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DENSITY AND CAPACITY OF BALLEANS GENERATED BY FILTERS

ЩІЛЬНІСТЬ ТА ЄМНІСТЬ БОЛЕАНІВ, ЗГЕНЕРОВАНИХ ФІЛЬТРАМИ

We consider a ballean $\mathbb{B} = (X, P, B)$ with an infinite support X and a free filter ϕ on X and define $B_{P\times\phi}(x, (\alpha, F))$ for every $\alpha \in P$ and $F \in \phi$. The ballean $(X, P \times \phi, B_{P\times\phi})$ will be called the *ballean-filter mix* of $\mathbb{B}$ and ϕ and denoted by $\mathbb{B}(B, \phi)$. It was introduced in [O. V. Petrenko, I. V. Protasov, *Balleans and filters*, Mat. Stud., **38**, № 1, 3 – 11 (2012)] and was used to construction of a non-metrizable Fréchet group ballean. In this paper some cardinal invariants are compared. In particular, we give a partial answer to the question: if we mix an ordinal unbounded ballean with a free filter of the subsets of its support, will the mix-structure's density be equal to its capacity, as it holds in the original balleans?

Розглядається болеан $\mathbb{B} = (X, P, B)$ з нескінченним супортом X і вільний фільтр ϕ на X та визначається $B_{P\times\phi}(x, (\alpha, F))$ для кожного $\alpha \in P$ та $F \in \phi$. Болеан $(X, P \times \phi, B_{P\times\phi})$ називають *болеан-фільтр міксом* для $\mathbb{B}$ і ϕ та позначають $\mathbb{B}(B, \phi)$. Таку термінологію було введено у статті [O. V. Petrenko, I. V. Protasov, *Balleans and filters*, Mat. Stud., **38**, № 1, 3 – 11 (2012)], де її застосовано для побудови болеана групи Фреше без метризації. У цій роботі порівнюються деякі кардинальні інваріанти. Зокрема, наведено часткову відповідь на питання: якщо є мікс ординально необмеженого болеана з вільним фільтром підмножин його супорту, то чи буде щільність мікс-структури рівною її ємності, як це має місце для оригінальних болеанів?

1. Introduction. Given sets X, P and a function $B\colon X \times P \to \mathcal{P}(X)$, a triple $\mathbb{B} = (X, P, B)$ is called *a ball structure* with *a support* X, *a set of radiuses* P and *a ball function* B. If $(x, \alpha) \in X \times P$, then $B(x, \alpha)$ is called *a ball of radius* α *around* x. Consistently if $A \subseteq X$ and $\alpha \in P$, then $\bigcup\{B(x, \alpha)\colon x \in A\}$ is called *a ball of radius* α *around the set* A.

If B is a ball function, then *a dual ball function* $B^\star$ is defined as follows: for every $x \in X$ and $\alpha \in P$, we put

$$B^\star(x, \alpha) = \{y \in X : x \in B(y, \alpha)\}.$$

The ball structure $\mathbb{B}^\star = (X, P, B^\star)$ is called *dual* to the structure $\mathbb{B} = (X, P, B)$.

A ball structure $\mathbb{B} = (X, P, B)$ is called *upper symmetric* if for any $\alpha, \beta \in P$ there exist α', $\beta' \in P$ such that

$$B(x, \alpha) \subseteq B^\star(x, \alpha') \quad \text{and} \quad B^\star(x, \beta) \subseteq B(x, \beta')$$

for every $x \in X$. A ball structure $\mathbb{B} = (X, P, B)$ is called *upper multiplicative* if, for any $\alpha, \beta \in P$, there exists $\gamma \in P$ such that

$$B(B(x, \alpha), \beta) \subseteq B(x, \gamma)$$

for every $x \in X$. A ball structure which is both upper symmetric and upper multiplicative is called a *ballean*. Note that if $\mathbb{B}$ is upper symmetric/multiplicative, then $\mathbb{B}^\star$ is upper symmetric/multiplicative.

Let $\mathbb{B} = (X, P, B)$ and $\mathbb{B}' = (X', P', B')$ be balleans. A mapping $f : X \to X'$ is called:

$\prec$-*mapping* if, for any $\alpha \in P$, there exists $\alpha' \in P'$ such that $f(B(x, \alpha)) \subseteq B'(f(x), \alpha')$ for every $x \in X$;

asymorphism (or *isomorphism* as in [3]) if f is a bijection and both f and f^{-1} are $\prec$-mappings.

