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A. Teymouri (Dep. Math., Central Tehran Branch, Islamic Azad Univ., Tehran, Iran),
A. Bodaghi (Dep. Math., Garmsar Branch, Islamic Azad Univ., Garmsar, Iran),
D. Ebrahimi Bagha (Dep. Math., Central Tehran Branch, Islamic Azad Univ., Tehran, Iran)

DERIVATIONS ON THE MODULE EXTENSION BANACH ALGEBRAS

ДИФЕРЕНЦІЮВАННЯ НА БАНАХОВИХ АЛГЕБРАХ РОЗШИРЕННЯ МОДУЛЯ

We correct some results presented in [M. Eshaghi Gordji, F. Habibian, A. Rejali, *Ideal amenability of module extension Banach algebras*, Int. J. Contemp. Math. Sci., **2**, № 5, 213 – 219 (2007)] and, using the obtained consequences, we find necessary and sufficient conditions for the module extension $\mathcal{A}\oplus X$ to be $(\mathcal{I}\oplus Y)$-weakly amenable, where $\mathcal{I}$ is a closed ideal of the Banach algebra $\mathcal{A}$ and Y is a closed $\mathcal{A}$-submodule of the Banach $\mathcal{A}$-bimodule X. We apply this result to the module extension $\mathcal{A}\oplus(X_1\dotplus X_2)$, where $X_1,\ X_2$ are two Banach $\mathcal{A}$-bimodules.

Виправлено деякі результати роботи [M. Eshaghi Gordji, F. Habibian, A. Rejali, *Ideal amenability of module extension Banach algebras*, Int. J. Contemp. Math. Sci., **2**, № 5, 213 – 219 (2007)] та за допомогою отриманих наслідків знайдено необхідні та достатні умови того, що розширення модуля $\mathcal{A}\oplus X$ буде $(\mathcal{I}\oplus Y)$-слабко аменабельним, де $\mathcal{I}$ — замкнений ідеал банахової алгебри $\mathcal{A}$, а Y — замкнений $\mathcal{A}$-субмодуль банахового $\mathcal{A}$-бімодуля X. Ці результати застосовано до розширення модуля $\mathcal{A}\oplus(X_1\dotplus X_2)$, де $X_1,\ X_2$ — банахові $\mathcal{A}$-бімодулі.

1. Introduction. Let $\mathcal{A}$ be a Banach algebra and X be a Banach $\mathcal{A}$-bimodule. Then X^* is a Banach $\mathcal{A}$-bimodule with module actions

$$\langle a\cdot x^*,x\rangle=\langle x^*,x\cdot a\rangle,\qquad \langle x^*\cdot a,x\rangle=\langle x^*,a\cdot x\rangle,\quad a\in A,\quad x\in X,\quad x^*\in X^*.$$

A *derivation* from a Banach algebra $\mathcal{A}$ into a Banach $\mathcal{A}$-bimodule X is a bounded linear mapping $D\colon\mathcal{A}\longrightarrow X$ such that $D(ab)=D(a)\cdot b+a\cdot D(b)$ for every $a,b\in A$. A derivation $D\colon\mathcal{A}\longrightarrow X$ is called *inner* if there exists $x\in X$ such that $D(a)=a\cdot x-x\cdot a=\delta_x(a)$ for $a\in\mathcal{A}$. A Banach algebra $\mathcal{A}$ is called *amenable* if every bounded derivation $D\colon\mathcal{A}\longrightarrow X^*$ is inner for every Banach $\mathcal{A}$-bimodule X, i.e., $H^1(\mathcal{A},X^*)=\{0\}$, where $H^1(\mathcal{A},X^*)$ is the first cohomology group from $\mathcal{A}$ with coefficients in X^*. This definition was introduced by B. E. Johnson in [10]. In addition, a Banach algebra $\mathcal{A}$ is *weakly amenable* if $H^1(\mathcal{A},\mathcal{A}^*)=\{0\}$. Bade, Curtis and Dales [1] introduced the notion of weak amenability for the first time for Banach algebras. They considered this concept only for commutative Banach algebras. Next, Johnson defined the weak amenability for arbitrary Banach algebras and showed that, for a locally compact group G, $L^1(G)$ is always weakly amenable [11]. In [7], Gorgi and Yazdanpanah introduced and studied the concept of ideal amenability for a Banach algebra. In fact, a Banach algebra $\mathcal{A}$ is called $\mathcal{I}$-weakly amenable if $H^1(\mathcal{A},\mathcal{I}^*)=\{0\}$ for a closed two-sided ideal $\mathcal{I}$ of $\mathcal{A}$, and ideally amenable if it $\mathcal{I}$-weakly amenable for every closed two-sided ideal $\mathcal{I}$ of $\mathcal{A}$. Weak amenability, and ideal amenability of module extension Banach algebras are investigated in [17] and [6], respectively. Furthermore, ideal amenability of the (projective) tensor product of Banach algebras is studied in [13]. An alternative notion of ideal amenability, namely, quotient ideal amenability for Banach algebras was introduced and investigated by the authors in [15]. Indeed, a Banach algebra $\mathcal{A}$ is said to be *quotient ideally amenable* if all derivations from $\mathcal{A}$ into its annihilators of all ideals are inner. For the ideal Connes-amenability of dual Banach algebras, we refer to [12].

The main motivation for this work is taken from [4–6] and [17]. In this paper, for a Banach algebra $\mathcal{A}$, we study the $(\mathcal{I}\oplus\mathcal{I})$-weakly amenability of module extension $\mathcal{A}\oplus\mathcal{I}$, where $\mathcal{I}$ is a closed ideal of $\mathcal{A}$. We apply this result to show that $\mathcal{B}(\mathcal{H})\oplus\mathcal{K}(\mathcal{H})$ is $(\mathcal{K}(\mathcal{H})\oplus\mathcal{K}(\mathcal{H}))$-weakly amenable, where $\mathcal{B}(\mathcal{H})$ and $\mathcal{K}(\mathcal{H})$ are bounded linear and compact operators on the infinite dimensional Hilbert space $\mathcal{H}$, respectively. Finally, we prove that under what conditions the module extension $\mathcal{A}\oplus(X_1\dot{+}X_2)$ can be $\mathcal{J}$-weakly amenable, where $X_1,\ X_2$ are two Banach $\mathcal{A}$-bimodules and $\mathcal{J}$ is a closed ideal of $\mathcal{A}\oplus(X_1\dot{+}X_2)$.

2. Module extension Banach algebra. We start this section with an example of Banach semigroup algebras which is $\mathcal{I}$-weakly amenable.

