end{bmatrix}.$$

Since H_ν is a homotopy, then for each $\nu \in [0,1]$ the Brouwer degree of $\deg(JVC, \Omega \cap \operatorname{Ker} L, 0)$, $\deg(G, \Omega \cap \operatorname{Ker} L, 0)$ and $\deg(\nu(JVC) + (1-\nu)G, \Omega \cap \operatorname{Ker} L, 0)$ are equal. Then, it is enough to find the Brouwer degree of one of them.

Take DJ_G as the determinant of the Jacobian of G. Since $(x,y)^\intercal \in \operatorname{Ker} L$, then Jacobian of G is

$$\begin{bmatrix} -e^x \int\limits_0^w b(s)ds & 0 \\ \begin{array}{c} \int\limits_0^w -f(s)E(s, e^x, e^x, e^y)ds + \\ + \int\limits_0^w \dfrac{(e^x)^2 f(s)\beta(s)}{(\alpha(s) + \beta(s)e^x + m(s)e^y)^2}ds \end{array} & -\int\limits_0^w \dfrac{e^x e^y f(s)m(s)}{(\alpha(s) + \beta(s)e^x + m(s)e^y)^2}ds \end{bmatrix}.$$

All the functions in Jacobian of G is positive, then $\operatorname{sign} DJ_G$ is always positive. Hence

$$\deg(JVC, \Omega \cap \operatorname{Ker} L, 0) = \deg(G, \Omega \cap \operatorname{Ker} L, 0) =$$

$$= \sum_{(x,y)^\intercal \in G^{-1}((0,0)^\intercal)} \operatorname{sign} DJ_G\big((x,y)^\intercal\big) \neq 0.$$

Thus, all the conditions of Theorem 1 are satisfied. Therefore, system (4) has at least one positive w-periodic solution.

If the given system (4) has at least one periodic solution, then for at least one solution of $y(t)$, $\exp(y(t))$ does not go to zero as t goes to infinity which means $y(t)$ does not go to extinction. Then by using Lemma 9, since systems (3) and (4) are equivalent, we get for all solutions of $y(t)$, $\exp(y(t))$ does not go to zero as t tends to infinity. Hence we are done.

Since systems (4) and (3) are equivalent, to if one of them has at least one w-periodic solution, then the other one also has.

***3.3. Some simple results obtained from Subsections* 3.1 *and* 3.2.**

***Remark* 1.** Assume that (5) and

$$\int\limits_0^w a(t) - \frac{c(t)}{m(t)} dt + \ln \prod_{i=1}^q (1+g_i) > 0 \tag{31}$$

are satisfied. Consider the system

$$\begin{aligned} &\tilde{v}'(t) = (a(t) - \frac{c(t)}{m(t)})\tilde{v}(t) - b(t)\tilde{v}^2(t), \quad t \neq t_k, \\ &\tilde{v}(t_k^+) = (1+g_k)\tilde{v}(t_k). \end{aligned} \tag{32}$$

By using system (32) and Lemma 2, we get

$$\int\limits_0^w a(t) - \frac{c(t)}{m(t)} dt + \ln \prod_{i=1}^q (1+g_i) = \int\limits_0^w b(t)\exp(v(t))dt.$$

Here, $\tilde{v}(t) = \exp(v(t))$. Therefore,

$$l_1 := \frac{\displaystyle\int_0^w a(t)dt + \ln \prod\nolimits_{i=1}^q (1+g_i)}{\displaystyle\int_0^w b(t)dt} \leq \tilde{v}(\xi_1),$$

where $\tilde{v}(\xi_1)$ is the supremum of $\tilde{v}$.

If we use system (32) and in this system take $\tilde{v}(t) = \exp(v(t))$, then

$$\int\limits_0^w |v'(t)|dt \leq \left[\int\limits_0^w a(t)dt + \int\limits_0^w b(t)\exp(u(t))\right] \leq$$

$$\leq \left[\int\limits_0^w a(t)dt + \int\limits_0^w a(t)dt + \ln \prod_{i=1}^q (1+g_i)\right].$$

By Lemma 2, supremum of $\tilde{v}(t)$, therefore supremum of $v(t)$ exists. Since $\tilde{v}(\xi_1)$ is the supremum of $\tilde{v}$, by the definition of $v(t)$, $v(\xi_1)$ is the supremum of $v(t)$ for $t \in [0, w]$, then there exists $t_1 \in [0, w]$ such that $v(\xi_1) \geq v(t_1) > l_1$. By using Lemma 2.4 in [4], we have

$$x(t) \geq v(t) \geq v(t_1) - \int\limits_0^w |v'(t)|dt \geq$$

$$\geq \ln\left(\frac{\int_0^w a(t)dt + \ln\prod_{i=1}^q (1+g_i)}{\int_0^w b(t)dt}\right) - \left(2\int\limits_0^w a(t)dt + \ln\prod_{i=1}^q (1+g_i)\right).$$

The following corollary is obtained from Lemma 5.