If $f: X \to X'$ is a $\prec$-mapping we will write $\mathbb{B} \prec \mathbb{B}'$. Balleans $\mathbb{B} = (X, P, B)$ and $\mathbb{B}' = (X', P', B')$ are called *asymorphic* if there exists an asymorphism $f: X \to X'$. We will write $\mathbb{B} = \mathbb{B}'$ if $X = X'$ and $\mathbb{B}$ and $\mathbb{B}'$ are asymorphic. Note that $\mathbb{B} = \mathbb{B}^\star$ if and only if a ball structure $\mathbb{B}$ is upper symmetric. Of course for every balleans we have $\mathbb{B} = \mathbb{B}^\star$. A ballean $\mathbb{B} = (X, P, B)$ is called *symmetric* if $B(x, \alpha) = B^\star(x, \alpha)$ for any two $x \in X$ and $\alpha \in P$. Recall that every ballean is asymorphic to some symmetric ballean [1].

Given $\mathbb{B} = (X, P, B)$, we define natural preordering $\leq$ on P by the rule: $\alpha \leq \beta$ iff $B(x, \alpha) \subseteq B(x, \beta)$ for every $x \in X$. A subset $P' \subseteq P$ is called *cofinal* if, for each $\alpha \in P$ there exists $\beta \in P'$ such that $\alpha \leq \beta$. A ballean $\mathbb{B} = (X, P, B)$ is called *connected* if, for every two points x and y of X, there exists $\alpha = \alpha(x, y) \in P$ such that $y \in B(x, \alpha)$. The connectedness is an equivalence relation. A connected ballean $\mathbb{B} = (X, P, B)$ is called *ordinal* if P contains a cofinal subset P' which is well-ordered by $\leq$. We put $\operatorname{cf}(\mathbb{B}) = \min\{\operatorname{card}(R) \colon R \subseteq P \wedge R \text{ is cofinal in } \mathbb{B}\}$.

Observe that if we replace P by its minimal cofinal subset P', we get an asymorphic ballean [4, p. 175]. Hence, we can replace P by a regular cardinal $\rho = \operatorname{cf}\operatorname{card}(P)$ and write $\mathbb{B}(X, \rho, B)$ in the place of $\mathbb{B}(X, P, B)$.

2. Examples.

***Example* 1.** Let (X, d) be a metric space. Put $B_d(x, \epsilon) = \{y \in X : d(x, y) \leq \epsilon\}$ for every $\epsilon > 0$. The ballean $\mathbb{B}(X, d) = (X, \mathbb{R}^+, B_d)$ is called a *metric* ballean. Note that every metric ballean is ordinal.

A filter ϕ on infinite set X is called *free* if $\bigcap \phi = \varnothing$. A free filter ϕ on infinite set X is called *uniform* if $\operatorname{card}(Y) = \operatorname{card}(X)$ for every $Y \in \phi$. If $\operatorname{card}(X) \geq \kappa \geq \omega$, then the free filter of all subsets $F \subseteq X$ such that $\operatorname{card}(X \setminus F) < \kappa$ will be denoted by $\mathcal{F}_\kappa(X)$.

***Example* 2.** Let X be a set and ϕ be a free filter on X. For any $x \in X$ and $F \in \phi$, we put

$$B_\phi(x, F) = \begin{cases} X \setminus F, & \text{if } x \notin F, \\ \{x\}, & \text{if } x \in F. \end{cases}$$

Then $\mathbb{B}(\phi) = (X, \phi, B_\phi)$ is a symmetric ballean.

Note that $\chi(\phi) = \operatorname{cf}(\mathbb{B}(\phi))$ for every free filter on an infinite set X [4, p. 23].

***Example* 3.** Let $\mathbb{B} = (X, P, B)$ be a ballean and ϕ be a free filter on X. We put

$$B_\phi(x, F) = \begin{cases} B(x, \alpha) \setminus F, & \text{if } x \notin F, \\ \{x\}, & \text{if } x \in F. \end{cases}$$

Then the ball structure $\mathbb{B}(B, \phi) = (X, P \times \phi, B_{P\times\phi})$ is a ballean. We will call it a *ballean-filter mix* of $\mathbb{B}$ and ϕ [2].

***Remark* 1.** Note that:

(a) if ϕ and ψ are free filters on X such that $\phi \subseteq \psi$, then $\mathbb{B}(B, \phi) \prec \mathbb{B}(B, \psi)$;

(b) $\mathbb{B}(B, \phi) \prec \mathbb{B}$ for every free filter ϕ on X;

(c) if $\mathbb{B} = (X, P, B)$, $\mathbb{B}' = (X, P', B')$ and $\mathbb{B} = \mathbb{B}'$, then $\mathbb{B}(B, \phi) = \mathbb{B}(B', \phi)$ for every free filter ϕ on X.