Let S be a non-empty set. Consider $l^1(S)=\Big\{f\in\mathbb{C}^S\colon \sum_{s\in S}\big|f(s)\big|<\infty\Big\}$ with the norm $\|\cdot\|_1$ given by $\|f\|_1=\sum_{s\in S}|f(s)|$ for $f\in l^1(S)$. We write δ_s for the characteristic function of $\{s\}$ when $s\in S$. Suppose that S is a semigroup. We define the convolution of two elements f and g of $l^1(S)$ by

$$(f*g)(s)=\sum_{uv=s}f(u)g(v),\quad s\in S,$$

where $\sum_{uv=s}f(u)g(v)=0$, when there are no elements $u,v\in S$ with $uv=s$. Then $\big(l^1(S),*,\|\cdot\|_1\big)$ becomes a Banach algebra that is called the *semigroup algebra* of S. Clearly, $l^1(S)$ is commutative if and only if S is Abelian. Moreover, the dual space of $l^1(S)$ is $l^\infty(S)$, with the duality

$$\langle f,\lambda\rangle=\sum_{s\in S}f(s)\lambda(s),\qquad f\in l^1(S),\quad \lambda\in l^\infty(S).$$

Let S be a semigroup and $E(S)=\{e\in S\colon e^2=e\}$ be the set of idempotents in S. We note that if $\mathcal{I}$ is an ideal in S, then $l^1(\mathcal{I})$ is a closed ideal in $l^1(S)$.

***Example* 2.1.** Let $\mathbb{N}$ be the commutative semigroup of positive integers. Consider $(\mathbb{N},\vee)$ with maximum operation $m\vee n=\max\{m,n\}$. Obviously, each element of $\mathbb{N}$ is an idempotent. It is easily verified that all ideals of $(\mathbb{N},\vee)$ are exactly the sets $\mathcal{I}_n=\{m\in\mathbb{N}\colon m\geq n\}$, and so $l^1(\mathcal{I}_n)$ are ideals of $l^1(\mathbb{N})$. Indeed, for any element $f=\sum_{r\in\mathcal{I}_n}\alpha_r\delta_r$ and $g=\sum_{s\in\mathbb{N}}\beta_s\delta_s$, we have

$$f*g=\left(\sum_{r\in\mathcal{I}_n}\alpha_r\delta_r\right)\left(\sum_{s\in\mathbb{N}}\beta_s\delta_s\right)=\sum_{r\vee s=t\in\mathcal{I}_n}(\alpha_r\beta_s)\delta_t\in l^1(\mathcal{I}_n).$$

Similarly, $g*f\in l^1(\mathcal{I}_n)$. Since $E(\mathbb{N})=\mathbb{N}$ and $\mathbb{N}$ is a commutative semigroup with maximum operation, by [3] (Proposition 10.5), $l^1(\mathbb{N})$ is weakly amenable, and thus $l^1(\mathbb{N})$ is $l^1(\mathcal{I}_n)$-weakly amenable by [1] (Theorem 1.5).

Let $\mathcal{A}$ and X be a Banach algebra and a $\mathcal{A}$-bimodule, respectively. Consider $\mathcal{A}\oplus X$ as a Banach space with the following norm:

$$\|(a,x)\|=\|a\|+\|x\|,\quad a\in\mathcal{A},\quad x\in x.$$

Then $\mathcal{A}\oplus X$ is a Banach algebra with product

$$(a_1,x_1)\cdot(a_2,x_2)=(a_1a_2,x_1\cdot a_2+a_1\cdot x_2).$$

$\mathcal{A}\oplus X$ is called a *module extension* Banach algebra. Since $(\mathcal{A}\oplus X)^* = (0+X)^{\perp}\dot{+}(\mathcal{A}\oplus 0)^{\perp}$, where $\dot{+}$ denotes the direct $\mathcal{A}$-bimodule l_∞-sum, and $(0\oplus X)^{\perp}$(resp., $(\mathcal{A}\oplus 0)^{\perp}$) is isometrically isomorphic to $\mathcal{A}^*$ (resp., X^*) as $\mathcal{A}$-bimodule. For convenience, we simply identify the corresponding terms and write

$$(\mathcal{A}\oplus X)^* = \mathcal{A}^*\dot{+}X^*.$$

Suppose that X and Y are $\mathcal{A}$-bimodules. Recall that an $\mathcal{A}$-module morphism from X to Y is a bounded linear map $T\colon X \longrightarrow Y$ such that

$$T(a\cdot x) = a\cdot T(x), \qquad T(x\cdot a) = T(x)\cdot a, \quad a\in\mathcal{A}, \quad x\in X.$$

Let $\mathcal{A}$ and X be as the above. It is shown (without proof) in [5] that $\mathcal{J}$ is a closed ideal in $\mathcal{A}\oplus X$ if and only if there exist a closed ideal $\mathcal{I}$ of $\mathcal{A}$ and a closed $\mathcal{A}$-submodule Y of X such that $\mathcal{J} = \mathcal{I}\oplus Y$ and $(\mathcal{I}\cdot X)\cup(X\cdot\mathcal{I})\subseteq Y$. We mention that one side of this result is not valid in general. In the next lemma we correct it and indicate the proof completely.

Lemma 2.1. *Let $\mathcal{A}$ be a Banach algebra and X be a Banach $\mathcal{A}$-bimodule.*

(i) *If $\mathcal{I}$ is a closed ideal of $\mathcal{A}$ and Y is a closed $\mathcal{A}$-submodule of X such that $(\mathcal{I}\cdot X)\cup(X\cdot\mathcal{I})\subseteq$ $\subseteq Y$, then $\mathcal{J} = \mathcal{I}\oplus Y$ is a closed ideal of $\mathcal{A}\oplus X$.*

(ii) *Let $\mathcal{J}$ is a closed ideal in $\mathcal{A}\oplus X$ and*

$$\mathcal{I} = \{a\in\mathcal{A} | (a,x)\in\mathcal{J} \quad \textit{for some} \quad x\in X\},$$

$$Y = \{x\in X | (a,x)\in\mathcal{J} \quad \textit{for some} \quad a\in\mathcal{A}\}.$$

Then $\mathcal{I}$ is an ideal of $\mathcal{A}$ and Y is $\mathcal{A}$-submodule of X. Moreover, if $\mathcal{A}$ has a approximate identity and left action $\mathcal{A}$ over X is zero, then $\mathcal{J} = \mathcal{I}\oplus Y$.

Proof. (i) Consider $(i,y)\in\mathcal{J} = \mathcal{I}\oplus Y$ and $(a,x)\in\mathcal{A}\oplus X$. By assumption, we have $(i,y)\cdot(a,x) = (ia, i\cdot x + y\cdot a)\in(\mathcal{I}\oplus Y)$, and thus $(i,y)\cdot(a,x)\subseteq\mathcal{J}$. Similarly, $(a,x)\cdot(i,y)\subseteq\mathcal{J}$. Since $\mathcal{I}$ and Y are closed, $\mathcal{J}$ is a closed ideal of $\mathcal{A}\oplus X$.