***Corollary* 1.** *In addition to* (31) *and* (5), *if the following inequality is also satisfied*:

$$\left(\frac{\int_0^w a(t) - \dfrac{c(t)}{m(t)}dt + \ln\prod_{i=1}^q (1+g_i)}{\int_0^w b(t)dt}\right) \times$$

$$\times \exp\left[-\left(2\int\limits_0^w a(t)dt + \ln\prod_{i=1}^q (1+g_i)\right)\right] \times$$

$$\times \left(\int\limits_0^w f(t)dt - \beta^u\left(\int\limits_0^w d(t)dt - \ln\prod_{i=1}^q (r_i)\right)\right) -$$

$$-\alpha^u\left(\int\limits_0^w d(t)dt - \ln\prod_{i=1}^q (r_i)\right) > 0,$$

then system (3) *has at least one* w*-periodic solution.*

This result is same with Theorem 2 in [16] for continuous case.

3.4. Some examples.

***Example* 1:**

$$x'(t) = (2\sin(2\pi t) + 3) - (0.2\sin(2\pi t) + 0.4)\exp(x) -$$

$$- \frac{(5 + 2\cos(2\pi t))\exp(y)}{(\sin(2\pi t) + 1.2) + (1 + 0.5\sin(2\pi t))\exp(x) + \exp(y)},$$

$$y'(t) = -(0.5\sin(2\pi t) + 1.5) + \frac{(0.8\cos(2\pi t) + 4.45)\exp(x)}{(\sin(2\pi t) + 1.2) + (1 + 0.5\sin(2\pi t))\exp(x) + \exp(y)}, \tag{33}$$

$$\Delta x(t_k^+) = \ln(1 + g_k),$$

$$\Delta y(t_k^+) = \ln(r_k).$$

Impulse points: $t_1 = 2k + 1/4$, $t_2 = 2k + 3/4$ for $k = 1, 2, 3, \ldots$ and $q = 2$. $g_1 = e^1 - 1$, $g_2 = e^1 - 1$, $r_1 = e^{0.4}$, $r_2 = e^{0.4}$.