***Example* 4.** Given an uncountable set X and a regular cardinal $\kappa < \mathrm{card}(X)$, we define $\mathrm{Seq}(\kappa)$ as a family of all subsets A of X such that $\mathrm{card}(A) < \kappa$. We put

$$P_{\mathrm{Seq}(\kappa)} = \{f : X \to \mathrm{Seq}(\kappa) : \forall_{x \in X} \, (x \in f(x) \wedge$$

$$\wedge \, \{y \in X : x \in f(y)\} \in \mathrm{Seq}(\kappa) \,) \, \}$$

and $B_{\mathrm{Seq}(\kappa)}(x, f) = f(x)$ for every $(x, f) \in X \times P_{\mathrm{Seq}(\kappa)}$. Then, thanks to regularity of κ, $\mathbb{B}(\mathrm{Seq}(\kappa)) = (X, P_{\mathrm{Seq}(\kappa)}, B_{\mathrm{Seq}(\kappa)})$ is a ballean.

3. Subsets of balleans. Given a ballean $\mathbb{B} = (X, P, B)$, a subset $V \subseteq X$ is called:

bounded if there exist $\alpha \in P$ and $x \in X$ such that $V \subseteq B(x, \alpha)$;

unbounded if V is not bounded;

large if there exists $\alpha \in P$ such that $X = B(V, \alpha)$;

thick if $X \setminus V$ is not large;

α-discrete for $\alpha \in P$ if $\{B(v, \alpha) : v \in V\}$ is disjont family.

The set of all, respectively, bounded, unbounded, large, thick subsets of ballean $\mathbb{B} = (X, P, B)$ will be denoted by, respectively, $\mathrm{Bound}(B)$, $\mathrm{Unbound}(B)$, $\mathrm{Large}(B)$, $\mathrm{Thick}(B)$. Note that every dense subset of a metric space (X, d) is large in $\mathbb{B}(X, d)$. However it is not vice versa. For example, the set of integers is large in $\mathbb{B}(\mathbb{R}, d)$ but it is not dense in $(\mathbb{R}, d)$, where d is an Euclidean metric.

It is also known that for every $\alpha \in P$ each maximal (with respect to inclusion) α-discrete subset of any ballean $\mathbb{B} = (X, P, B)$ is large in it [1, 4].

4. Density. We define a *density* of a ballean $\mathbb{B}$ as follows:

$$\mathrm{den}(\mathbb{B}) = \min \{\mathrm{card}(Y) : Y \subseteq X \wedge Y \text{ is large in } \mathbb{B}\}.$$

Recall [1] that if X is infinite and filter ϕ is free, then ballean $\mathbb{B}(\phi)$ is unbounded and connected (such a ballean is also called a *conun*) and $\phi = \mathrm{Large}(\mathbb{B}(\phi))$.

***Remark* 2.** Note that $\mathrm{Large}(\mathbb{B}(B, \phi)) \subseteq \phi \cap \mathrm{Large}(\mathbb{B})$. Indeed, if $Y \subseteq X$ is large in $\mathbb{B}(B, \phi)$, then there exist $F \in \phi$ and $\alpha \in P$ such that $B(Y, (\alpha, F)) = X$. Hence $F \subseteq Y$ and $B(Y \setminus F, \alpha) = X \setminus F$. So, $Y \in \phi \cap \mathrm{Large}(\mathbb{B})$.

***Corollary* 1.** *If ϕ is a free filter on an infinite set X, then*

$$\mathrm{den}(\mathbb{B}(\phi)) \, \mathrm{den}(\mathbb{B}) \leq \mathrm{den}(\mathbb{B}(B, \phi))$$

for every ballean $\mathbb{B} = (X, P, B)$. So, if ϕ is uniform, then

$$\mathrm{den}(\mathbb{B}(B, \phi)) = \mathrm{den}(\mathbb{B}(\phi)) = \mathrm{card}(X).$$

Proposition 1. *Let $\mathbb{B} = (X, P, B)$ be a connected ballean with infinite support X and let ϕ be a free filter on X containing $\mathcal{F}_\omega(X)$ and such that $X \setminus F$ is bounded for every $F \in \phi$. Then* $\mathrm{Large}(\mathbb{B}(B, \phi)) = \phi$.