(ii) Let $b\in\mathcal{I}$ and $a\in\mathcal{A}$. Then there exists $x\in X$ such that $(b,x)\in\mathcal{J}$. We obtain

$$(a,0)\cdot(b,x) = (ab, a\cdot x)\in\mathcal{J},$$

$$(b,x)\cdot(a,0) = (ba, x\cdot a)\in\mathcal{J}.$$

The above relations imply that $ab, ba\in\mathcal{I}$, which means $\mathcal{I}$ is an ideal of $\mathcal{A}$. Similarly, one can show that Y is a $\mathcal{A}$-submodule of X. Now, suppose that (a_α) is an approximate identity for $\mathcal{A}$. By Cohen's factorization theorem (a_α) is an approximate identity for X. It is obvious that $\mathcal{J}\subseteq\mathcal{I}\oplus Y$. Let $y\in Y$ and $a\in\mathcal{I}$. In this case, there exist $y_0\in X$ and $a_0\in\mathcal{A}$ such that $(a,y_0),(a_0,y)\in\mathcal{J}$. We have

$$\big\|(a_\alpha,0)\cdot(a,y_0) - (a,0)\big\| = \big\|(a_\alpha a, a_\alpha\cdot y_0) - (a,0)\big\| = \big\|a_\alpha a - a\big\| \to 0.$$

Hence, $(a,0)\in\mathcal{J}$. Similarly, $(a_0,0)\in\mathcal{J}$, and so $(a,y) = (a,0) + (a_0,y) - (a_0,0)\in\mathcal{J}$. Therefore, $\mathcal{I}\oplus Y\subseteq\mathcal{J}$.

Lemma 2.1 is proved.

Let $\mathcal{A}$ be a Banach algebra, X be a Banach $\mathcal{A}$-bimodule, and $\mathcal{I}\oplus Y$ be a closed ideal of $\mathcal{A}\oplus X$. In view of [17] and that

$$(\mathcal{I}\cdot X)\cup(X\cdot\mathcal{I})\subseteq Y \tag{2.1}$$

the module actions are successively defined as follows:

first, for $x\in X,\ F\in Y^*$, define $x\cdot F,\ F\cdot x\in\mathcal{I}^*$ by

$$\langle x\cdot F,i\rangle=\langle F,i\cdot x\rangle,\qquad \langle F\cdot x,i\rangle=\langle F,x\cdot i\rangle,\quad i\in\mathcal{I}; \tag{2.2}$$

for $a\in\mathcal{A}$ and $u\in\mathcal{I}^*$, define $a\cdot u,u\cdot a\in\mathcal{I}^*$ via

$$\langle a\cdot u,i\rangle=\langle u,ia\rangle,\qquad \langle u\cdot a,i\rangle=\langle u,ai\rangle,\quad i\in\mathcal{I}; \tag{2.3}$$

also, for $a\in\mathcal{A}$ and $F\in Y^*$, define $a\cdot F,\ F\cdot a\in Y^*$ through

$$\langle a\cdot F,y\rangle=\langle F,y\cdot a\rangle,\qquad \langle F\cdot a,y\rangle=\langle F,a\cdot y\rangle,\quad y\in Y. \tag{2.4}$$

Throughout this paper, we assume that $\mathcal{I}$ is a closed ideal of Banach algebra $\mathcal{A}$ and Y is a closed $\mathcal{A}$-submodule of Banach $\mathcal{A}$-bimodule X such that condition (2.1) holds unless otherwise stated explicitly.

Lemma 2.2. *Suppose that X is a Banach $\mathcal{A}$-bimodule and $\mathcal{I}\oplus Y$ is a closed ideal of $\mathcal{A}\oplus X$. Then $(\mathcal{A}\oplus X)$-bimodule actions on $\mathcal{I}^*\dot{+}Y^*$ are given by the following formulas*:

$$(a,x)\cdot(u,F)=(a\cdot u+x\cdot F,a\cdot F), \tag{2.5}$$

$$(u,F)\cdot(a,x)=(u\cdot a+F\cdot x,F\cdot a). \tag{2.6}$$

Proof. For $(i,y)\in(\mathcal{I}\oplus Y)$, $(u,F)\in(\mathcal{I}\oplus Y)^*$, by using relations (2.2), (2.3) and (2.4), we have

$$\langle(a,x)\cdot(u,F),(i,y)\rangle=\langle(u,F),(i,y)\cdot(a,x)\rangle=$$

$$=\langle(u,F),(ia,i\cdot x+y\cdot a)\rangle=\langle u,ia\rangle+\langle F,i\cdot x+y\cdot a\rangle=$$

$$=\langle u,ia\rangle+\langle F,i\cdot x\rangle+\langle F,y\cdot a\rangle=\langle a\cdot u,i\rangle+\langle x\cdot F,i\rangle+\langle a\cdot F,y\rangle=$$

$$=\langle a\cdot u+x\cdot F,i\rangle+\langle a\cdot F,y\rangle=\langle(a\cdot u+x\cdot F,a\cdot F),(i,y)\rangle.$$

Hence, equality (2.5) holds. The accuracy of relation (2.6) can be obtained similarly.

Lemma 2.2 is proved.

We wish to find necessary and sufficient conditions for a module extension Banach algebra $\mathcal{A}\oplus X$ to be $\mathcal{I}\oplus Y$-weakly amenable. However, to achieve our purposes in this paper, we need three upcoming lemmas which were presented as Lemmas 2.1 and 2.3 of [5]. We include them without proof.

Lemma 2.3. *Suppose that $\Gamma\colon X\longrightarrow\mathcal{I}^*$ is a continuous $\mathcal{A}$-bimodule morphism. Then $\bar{\Gamma}\colon\mathcal{A}\oplus X\longrightarrow(\mathcal{I}\oplus Y)^*$ defined by $\bar{\Gamma}((a,x))=(\Gamma(x),0)$ is a continuous derivation. The derivation $\bar{\Gamma}$ is inner if and only if there exists $F\in Y^*$ such that $a\cdot F-F\cdot a=0$ and $\Gamma(x)=x\cdot F-F\cdot x$ for $a\in\mathcal{A}$ and $x\in X$.*

Lemma 2.4. *Suppose that $D\colon\mathcal{A}\longrightarrow\mathcal{I}^*$ is a continuous derivation. Then $\bar{D}\colon\mathcal{A}\oplus X\longrightarrow$ $\longrightarrow(\mathcal{I}\oplus Y)^*$ defined by $\bar{D}((a,x))=(D(a),0)$ is also a continuous derivation, D is inner if and only if $\bar{D}$ is inner.*

Lemma 2.5. *Suppose that* $T\colon X \longrightarrow Y^*$ *is a continuous* $\mathcal{A}$*-bimodule morphism, satisfying* $x \cdot T(y) + T(x) \cdot y = 0$ *for all* $x, y \in X$. *Then* $\bar{T}\colon \mathcal{A} \oplus X \longrightarrow (\mathcal{I} \oplus Y)^*$ *defined by* $\bar{T}((a, x)) = (0, T(x))$ *is a continuous derivation,* $\bar{T}$ *is inner if and only if* $T = 0$.