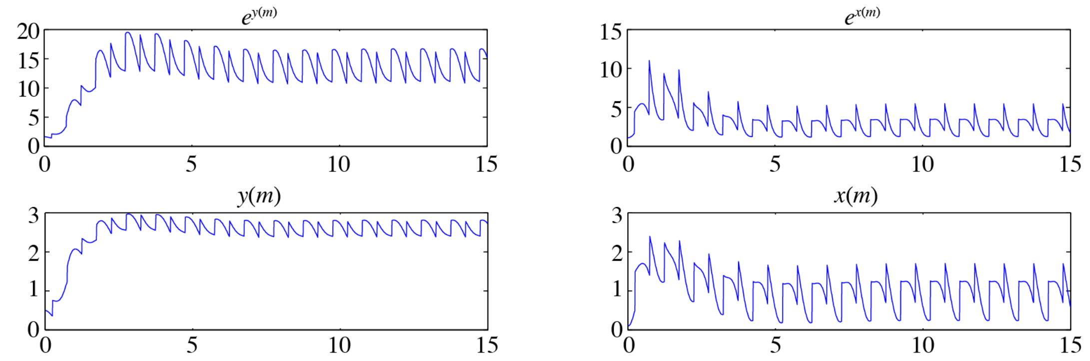

Fig. 1

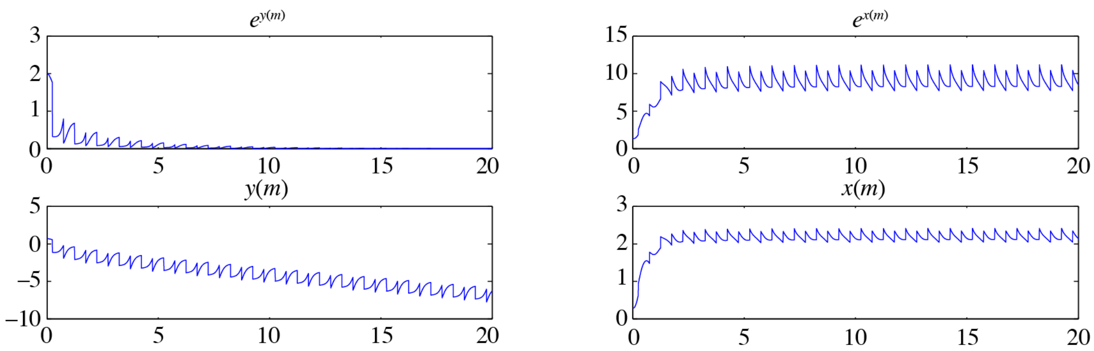

Fig. 2

First consider the system

$$x'(t) = (2\sin(2\pi t) + 3) - (0.2\sin(2\pi t) + 0.4)\exp(x),$$

$$\Delta x(t_k^+) = \ln(1 + g_k),$$

$$g_1 = e^1 - 1, \quad g_2 = e^1 - 1.$$

Then, by using the program Mathlab, $x^* > 6.5$ can be found. Then, by doing some simple calculations, it is easy to find that system (33) satisfies inequality (10) and by Lemma 8 system (33) has at least one 1-periodic solution and Fig. 1 $(x(0) = 0.1,\ y(0) = 0.5)$ also supports this result.

In Example 1, system (33) if we take $g_1 = e^{0.3} - 1$, $g_2 = e^{0.3} - 1$, $r_1 = e^{-1.7}$, $r_2 = e^{-1.7}$, then the inequality (9) is satisfied and by Lemma 5 we obtain the Fig. 2 $(x(0) = 0.3,\ y(0) = 0.7)$.

This result shows us the importance of the impulses. When we take impulses as $g_1 = e^{0.3} - 1$, $g_2 = e^{0.3} - 1$, $r_1 = e^{-1.7}$, $r_2 = e^{-1.7}$; although the system without impulses is same, since system (33) does not satisfies the inequality (10), predator goes to extinction.

The following example is for Corollary 1.

***Example* 2:**

$$x' = (0.2\sin(2\pi t) + 0.3) - (0.2\sin(2\pi t) + 0.2)\exp(x) -$$

$$- \frac{(0.1 + 0.1\cos(2\pi t))\exp(y)}{(0.5\sin(2\pi t) + 0.7) + (1 + 0.5\cos(2\pi t))\exp(x) + \exp(y)}, \quad t \neq t_k,$$

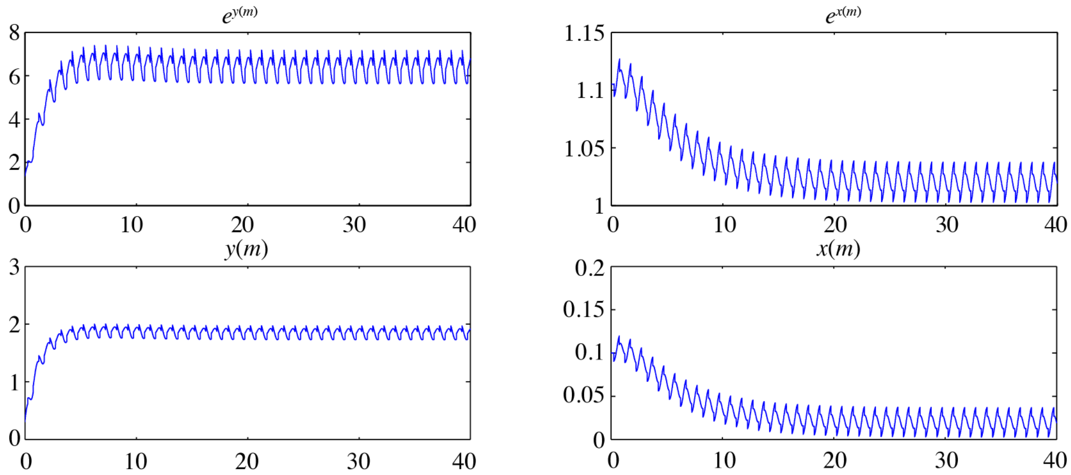

Fig. 3

$$y'(t) = -(0.3\sin(2\pi t) + 1) +$$

$$+\frac{(4\cos(2\pi t) + 6.5)\exp(x)}{(0.5\sin(2\pi t) + 0.7) + (1 + 0.5\cos(2\pi t))\exp(x) + \exp(y)}, \quad t \neq t_k,$$

$$\Delta x(t_k) = \ln(1 + g_k),$$

$$\Delta y(t_k) = \ln(r_k).$$

Impulse points: $t_1 = 2k + 1/4,\ t_2 = 2k + 3/4$, and $q = 2,\ g_1 = e^{-0.01} - 1,\ g_2 = e^{-0.01} - 1$, $p_1 = e^{0.1},\ p_2 = e^{0.1}$.