Proof. Take $F \in \phi$ and an arbitrary $y \in F$. Put $G = F \setminus \{y\}$ and choose $x \in X$ and $\alpha \in P$ such that $X \setminus G \subseteq B(x, \alpha)$. By connectedness of $\mathbb{B}$ there exists $\beta \in P$ such that $x \in B(y, \beta)$. By upper multiplicative condition we can find $\gamma \in P$ such that $X \setminus G \subseteq B(y, \gamma)$. So $B_{P \times \phi}(F, (\gamma, G)) = X$. Hence, $F \in \mathrm{Large}(\mathbb{B}(B, \phi))$.

***Corollary* 2.** *For every connected and bounded ballean* $\mathbb{B} = (X, P, B)$ *with an infinite support* X *and every free filter* ϕ *containing* $F_\omega(X)$ *the following equality holds*:

$$\mathrm{den}(\mathbb{B}(B, \phi)) = \min\{\mathrm{card}(F) : F \in \phi\}.$$

Proposition 2. *Let* ϕ *be a free filter on an infinite set* X *containing* $\mathcal{F}_\omega(X)$ *and let* $\mathbb{B} = (X, P, B)$ *be a bounded ballean. Then*

$$\mathrm{den}(\mathbb{B}(\phi)) = \mathrm{den}(\mathbb{B}(B, \phi)) > \mathrm{den}(\mathbb{B}).$$

Proof. Let $Y \in \phi$ and choose $x_0 \in X$ and $\alpha \in P$ such that $B(x_0, \alpha) = X$. We consider two cases:

Case 1: Assume that $x_0 \in Y$. Then Y is large in $\mathbb{B}(B, \phi)$. Indeed, if $x_0 \in Y$, then

$$B_{P\times\phi}(Y, (\alpha, Y \setminus \{x_0\})) = X.$$

Case 2: If $x_0 \notin Y$, then $Y \cup \{x_0\}$ appears large in $\mathbb{B}(B, \phi)$, since

$$B_{P\times\phi}(Y \cup \{x_0\}, (\alpha, Y)) = Y \cup (B(x_0, (\alpha, Y)) \setminus Y) = X.$$

Since $\mathrm{card}(Y) = \mathrm{card}(Y \cup \{x_0\})$, so $\mathrm{den}(\mathbb{B}(B, \phi)) \leq \mathrm{den}(\mathbb{B}(\phi))$. Of course, since $\mathbb{B} = (X, P, B)$ is bounded, we have $\mathrm{den}(\mathbb{B}) = 1$. By Corollary 1 the proof is complete.

***Example* 5.** If d is the Euclidean metric and $\phi = \mathcal{F}_\omega(\mathbb{R})$, then $\aleph_0 = \mathrm{den}(\mathbb{B}(\mathbb{R}, d)) < \mathrm{den}(\mathbb{B}(\phi)) = \mathrm{den}(\mathbb{B}(B_d, \phi)) = \mathrm{card}(\mathbb{R})$ although $\mathbb{B}(\mathbb{R}, d)$ is not bounded.

Given a ballean $\mathbb{B} = (X, P, B)$, a set $Y \subseteq X$ *has asymptotically isolated* α*-balls* for some $\alpha \in P$ if, for every $\beta > \alpha$, there exists $y \in Y$ satisfing $B(y, \alpha) = B(y, \beta)$. If there exist $\alpha \in P$ and a set $Y \subseteq X$ which has asymptotically isolated α-balls, then we say that $\mathbb{B}$ *has asymptotically isolated balls*. If $\phi_\alpha = \{x \in X : B(x, \alpha) = \{x\}\}$ is nonempty for every $\alpha \in P$, we will say that $\mathbb{B}$ *has asymptotically isolated* 0*-balls*.

***Example* 6.** If $\mathbb{B}$ is unbounded and has asymptotically isolated 0-balls, then $\mathcal{B} = \{\phi_\alpha : \alpha \in P\}$ is a base for some filter ϕ_0 on X. Then $\mathbb{B} \prec \mathbb{B}(B, \phi_0)$ [2]. So, for every free filter on X such that $\phi_0 \subseteq \phi$, we have

$$\mathbb{B} = \mathbb{B}(B, \phi_0) \prec \mathbb{B}(B, \phi) \prec \mathbb{B}.$$

Hence

$$\mathrm{den}\,\mathbb{B} = \mathrm{den}\,\mathbb{B}(B, \phi_0) = \mathrm{den}\,\mathbb{B}(B, \phi).$$

5. Capacity. The *capacity* of a ballean $\mathbb{B} = (X, P, B)$ is determined by its thick subsets. Namely

$\mathrm{cap}(\mathbb{B}) = \sup\{\mathrm{card}(\mathcal{F}) : \mathcal{F}$ is a disjoint family of thick subsets of $X\}$.