In light of the above lemmas, we correct the proof of Theorem 2.4 from [5], and so we bring its proof for the sake of completeness.

Theorem 2.1. *Suppose that* X *is a Banach* $\mathcal{A}$*-bimodule and* $\mathcal{I} \oplus Y$ *is a closed ideal of* $\mathcal{A} \oplus X$. *Then the module extension Banach algebra* $\mathcal{A} \oplus X$ *is* $(\mathcal{I} \oplus Y)$*-weakly amenable if and only if the following conditions hold*:

(i) $H^1(\mathcal{A}, \mathcal{I}^*) = \{0\}$;

(ii) *the only continuous derivation* $D\colon \mathcal{A} \longrightarrow Y^*$ *for which there is a continuous operator* $K\colon X \longrightarrow \mathcal{I}^*$ *such that* $K(a \cdot x) = D(a) \cdot x + a \cdot K(x)$ *and* $K(x \cdot a) = x \cdot D(a) + K(x) \cdot a$ $(a \in \mathcal{A})$ *are the inner derivations*;

(iii) *for every continuous* $\mathcal{A}$*-bimodule morphism* $\Gamma\colon X \longrightarrow \mathcal{I}^*$, *there exists* $F \in Y^*$ *such that* $a \cdot F - F \cdot a = 0$ *for* $a \in \mathcal{A}$ *and* $\Gamma(x) = x \cdot F - F \cdot x$ *for* $x \in X$;

(iv) *the only continuous* $\mathcal{A}$*-bimodule morphism* $T\colon X \longrightarrow Y^*$ *for which* $x \cdot T(y) + T(x) \cdot y = 0$ $(x, y \in X)$ *in* $\mathcal{I}^*$ *is* $T = 0$.

Proof. Denote by Δ_1 the projection from $(\mathcal{I} \oplus Y)^*$ onto $\mathcal{I}^*$ with kernel Y^*. Let Δ_2 be the projection $\mathrm{id} - \Delta_1 \colon (\mathcal{I} \oplus Y)^* \longrightarrow Y^*$ and let $\tau_1 \colon A \longrightarrow (\mathcal{A} \oplus X)$ and $\tau_2 \colon X \longrightarrow (\mathcal{A} \oplus X)$ be the inclusion mappings (i.e., $\tau_1(a) = (a, 0)$ and $\tau_2(x) = (0, x)$). Then Δ_1, Δ_2 are $\mathcal{A}$-bimodule morphisms and τ_1, τ_2 are algebra homomorphisms. We now proceed to prove the sufficiency. Suppose that conditions (i)–(iv) hold. Assume that $D\colon \mathcal{A} \oplus X \longrightarrow (\mathcal{I} \oplus Y)^*$ is a continuous derivation. Then $D \circ \tau_1 \colon \mathcal{A} \longrightarrow (\mathcal{I} \oplus Y)^*$ is a continuous derivation. This implies that $\Delta_1 \circ D \circ \tau_1 \colon \mathcal{A} \longrightarrow \mathcal{I}^*$ and $\Delta_2 \circ D \circ \tau_1 \colon \mathcal{A} \longrightarrow Y^*$ are continuous derivations. By condition (i), $\Delta_1 \circ D \circ \tau_1$ is inner. It follows from Lemma 2.4 that $\overline{\Delta_1 \circ D \circ \tau_1} \colon \mathcal{A} \oplus X \longrightarrow (\mathcal{I} \oplus Y)^*$ defined by

$$\overline{\Delta_1 \circ D \circ \tau_1}((a, x)) = \big(\Delta_1 \circ D \circ \tau_1(a), 0\big), \quad (a, x) \in (\mathcal{A} \oplus X),$$

is also inner.

Claim 1: $\Delta_2 \circ D \circ \tau_2 \colon X \longrightarrow Y^*$ is trivial.

Let $T = \Delta_2 \circ D \circ \tau_2$. By condition (iv), it suffices to show that T is an $\mathcal{A}$-bimodule morphism satisfying $x \cdot T(y) + T(x) \cdot y = 0,\ x, y \in X$. We have

$$0 = D(0,0) = D\big((0,x) \cdot (0,y)\big) = D\big((0,x)\big) \cdot (0,y) + (0,x) \cdot D((0,y)) =$$
$$= \big(0, \Delta_2 \circ D \circ \tau_2(x)\big) \cdot (0,y) + (0,x) \cdot \big(0, \Delta_2 \circ D \circ \tau_2(y)\big) =$$
$$= \big([\Delta_2 \circ D \circ \tau_2(x)]y, 0\big) + \big(x[\Delta_2 \circ D \circ \tau_2(y)], 0\big).$$

Thus, $x \cdot T(y) + T(x) \cdot y = 0$. On the other hand,

$$\big(0, T(a \cdot x)\big) = \Delta_2 \circ D\big((0, a \cdot x)\big) = \Delta_2 \circ D\big((a,0) \cdot (0,x)\big) =$$
$$= \Delta_2\big(D((a,0)) \cdot (0,x) + (a,0) \cdot D(0,x)\big) =$$
$$= \Delta_2\big((a,0) \cdot D(0,x)\big) = \Delta_2\big(aD \circ \tau_2(x)\big) = a \cdot T(x).$$

Similarly, $T(x \cdot a) = T(x) \cdot a$ and so T is an $\mathcal{A}$-bimodule morphism. This proves the claim 1.

Let $K = \Delta_1 \circ D \circ \tau_2 \colon X \longrightarrow \mathcal{I}^*$ and $D_1 = \Delta_2 \circ D \circ \tau_1 \colon \mathcal{A} \longrightarrow Y^*$.

Claim 2: $K(a\cdot x)=D_1(a)\cdot x+a\cdot K(x)$ and $K(x\cdot a)=x\cdot D_1(a)+K(x)\cdot a$ for $a\in A$ and $x\in X$. We have

$$(K(a\cdot x),0)=D\big((0,a\cdot x)\big)=D\big((a,0)\cdot(0,x)\big)=$$
$$=D((a,0))\cdot(0,x)+(a,0)\cdot D\big((0,x)\big)=$$
$$=(0,\Delta_2\circ D\circ\tau_1(a))\cdot(0,x)+(a,0)\cdot(\Delta_1\circ D\circ\tau_2(x),0)=$$
$$=\big([\Delta_2\circ D\circ\tau_1(a)]x,0\big)+\big(a[\Delta_1\circ D\circ\tau_2(x)],0\big)=$$
$$=(D_1(a)\cdot x,0)+(a\cdot K(x),0).$$