Example 2 satisfies the condition of Corollary 1, therefore it has at least one w-periodic solution and Fig. 3 $(x(0) = 0.1,\ y(0) = 0.3)$ supports this result.

3.5. Global attractivity of the solutions.

Theorem 3. *If inequalities* (2), (5) *and* (10) *are satisfied*, *then the* w*-periodic solution of the system* (3) *is globally attractive* (*globally asymptotically stable*).

Proof. Proof is very similar to the proof of Theorem 4.4 in [22]. To get the result, we apply Lemma 1. Let us consider the following ordinary differential equation:

$$\begin{gathered} z'(t) = F(t, z(t)), \\ z(t_k^+) - z(t_k) = I_k(z(t_k)), \\ z(0) = \phi. \end{gathered} \tag{34}$$

Here, $F \in C([0,\infty) \times \mathbb{R}^2, \mathbb{R}^2)$, $\phi \in \mathbb{R}^2$, $F(t + w, u) = F(t, u)$, $I_k \in C(\mathbb{R}^2, \mathbb{R}^2)$ and there exists an integer q such that $I_{k+q} = I_k,\ t_{k+q} = t_k + w$. Then, the operator that solves system (34) can be written as

$$\hat{T}(t)z = ze^{-\lambda t} + \int\limits_0^t e^{-\lambda(t-s)}\big[F(s, \hat{T}(s)z) + \lambda\hat{T}(s)z\big]ds + \sum_{0<t_k<t} e^{-\lambda(t-t_k)} I_k(\hat{T}(t_k)z),$$

where λ is a positive constant. It is obvious that $T(0) = I$. Also, we can verify that

$$u(s) = \begin{cases} T(s)z, & 0 \leq s \leq w, \\ T(s-w)T(w)z, & w \leq s \leq t+w, \end{cases}$$

is the solution of system (34) with the initial value $u(0) = z$, where $s \in [0, t+w]$. By uniqueness theorem, system (34) has a unique solution, therefore $T(t+w)z = u(t+w) = T(t)T(w)z$. This is true when $t \neq t_k$. For $t = t_k$,

$$T(t_k^+ + w)z = T(t_k + w)z + I_k(T(t_k + w)z) =$$

$$= T(t_k)T(w)z + I_k(T(t_k)T(w)z) = T(t_k^+)T(w)z.$$

To apply Lemma 1, let $S = T(w)$, $S^2 = SoS = T(w)oT(w) = T(2w)$. Here, the considered system (34) is a periodic system, therefore we can apply Arzela – Ascoli theorem for impulsive differential equations and the result from [1]. Hence, we obtain that $T(t)$ is a compact operator.

If we take $X_i^+ = \{z_i : z_i \in \mathbb{R},\ z_i \geq 0\}$ for $i = 1, 2$ and $X_{i_0}^+ = \{z_i : z_i \in \mathbb{R},\ z_i > 0\}$ for $i = 1, 2$, then $X = X_1^+ \times X_2^+$, $X = X_{1_0}^+ \times X_{2_0}^+$ and $\delta X_0 = X/X_0$. When system (3) satisfies inequality (2), (5), (10), the system becomes permanent. Therefore, S satisfies the conditions of Lemma 1. Hence, S admits a global attractor which means the system has globally asymptotically stable or globally attractive w-periodic solution.

***Corollary* 2.** *Assume that all the coefficient functions in system* (3) *are bounded*, *positive*, *w-periodic*, *from $PC(\mathbb{R}, \mathbb{R}^2)$. Then there exists globally attractive w-periodic solution for system* (3) *if and only if inequalities* (2), (5), *and* (10) *are satisfied*.

Proof is immediate from Theorems 2 and 3.

Example 1 satisfies all the inequalities (2), (5) and (10), therefore it has a w-periodic, globally asymptotically stable solution.

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