Recall [4, p. 175], [1] (Theorem 3.1) that $\mathrm{cap}(\mathbb{B}) \leq \mathrm{den}(\mathbb{B})$ for every ballean $\mathbb{B}$ and $\mathrm{cap}(\mathbb{B}) = \mathrm{den}(\mathbb{B})$ for every ordinal ballean $\mathbb{B}$.

***Remark* 3.** Note that $\mathrm{Thick}(\mathbb{B}(B, \phi)) \supseteq \phi \cup \mathrm{Thick}(\mathbb{B})$ and

$$\mathrm{cap}(\mathbb{B})\,\mathrm{cap}(\mathbb{B}(\phi)) \leq \mathrm{cap}(\mathbb{B}(B, \phi))$$

for every free filter ϕ and every ballean $\mathbb{B} = (X, P, B)$.

***Remark* 4.** Note that if ϕ is a free filter on an infinite X and $\mathbb{B} = (X, \rho, B)$ is ordinal and unbounded and such that $\mathrm{den}(\mathbb{B}) = \mathrm{den}(\mathbb{B}(B, \phi))$, then

$$\mathrm{cap}(\mathbb{B}) \le \mathrm{cap}(\mathbb{B}(B, \phi)) \le \mathrm{den}(\mathbb{B}(B, \phi)) = \mathrm{den}(\mathbb{B}) = \mathrm{cap}(\mathbb{B}).$$

***Remark* 5.** If ϕ is a free uniform ultrafilter on an infinite set X of the cardinality κ, then $cap(\mathbb{B}(\phi)) = 1$ and if also $\mathrm{den}(\mathbb{B}) < \kappa$, then $\mathrm{cap}(\mathbb{B}(B, \phi)) < \mathrm{den}(\mathbb{B}(B, \phi)) = \kappa$.

***Example* 7.** Note that if $\phi = \mathcal{F}_\omega(\mathbb{R})$, then $\phi \subseteq \mathrm{Thick}(\mathbb{B}(\mathbb{R}, d))$. Indeed, if $Y \in \phi$ (and, hence, Y is large in $\mathbb{B}(\phi) = (\mathbb{R}, \phi, B_\phi)$), then $\mathbb{R} \setminus Y$ is finite. So $\bigcup \{B(y, \epsilon) : y \in R \setminus Y\} \ne \mathbb{R}$ for every $\epsilon > 0$. This means $\mathbb{R} \setminus Y$ cannot be large in $\mathbb{B}(\mathbb{R}, d)$. Hence, $Y \in \mathrm{Thick}(\mathbb{B}(\mathbb{R}, d))$.

Recall [1] that if $\mathrm{card}(X) = \kappa$ is regular, then, since $\mathrm{Large}(\mathbb{B}(\mathrm{Seq}(\kappa))) = [X]^\kappa$, we have

$$1 = \mathrm{cap}(\mathbb{B}(\mathrm{Seq}(\kappa))) < \mathrm{den}(\mathbb{B}(\mathrm{Seq}(\kappa))) = \kappa.$$

Proposition 3. *Let X be an uncountably infinite set and let $\kappa < \mathrm{card}(X)$ be a regular cardinal. Then*

$$\mathrm{den}(\mathbb{B}(\mathrm{Seq}(\kappa)) = \mathrm{den}(\mathbb{B}(\mathcal{F}_\kappa)) = \mathrm{den}(\mathbb{B}(B_{\mathrm{Seq}(\kappa)}, \mathcal{F}_\kappa)) = \mathrm{card}(X)$$

and

$$\mathrm{cap}(\mathbb{B}(\mathrm{Seq}(\kappa))) = \mathrm{cap}(\mathbb{B}(\mathcal{F}_\kappa)) = \mathrm{cap}(\mathbb{B}(B_{\mathrm{Seq}(\kappa)}, \mathcal{F}_\kappa)) = 1.$$

Proof. To prove the statement we will note that $\mathbb{B}(\mathrm{Seq}(\kappa)) = \mathbb{B}(\mathcal{F}_\kappa) = \mathbb{B}(B, \mathcal{F}_\kappa)$. Indeed, put $\mathbb{B}_1 = \mathbb{B}(\mathrm{Seq}(\kappa))$, $\mathbb{B}_2 = \mathbb{B}(\mathcal{F}_\kappa)$ and $\mathbb{B}_3 = \mathbb{B}(B_{\mathrm{Seq}(\kappa)}, \mathcal{F}_\kappa)$. We will show that the identity mapping is an asymorphism in each of the following three cases.