Similarly, for every $a\in\mathcal{A}$ and $x\in X$, we obtain $(0,K(x\cdot a))=(0,x\cdot D_1(a)+K(x)\cdot a)$, and, hence, by condition (ii), $D_1=\Delta_2\circ D\circ\tau_1$ is inner. Now, suppose that $F\in Y^*$ satisfies $D_1(a)=a\cdot F-F\cdot a$ for $a\in\mathcal{A}$. Let $K_1\colon X\longrightarrow\mathcal{I}^*$ be defined by $K_1(x)=x\cdot F-F\cdot x$ for $x\in X$. Then $K-K_1\colon X\longrightarrow\mathcal{I}^*$ is a continuous $\mathcal{A}$-bimodule morphism. In fact, from claim 2, for every $a\in\mathcal{A}$ and $x\in X$, we get

$$(K-K_1)(a\cdot x)=K(a\cdot x)-K_1(a\cdot x)=$$
$$=(D_1(a)\cdot x+a\cdot K(x))-((a\cdot x)\cdot F-F\cdot(a\cdot x))=$$
$$=(a\cdot F-F\cdot a)\cdot x+a\cdot K(x)-(a\cdot x\cdot F-F\cdot a\cdot x)=$$
$$=a(F\cdot x-x\cdot F)+a\cdot K(x)=a\cdot(K-K_1)(x).$$

Similarly, $K-K_1$ is a right $\mathcal{A}$-bimodule morphism. From the condition (iii), there is a $G\in Y^*$ such that $a\cdot G-G\cdot a=0$ for $a\in\mathcal{A}$ and $(K-K_1)(x)=x\cdot G-G\cdot x$ for all $x\in X$. By Lemma 2.3, we see that

$$\overline{K-K_1}\colon\mathcal{A}\oplus X\longrightarrow(\mathcal{I}\oplus Y)^*,$$
$$(a,x)\mapsto(K-K_1(x),0)$$

is an inner derivation. Using claim 1, we arrive at

$$D((a,x))=\big(\Delta_1\circ D\circ\tau_1(a)+K(x),D_1(a)\big)=$$
$$=\overline{\Delta_1\circ D\circ\tau_1}((a,x))+(\overline{K-K_1})((a,x))+\big(K_1(x),D_1(a)\big).$$

Since

$$\big(K_1(x),D_1(a)\big)=(x\cdot F-F\cdot x,a\cdot F-F\cdot a)=$$
$$=(a,x)\cdot(0,F)-(0,F)\cdot(a,x)$$

for $a\in\mathcal{A}$ and $x\in X$, it gives an inner derivation from $\mathcal{A}\oplus X$ into $(\mathcal{I}\oplus Y)^*$. Hence, as a sum of three inner derivation, D is inner. Therefore, $\mathcal{A}\oplus X$ is $(\mathcal{I}\oplus Y)$-weakly amenable.

Now, we prove the necessity. Suppose that $\mathcal{A}\oplus X$ is $(\mathcal{I}\oplus Y)$-weakly amenable. Let $D\colon\mathcal{A}\longrightarrow\mathcal{I}^*$ be a continuous derivation with the property given in condition (ii), we define $\bar{D}\colon$

$\mathcal{A} \oplus X \longrightarrow (\mathcal{I} \oplus Y)^*$ via

$$\bar{D}((a,x)) := (K(x), D(a)), \quad (a,x) \in (\mathcal{A} \oplus X).$$

Then $\bar{D}$ is a continuous derivation which is inner. Therefore, there exists $(u, F) \in (\mathcal{I} \oplus Y)^*$ such that

$$\bar{D}((a,x)) = (a,x) \cdot (u,F) - (u,F) \cdot (a,x).$$

Once more, for some $u \in \mathcal{I}^*$, we have $\big(K(x), D(a)\big) = (x \cdot F - F \cdot x, a \cdot F - F \cdot a)$, and thus $D(a) = a \cdot F - F \cdot a$. This means that D is inner, and condition (ii) holds. Moreover, conditions (i) and (iv) hold by Lemmas 2.4 and 2.5, respectively. Furthermore, condition (iii) holds by Lemma 2.3.

Theorem 2.1 is proved.

Here, we make some comments on condition (iii) in Theorem 2.1 as follows:

Remark **2.1.** The condition (iii) in Theorem 2.1 is equivalent to:

(iii)$'$ there is no non-zero continuous $\mathcal{A}$-bimodule morphism $\Gamma \colon X \longrightarrow \mathcal{I}^*$.

To prove this, suppose that (iii) holds. Taking $F = 0 \in Y^*$, we see that condition (iii) holds. Conversely, assume that condition (iii) holds and $\Gamma \colon X \longrightarrow \mathcal{I}^*$ is a continuous $\mathcal{A}$-bimodule morphism. Then there is an $F \in Y^*$ with $a \cdot F - F \cdot a = 0$ for all $a \in \mathcal{A}$ and $\Gamma(x) = x \cdot F - F \cdot x$ for all $x \in X$. Hence,

$$\langle \Gamma(x), a \rangle = \langle x \cdot F - F \cdot x, a \rangle = \langle F \cdot a - a \cdot F, x \rangle = 0, \quad a \in \mathcal{A}, \quad x \in X.$$

Therefore, $\Gamma(x) = 0$ for all $x \in X$. This shows that $\Gamma = 0$.

Proposition 2.1. *If condition* (iv) *of Theorem* 2.1 *holds and* $\mathcal{A}$ *has a bounded approximate identity in* $\mathcal{I}$, *then* $\operatorname{span}(\mathcal{I} \cdot X + X \cdot \mathcal{I})$ *is dense in* Y *and there is no non-zero continuous* $\mathcal{A}$-*bimodule morphism* $T \colon X \longrightarrow Y^*$ *satisfying* $\langle T(y), x \rangle + \langle T(x), y \rangle = 0$ *for all* $x, y \in Y$.

Proof. Suppose the assertion is false. Take $0 \neq f \in Y^*$ such that $f(\mathcal{I} \cdot X + X \cdot \mathcal{I}) = 0$. Let $F \in X^*$ be a Hahn – Banach extension of f on X. Define $T \colon X \longrightarrow Y^*$ by $T(x) := \langle F, x \rangle f$. By assumption, $\mathcal{A}$ has a bounded approximate identity in $\mathcal{I}$, say (a_α). Then, for $y \in Y$ and $a \in \mathcal{A}$, we have

$$T(a \cdot x) = \langle F, a \cdot x \rangle f = \langle x \cdot F, a \rangle f = \big\langle x \cdot F, \lim_\alpha (a_\alpha a) \big\rangle f = \lim_\alpha \langle x \cdot F, a_\alpha a \rangle f =$$

$$= \lim_\alpha \langle F, a_\alpha a \cdot x \rangle f = 0.$$

Moreover,

$$\langle a \cdot T(x), y \rangle = \langle a \cdot \langle F, x \rangle f, y \rangle = \langle F, x \rangle \langle f, y \cdot a \rangle = \langle F, x \rangle \big\langle f, y \cdot \lim_\alpha (a a_\alpha) \big\rangle =$$

$$= \lim_\alpha \langle F, x \rangle \langle f, y \cdot a a_\alpha \rangle = 0.$$