Case 1: $\mathbb{B}_1 \prec \mathbb{B}_2$. Consider $x \in X$ and $f \in P_{\mathrm{Seq}(\kappa)}$ and put

$$F = X \setminus f(x).$$

Then $B_1(x, f) = f(x) = B_2(x, F)$.

Case 2: $\mathbb{B}_2 \prec \mathbb{B}_3$. Take $G \in \mathcal{F}_\kappa$ and $x \in X$ and put

$$f_F(x) = \begin{cases} \{x\}, & x \in F, \\ X \setminus F, & x \notin F. \end{cases}$$

Then $B_2(x, F) = B_3(x, (f_F, F))$ for every $x \in X$.

Case 3: $\mathbb{B}_3 \prec \mathbb{B}_1$. If $F \in \mathcal{F}_\kappa$, $f \in P_{\mathrm{Seq}(\kappa)}$ and $x \in X$, then we define $g(x)$ as follows:

$$g(x) = \begin{cases} \{x\}, & x \in F, \\ f(x) \setminus F, & x \notin F. \end{cases}$$

Then $B_3(x, (f, F)) = B_1(x, g)$. So $\mathbb{B}_1$, $\mathbb{B}_2$ and $\mathbb{B}_2$ are pairwise asymorphic.

Observe that $\mathrm{den}(\mathbb{B}(B, \mathcal{F}_\kappa)) = \min\{F : F \in \mathcal{F}_\kappa\} = \mathrm{card}(X)$.

To see that $\mathrm{cap}(\mathbb{B}(\mathrm{Seq}(\kappa))) = 1$ suppose $A \in \mathrm{Thick}(\mathbb{B}(\mathrm{Seq}(\kappa)))$. Then $\mathrm{card}(A) = \mathrm{card}(X)$, since otherwise we could find a bijection $f : X \setminus A \to X$ and define $g(x) = \{f(x), x\}$ for every $x \in X$. Then $\mathbb{B}_{\mathrm{Seq}(\kappa)}(X \setminus A, g) = X$ and hence $X \setminus A$ would be large in $\mathbb{B}(\mathrm{Seq}(\kappa))$, so we would get a contradiction.

So there exists a bijection $f : A \to X$ and $g \in P_{\mathrm{Seq}(\kappa)}$ (defined as above) such that $\mathbb{B}_{\mathrm{Seq}(\kappa)}(A, g) = X$. Hence $A \in \mathrm{Large}(\mathbb{B}(\mathrm{Seq}(\kappa)))$. So since $\mathrm{Thick}(\mathbb{B}(\mathrm{Seq}(\kappa))) \subseteq \mathrm{Large}(\mathbb{B}(\mathrm{Seq}(\kappa)))$ we have $\mathrm{cap}(\mathbb{B}(\mathrm{Seq}(\kappa))) = 1$.

***Example* 8.** If $\phi = \mathcal{F}_\omega(\mathbb{R})$, then density and capacity of $\mathbb{B}(\mathbb{B}(\mathbb{R}, d), \phi)$ are equal and uncountable. Indeed, if $A \in \mathrm{Large}(\mathbb{B}\,(\mathbb{B}(\mathbb{R}, d), \phi))$, then $A \in \phi$, so A is uncountable. For estimating the capacity we put $A_x = \{x + k : k \in \mathbb{Z}\}$ for every $x \in \mathbb{R}$. Then $\{A_x : x \in [0, 1)\}$ is a family of pairwise disjoint thick subsets of $\mathbb{B}\,(\mathbb{B}(\mathbb{R}, d), \phi)$. Hence

$$\mathrm{cap}\,(\mathbb{B}\,(\mathbb{B}(\mathbb{R}, d), \phi)) = \mathrm{den}\,(\mathbb{B}\,(\mathbb{B}(\mathbb{R}, d), \phi)) > \mathrm{den}(\mathbb{B}(\mathbb{R}, d)) = \aleph_0.$$

Recall that $\rho \leq \mathrm{den}(\mathbb{B})$ for every unbounded ordinal ballean $\mathbb{B} = (X, \rho, B)$ [1] (proof of the Theorem 3.1).