In the above relation, we have used the fact that $\mathcal{I} \cdot X + X \cdot \mathcal{I} \subseteq Y$. Similarly, $T(x) \cdot a = 0$, and thus T is a non-zero continuous $\mathcal{A}$-bimodule morphism and $\mathcal{I} \cdot T(x) = T(x) \cdot \mathcal{I} = \{0\}$. In other words, for all $x, y \in X, i \in \mathcal{I}$, we get

$$\langle i \cdot T(x), y \rangle = \langle i \langle F, x \rangle f, y \rangle = \langle F, x \rangle \langle f, y \cdot i \rangle = 0.$$

The equality $T(x) \cdot \mathcal{I} = \{0\}$ can be shown similarly. Since $T(x) \subset (\mathcal{I} \cdot X)^{\perp} \bigcap (X \cdot \mathcal{I})^{\perp}$, one can check that $x \cdot T(y) = T(x) \cdot y = 0$ in $\mathcal{I}^*$ for all $x, y \in X$. This leads to a contradiction with the condition (iv) of Theorem 2.1. We now assume an $\mathcal{A}$-bimodule morphism $T\colon X \longrightarrow Y^*$ satisfies $\langle T(y), x\rangle + \langle T(x), y\rangle = 0$ for $x, y \in Y$. Then, for any $i \in \mathcal{I}$,

$$\langle x \cdot T(y) + T(x) \cdot y, i\rangle = \langle T(y), i \cdot x\rangle + \langle T(i \cdot x), y\rangle = 0.$$

This show that $x \cdot T(y) + T(x) \cdot y = 0$ for all $x, y \in Y$, and therefore $T = 0$.

Proposition 2.1 is proved.

Let X_0 be an $\mathcal{A}$-bimodule with trivial right module action, i.e., $X_0\mathcal{A} = \{0\}$ such that $\mathcal{A}$ has a bounde approximate identity on $\mathcal{I}$. Suppose that Y_0 is a submodule of X_0 and $\mathcal{I}$ is a closed ideal of $\mathcal{A}$. We, firstly, observe that conditions (iii) and (iv) in Theorem 2.1 are reduced, respectively, to:

(iii)$'_0$ for any continuous $\mathcal{A}$-bimodule morphism $\Gamma\colon X_0 \longrightarrow \mathcal{I}^*$ there is $F \in {Y_0}^*$ such that $F \cdot a = 0$ for $a \in \mathcal{A}$ and $\Gamma(x) = x \cdot F$ for $x \in X_0$;

(iv)$'_0$ $\mathcal{I}X_0$ is dense in Y_0.

Indeed, the equivalence of (iii) and (iii)$'_0$ in this case is clear. Now, if (iv) holds for $X = X_0$, then Proposition 2.1 necessitates that span$(\mathcal{I} \cdot X_0)$ is dense in Y_0, and so (iv)$'_0$ holds. Conversely, with having the condition (iv)$'_0$, any $\mathcal{A}$-bimodule morphism $T\colon X_0 \longrightarrow Y_0^*$ is trivial, because the left $\mathcal{A}$-module action on Y_0^* is trivial.

Suppose that $\mathcal{A}$ has a bounded approximate identity on $\mathcal{I}$. From Proposition 1.5 in [10], condition (ii) in Theorem 2.1 always holds for $X = X_0$. This verifies the following consequence.

Theorem 2.2. *Suppose that $\mathcal{A}$ is $\mathcal{I}$-weakly amenable Banach algebra with a bounded approximate identity on $\mathcal{I}$. Then $\mathcal{A} \oplus X_0$ is $(\mathcal{I} \oplus Y)$-weakly amenable if and only if $\mathcal{I}X_0$ is dense in Y_0.*

By a similar way, Theorem 2.2 is valid when X_0 is a $\mathcal{A}$-bimodule that left module action is trivial.

The rest of this section we will be concerned with the two cases $X = \mathcal{I}$ and $X = \mathcal{A}^*$ as Banach $\mathcal{A}$-bimodules for which $\mathcal{I}$ is a closed ideal of $\mathcal{A}$. Firstly, we note that if $\mathcal{A}$ is not ideally amenable, then there is an ideal $\mathcal{I}_1$ of $\mathcal{A}$ such that $H^1(\mathcal{A}, {\mathcal{I}_1}^*) \neq \{0\}$. It follows from Theorem 2.1 that, for such $\mathcal{I}$, the module extension $\mathcal{A} \oplus \mathcal{I}$ is not $(\mathcal{I}_1 \oplus Y)$-weakly amenable. For the ideal amenability of $\mathcal{A} \oplus \mathcal{A}^*$, we have the following result.

Proposition 2.2. *For any Banach algebra $\mathcal{A}$, $\mathcal{A} \oplus \mathcal{A}^*$ is never ideally amenable.*

Proof. Since the identity mapping from $X = \mathcal{A}^*$ onto $\mathcal{I}^*$ is a non-zero continuous $\mathcal{A}$-bimodule morphism, Remark 2.1 implies that condition (iii) of Theorem 2.1 does not holds.

Proposition 2.2 is proved.

We now consider the case $X = \mathcal{I}$ and $\mathcal{I} \oplus \mathcal{I}$ be a closed ideal of $\mathcal{A} \oplus \mathcal{I}$. We show that under which conditions $H^1(\mathcal{A} \oplus \mathcal{I}, \mathcal{I}^* \dot{+} \mathcal{I}^*) = \{0\}$. In other words, we show when $\mathcal{A} \oplus \mathcal{I}$ is $(\mathcal{I} \oplus \mathcal{I})$-weakly amenable. Note that for $X = \mathcal{I}$ conditions (iii) and (iv) in Theorem 2.1 hold if and only if there is no non-zero $\mathcal{A}$-bimodule morphism T from $\mathcal{I}$ into $\mathcal{I}^*$. Besides, we see in the case $X = \mathcal{I}$ that conditions (i) and (ii) in Theorem 2.1 are the same.

In light of Theorem 5.4 of [17], we have the upcoming result.