Proposition 4. *If $\mathbb{B} = (X, \rho, B)$ is an unbounded ordinal ballean and $\mathrm{card}(X) = \kappa > \aleph_0$ is such that $\mathrm{den}(\mathbb{B}) < \kappa$, then*

$$\mathrm{cap}(\mathbb{B}(B, \mathcal{F}_\rho(X))) = \mathrm{den}(\mathbb{B}(B, \mathcal{F}_\rho(X))) = \mathrm{card}(X)$$

and there exists a disjoint family $\mathcal{A} \subseteq \mathrm{Thick}(\mathbb{B}(B, \mathcal{F}_\rho(X)))$ such that $\mathrm{card}(\mathcal{A}) = \kappa$.

Proof. Let $\kappa = \mathrm{card}(X)$ and $\mathcal{P} = \{X_\tau : \tau < \rho\}$ be a disjoint family of sets, such that $\bigcup \mathcal{P} = X$ and $\mathrm{card}(X_\tau) = \kappa$ for every $\tau < \rho$. Let $\{x^\tau_\lambda : \lambda < \kappa\}$ be an enumeration of X_τ for every $\tau < \rho$. Then $Z_\lambda = \{x^\tau_\lambda : \tau < \rho\} \in \mathrm{Thick}(\mathbb{B}(B, \mathcal{F}_\rho(X))$ for every $\lambda < \kappa$. Indeed, suppose there exist $\lambda < \kappa$, $F \in \mathcal{F}_\rho(X)$ and $\alpha < \rho$ such that $B_{\rho\times\mathcal{F}_\rho(X)}(X \setminus Z_\lambda, (\alpha, F)) = X$. Then $S = F \setminus (X \setminus Z_\lambda) \neq \varnothing$. Hence $B_{\rho\times\mathcal{F}_\rho(X)}(X \setminus Z_\lambda, (\alpha, F)) \subseteq X \setminus S \neq X$, a contradition.

So, $\mathrm{cap}(\mathbb{B}(B, \mathcal{F}_\rho(X))) = \kappa$.

Theorem 1. *Let $\mathbb{B} = (X, \rho, B)$ be an ordinal unbounded ballean with an uncountable support X and let ϕ be a free filter on X containing $\mathcal{F}_\omega(X)$ and such that $X \setminus F \in \mathrm{Bound}(\mathbb{B})$ for every $F \in \phi$. Then $\mathrm{cap}(\mathbb{B}(B, \phi)) = \mathrm{den}(\mathbb{B}(B, \phi))$ and there exists a disjoint family $\mathcal{A} \subseteq$ $\subseteq \mathrm{Thick}(\mathbb{B}(B, \mathcal{F}_\rho(X)))$ of the cardinality $\mathrm{den}(\mathbb{B}(B, \phi))$.*

Proof. Assume that $\kappa = \mathrm{den}(\mathbb{B}(B, \phi))$. Let $F \in \mathrm{Large}(\mathbb{B}(B, \phi))$ be such that $\mathrm{card}(F) = \kappa$. We choose $\alpha_1 < \rho$ and $G \in \phi$ such that $X = B_{\rho\times\phi}(F, (\alpha_1, G))$ and take an arbitrary $x_0 \in X \setminus G$. Since $\rho \leq \mathrm{den}(\mathbb{B})$, we shall consider two cases:

Case 1: $\rho < \mathrm{cf}\,\kappa$. Inductively assume that, for some $\beta < \rho$, we have just defined a family $\{Y_\alpha \subseteq F : \alpha < \beta\}$ and a strictly increasing sequence $\{\gamma_\alpha : \alpha < \beta\}$ such that, for each $\alpha < \beta$, the following conditions hold:

(i) $\mathrm{card}(Y_\alpha) = \kappa$;

(ii) $\gamma_\alpha > \alpha$;

(iii) $Y_\alpha \subseteq B(x_0, \gamma_\alpha) \setminus B(x_0, \alpha)$.