Theorem 2.3. *Let $\mathcal{I}$ be a closed ideal of a Banach algebra $\mathcal{A}$ such that $\mathcal{A}$ is ideally amenable and $\mathcal{A}$ has a bounded approximate identity on $\mathcal{I}$. Then $\mathcal{A} \oplus \mathcal{I}$ is $(\mathcal{I} \oplus \mathcal{I})$-weakly amenable if and only if* $\mathrm{span}\{ij - ji; i, j \in \mathcal{I}\}$ *is dense in $\mathcal{I}$.*

Proof. If span $\{ij - ji; i, j \in \mathcal{I}\}$ is not dense in $\mathcal{I}$, then there is a non-zero linear functional f in $\mathcal{I}^*$ such that $\langle f, ij - ji\rangle = 0$ for all $i, j \in \mathcal{I}$. This means that $i \cdot f = f \cdot i$ for $i \in \mathcal{I}$. Define $T\colon \mathcal{I} \longrightarrow \mathcal{I}^*$ via $T(i) = i \cdot f = f \cdot i$. By assumption, $\mathcal{A}$ has a bounded approximate identity in $\mathcal{I}$, say (a_α). For every $j \in \mathcal{I}$ and $a \in \mathcal{A}$, we have

$$T(a \cdot i) = (a \cdot i) \cdot f = \big(\lim_{\alpha}(aa_\alpha) \cdot i\big) \cdot f = \lim_{\alpha}(aa_\alpha \cdot i) \cdot f = 0.$$

Moreover,

$$\langle a \cdot T(i), j\rangle = \langle i \cdot f, j \cdot a\rangle = \langle f, (j \cdot a) \cdot i\rangle = 0.$$

Similarly, $T(i) \cdot a = 0$, and hence T is a non-zero continuous $\mathcal{A}$-bimodule morphism and, clearly, $x \cdot T(y) + T(x) \cdot y = 0$ on $\mathcal{I}^*$ for all $x, y \in \mathcal{I}$. Since $\mathcal{A}$ is ideally amenable and $\mathcal{I}$ has a bounded approximate identity, $\mathcal{I}$ is weakly amenable by [7] (Theorem 1.9). Now, Proposition 1.3 from [2] implies that $\mathcal{I}^2$, the linear span of all product elements ij, $i, j \in \mathcal{I}$, is dense in $\mathcal{I}$, and so there are $i, j \in \mathcal{I}$ such that $\langle f, ij\rangle \neq 0$. This shows that $T \neq 0$. Therefore, in this case $\mathcal{A} \oplus \mathcal{I}$ is not $(\mathcal{I} \oplus \mathcal{I})$-weakly amenable.

For the converse, assume that span$\{ij - ji; i, j \in \mathcal{I}\}$ is dense in $\mathcal{I}$. Then, for any given continuous $\mathcal{A}$-bimodule morphism $T : \mathcal{I} \longrightarrow \mathcal{I}^*$, we have $T(a) = a \cdot f = f \cdot a$, where f is weak* cluster point of $(T(a_\alpha))$. This means that $f(ij - ji) = 0$ for all $i, j \in \mathcal{I}$. It follows that $f = 0$ and hence $T = 0$. Thus, the conditions (iii) and (iv) in Theorem 2.1 hold. The other two conditions hold automatically. Therefore, $\mathcal{A} \oplus \mathcal{I}$ is $(\mathcal{I} \oplus \mathcal{J})$-weakly amenable by Theorem 2.1.

Theorem 2.3 is proved.

From Theorem 2.3 we have immediately the next direct consequence.

***Corollary* 2.1.** *For any commutative Banach algebra $\mathcal{A}$ which has a bounded approximate identity in $\mathcal{I}$, $\mathcal{A} \oplus \mathcal{I}$ is not $(\mathcal{I} \oplus \mathcal{I})$-weakly amenable.*

Consider the algebra $\mathcal{A}(X)$ of approximable operators on a Banach space X. It is well-known that $\mathcal{A}(X)$ is the closure in $\mathcal{B}(X)$ of ideal of continuous finite-rank operators on X, where $\mathcal{B}(X)$ denotes the Banach algebra of all bounded linear operator on X. We also denote the algebra of compact operator on a Banach space X by $\mathcal{K}(X)$.

***Example* 2.2.** (i) For a Banach space X, it is shown in [8] that $\mathcal{A}(X) = \mathcal{K}(X)$ if $\mathcal{A}(X)$ are amenable. Consider $l^p = l^p(\mathbb{N})$, $1 < p < \infty$, which is a reflexive Banach space. By [10], $\mathcal{K}(l^p)$ is amenable for $1 < p < \infty$. It is also proved in [8] (Theorem 6.9) that $\mathcal{A}(X)$ is amenable for $X = l^p \oplus l^q$ $(1 < p, q < \infty)$ if and only if either $p = q$ or one of p or q is 2. Therefore, $\mathcal{A}(l^2) = \mathcal{K}(l^2)$ is amenable. Thus, $\mathcal{A}(l_2)$ as a closed ideal is $\mathcal{B}(l^2)$ and has a bounded approximate identity. Since l^2 is a Hilbert space, by [16] $\mathcal{B}(l^2)$ is a C^*-algebra, and thus it is ideally amenable [7] (Corollary 2.2). In fact, $H^1(\mathcal{B}(l^2), \mathcal{A}(l^2)^*) = 0$. It is shown in [14] that $\mathrm{span}\{ab - ba, a, b \in \mathcal{A}(l^2)\}$ is dense in $\mathcal{A}(l_2)$. Therefore, it follows from Theorem 2.3 that $\mathcal{B}(l^2) \oplus \mathcal{A}(l^2)$ is $(\mathcal{A}(l^2) \oplus \mathcal{A}(l^2))$-weakly amenable.

(ii) Let $\mathcal{B}(\mathcal{H})$ bounded linear operators on the infinite dimensional Hilbert space $\mathcal{H}$. It is well-known that $\mathcal{B}(\mathcal{H})$ has exactly two non-zero closed ideals $\mathcal{K}(\mathcal{H})$ and $\mathcal{B}(\mathcal{H})$. According to a classical result due to Halmos, every element in $\mathcal{B}(\mathcal{H})$ can be written as a sum of two commutators ([9] (Theorem 8) and [14] (Theorem 1)). On the other hand, $\mathcal{B}(\mathcal{H})$ and $\mathcal{K}(\mathcal{H})$ have an identity and as C^*-algebras which are ideally amenable by [7] (Corollary 2.2). Now, Theorem 2.3 implies that $\mathcal{B}(\mathcal{H}) \oplus \mathcal{K}(\mathcal{H})$ is $(\mathcal{K}(\mathcal{H}) \oplus \mathcal{K}(\mathcal{H}))$-weakly amenable.

3. Derivations on $\mathcal{A} \oplus (X_1 \dot{+} X_2)$. Suppose that X_1 and X_2 are two Banach $\mathcal{A}$-bimodules. We denote by $X_1 \dot{+} X_2$ the direct module sum of X_1 and X_2, i.e., the l_1 direct sum of X_1 and X_2 with the module actions given by

$$a \cdot (x_1, x_2) = (a \cdot x_1, a \cdot x_2), \qquad (x_1, x_2) \cdot a = (x_1 \cdot a, x_2 \cdot a), \quad a \in \mathcal{A}, \quad x_1 \in X_1, \quad x_2 \in X_2.$$

For this module actions we have the following equality:

$$(x_1, x_2) \cdot (f_1^*, f_2^*) = x_1 \cdot f_1^* + x_2 \cdot f_2^*, \qquad (x_1, x_2) \in X_1 \dot{+} X_2, \quad (f_1^*, f_2^*) \in (X_1 \dot{+} X_2)^*.$$

In this section, we investigate the $\mathcal{J}$-ideal amenability for Banach algebra $\mathcal{A} \oplus (X_1 \dot{+} X_2)$. In analogy with Lemma 2.1, we have the next lemma for $\mathcal{A} \oplus (X_1 \dot{+} X_2)$. Since the proof is similar, is omitted.

Lemma 3.1. *Let $\mathcal{A}$ be a Banach algebra and X_1, X_2 be two Banach $\mathcal{A}$-bimodules. If $\mathcal{I}$ is a closed ideal of $\mathcal{A}$, and Y_1, Y_2 are closed $\mathcal{A}$-submodules of X_1, X_2, respectively, then $\mathcal{I} \oplus (Y_1 \dot{+} Y_2)$ is a closed ideal of $\mathcal{A} \oplus (X_1 \dot{+} X_2)$ provided that $(\mathcal{I} \cdot X_1) \bigcup (X_1 \cdot \mathcal{I}) \subseteq Y_1$ and $(\mathcal{I} \cdot X_2) \bigcup (X_2 \cdot \mathcal{I}) \subseteq Y_2$.*

The idea of proof of the next result is taken from [17] (Lemma 7.1), but we bring its proof for the sake of completeness.

Theorem 3.1. *Suppose that $\mathcal{A} \oplus X_1$ is $(\mathcal{I} \oplus Y_1)$-weakly amenable and $\mathcal{A} \oplus X_2$ is $(\mathcal{I} \oplus Y_1)$-weakly amenable. Then the following are equivalent*:

(i) *$\mathcal{A} \oplus (X_1 \dot{+} X_2)$ is $(\mathcal{I} \oplus (Y_1 \dot{+} Y_2))$-weakly amenable*;

(ii) *there is no non-zero continuous $\mathcal{A}$-bimodule morphism $\Gamma \colon Y_1 \longrightarrow Y_2^*$*;

(iii) *there is no non-zero continuous $\mathcal{A}$-bimodule morphism $\Lambda \colon Y_2 \longrightarrow Y_1^*$.*

Proof. (i) $\Rightarrow$ (ii). Assume that $\mathcal{A} \oplus (X_1 \dot{+} X_2)$ is $(\mathcal{I} \oplus (Y_1 \dot{+} Y_2))$-weakly amenable. Let $\Gamma \colon Y_1 \longrightarrow Y_2^*$ be a continuous $\mathcal{A}$-bimodule morphism and $T \colon Y_1 \dot{+} Y_2 \longrightarrow (Y_1 \dot{+} Y_2)^*$ be the continuous $\mathcal{A}$-bimodule morphism defined through

$$T((y_1, y_2)) = (-\Gamma^*(y_2), \Gamma(y_1)), \quad (y_1, y_2) \in (Y_1 \dot{+} Y_2).$$

For each $(y_1, y_2), (z_1, z_2) \in (Y_1 \dot{+} Y_2)$ and $i \in \mathcal{I}$, we get

$$\big\langle (y_1, y_2) \cdot T\big((z_1, z_2)\big) + T\big((y_1, y_2)\big) \cdot (z_1, z_2), i \big\rangle =$$

$$= \big\langle -y_1 \cdot \Gamma^*(z_2) + y_2 \cdot \Gamma(z_1), i \big\rangle + \big\langle -\Gamma^*(y_2) \cdot z_1 + \Gamma(y_1) \cdot z_2, i \big\rangle =$$

$$= \big\langle -\Gamma(y_1) \cdot z_2 + y_2 \cdot \Gamma(z_1), i \big\rangle + \big\langle -y_2 \cdot \Gamma(z_1) + \Gamma(y_1) \cdot z_2, i \big\rangle = 0.$$

Thus, $(y_1, y_2) \cdot T\big((z_1, z_2)\big) + T\big((y_1, y_2)\big) \cdot (z_1, z_2) = 0$. It follows from condition (iv) of Theorem 2.1 that $T = 0$. This implies that $\Gamma = 0$.

(ii) $\Rightarrow$ (iii). Suppose that $\Lambda \colon Y_2 \longrightarrow Y_1^*$ is a continuous $\mathcal{A}$-bimodule morphism. It is easily verified that the mapping $\Gamma \colon Y_1 \longrightarrow Y_2^*$ defined by $\Gamma = \Lambda^*|_{Y_1}$ is a continuous $\mathcal{A}$-bimodule morphism. Consequently, $\Gamma = 0$. This implies that $\Lambda^* = 0$. Since Λ^* is weak*-weak* continuous and Y_1 is weak* dense in Y_1^{**}, we have $\Lambda = 0$.

(iii) $\Rightarrow$ (ii). The proof is similar to the preceding implication.

(ii) + (iii) $\Rightarrow$ (i). Since $\mathcal{A} \oplus X_1$ is $(\mathcal{I} \oplus Y_1)$-weakly amenable and $\mathcal{A} \oplus X_2$ is $(\mathcal{I} \oplus Y_1)$-weakly amenable, conditions (i) – (iii) of Theorem 2.1 hold automatically for $X = X_1 \dot{+} X_2$ and $Y = Y_1 \dot{+} Y_2$. For condition (iv), assume that $T \colon X \longrightarrow Y^*$ is a continuous $\mathcal{A}$-bimodule morphism fulfilling

$$(x_1, x_2) \cdot T\big((y_1, y_2)\big) + T\big((x_1, x_2)\big) \cdot (y_1, y_2) = 0, \quad (x_1, x_2), \quad (y_1, y_2) \in X.$$

Let $P_i : Y^* \longrightarrow Y_i^*$ be the natural projections, $\iota : Y \longrightarrow X$ and $\tau_i : Y_i \longrightarrow Y$ be the natural embedding for $i = 1, 2$. Choosing $x_2 = y_2 = 0$ and $x_1 = y_1 = 0$, we arrive at

$$x_1 \cdot P_1 \circ T \circ \iota \circ \tau_1(y_1) + P_1 \circ T \circ \iota \circ \tau_1(x_1) \cdot y_1 = 0,$$

$$x_2 \cdot P_2 \circ T \circ \iota \circ \tau_2(y_2) + P_2 \circ T \circ \iota \circ \tau_2(x_2) \cdot y_2 = 0$$

for all $x_i, y_i \in Y_i$ and $i = 1, 2$. Applying condition (iv) of Theorem 2.1 to the $\mathcal{A} \oplus X_i$ is $(\mathcal{I} \oplus Y_i)$-weakly amenable, for $i = 1, 2$, we have $P_i \circ T \circ \iota \circ \tau_i = T|_{Y_i} = 0$. Moreover, the parts (ii) and (iii) imply that $P_1 \circ T \circ \iota \circ \tau_2 : Y_2 \longrightarrow Y_1^*$ and $P_2 \circ T \circ \iota \circ \tau_1 : Y_1 \longrightarrow Y_2^*$ are trivial. These show that $T = 0$, that is, the condition (iv) of Theorem 2.1 holds.

Theorem 3.1 is proved.

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