Consider $\beta = \alpha + 1$ for some $\alpha < \kappa$. Let $Z = F \setminus B(x_0, \gamma_\alpha)$. Observe that $\mathrm{card}(Z) = \kappa$. Indeed, on the contrary $L = Z \cup \{x_0\} \in \mathrm{Large}(\mathbb{B}(B, \phi))$. To see this we use the upper multiplicative condition and find an $\alpha_2 < \rho$ such that $B(B(x, \alpha_0), \alpha_1) \subseteq B(x, \alpha_2)$ for every $x \in X$. So we obtain the following:

$$X = B_{\rho\times\phi}(F, (\alpha_1, G)) \subseteq B_{\rho\times\phi}(Z, (\alpha_1, G)) \cup B_{\rho\times\phi}(B(x_0, \alpha_0), (\alpha_1, G)) \subseteq$$

$$\subseteq B_{\rho\times\phi}(Z, (\alpha_1, G)) \cup B_{\rho\times\phi}(x_0, (\alpha_2, G)) \subseteq$$

$$\subseteq B_{\rho\times\phi}(Z \cup \{x_0\}, (\max(\alpha_2, \alpha_1), G)).$$

So, $Z \cup \{x_0\} \in \mathrm{Large}(\mathbb{B}(B, \phi))$ and its cardinality is smaller then κ. A contradiction.

Since $\rho < \mathrm{cf}\,\kappa$ we can choose $\gamma_\beta > \gamma_\alpha$ such that $\mathrm{card}(Z \cap B(x_0, \gamma_\beta)) = \kappa$. Put $Y_\beta =$ $= Z \cap B(x_0, \gamma_\beta)$.

If β is a limit cardinal then, by regularity of ρ, there exists $\gamma < \rho$ such that $\tau > \gamma_\alpha$ for each $\alpha < \beta$. Then take $Z = F \setminus B(x_0, \tau)$ and again choose $\gamma_\tau > \gamma$ such that $\mathrm{card}(Z \cap B(x, \gamma_\tau)) = \kappa$ and put $Y_\beta = Z \cap B(x, \gamma_\tau) \subseteq F$.

Let $\left\{y_\alpha^\lambda : \lambda < \kappa\right\}$ be an enumeration of Y_α for every $\alpha < \rho$. Then $T_\lambda = \left\{y_\alpha^\lambda : \alpha < \rho\right\} \notin$ $\notin \mathrm{Bound}(\mathbb{B})$ for every $\lambda < \kappa$. Indeed, if there exist $x \in X$ and $\alpha_0 < \rho$ such that $T_\lambda \subseteq B(x, \alpha_0)$, then by connectedness and upper multiplicative condition we can find $\alpha_1 < \rho$ satisfying $T_\lambda \subseteq$ $\subseteq B(x_0, \alpha_1)$. But $y_\alpha^\lambda \notin B(x_0, \alpha_1)$ for $\alpha > \alpha_1$. A contradiction.
So, $T_\lambda \in \mathrm{Thick}(\mathbb{B}(B, \phi))$ for every $\lambda < \kappa$.

Case 2: Now assume $\mathrm{cf}\,\kappa \leq \rho \leq \mathrm{den}(\mathbb{B})$. Let $g : \rho \to \kappa$ be an injection holding $g(\rho)$ cofinal in κ. Inductively assume that for some $\beta < \rho$ we have already defined a family $\{Y_\alpha \subseteq F : \alpha < \beta\}$ and a strictly increasing sequence $\{\gamma_\alpha : \alpha < \beta\}$ such that following conditions hold:

(i) $\gamma_\alpha > \alpha$;

(ii) $Y_\alpha \subseteq B(x, \gamma_\alpha) \setminus B(x, \alpha)$;

(iii) $\mathrm{card}(Y_\alpha) = g(\alpha)$.

Consider $\beta = \alpha + 1$ for some $\alpha < \rho$. Put $Z = F \setminus B(x_0, \gamma_\alpha)$. Since $\mathrm{card}(Z) = \kappa$ (compare with Case 1) and $\rho < \kappa$ there exists a $\gamma_\beta > \gamma_\alpha$ such that $\mathrm{card}(B(x_0, \gamma_\beta) \cap Z) \geq g(\beta)$. We choose $Y_\beta \subseteq B(x_0, \gamma_\beta) \cap Z$ of the cardinality $g(\beta)$.

If β is a limit cardinal then, by regularity of ρ, there exists $\gamma < \rho$ such that $\tau > \gamma_\alpha$ for each $\alpha < \beta$. Then take $Z = F \setminus B(x_0, \tau)$ and choose $\gamma_\tau > \gamma$ such that $\mathrm{card}(Z \cap B(x, \gamma_\tau)) = g(\beta)$. Put $Y_\beta = Z \cap B(x_0, \gamma_\tau)$.

Let $\left\{y_\alpha^\lambda : \lambda < g(\alpha)\right\}$ be an enumeration of Y_α for every $\alpha < \rho$ and let define

$$T_\lambda^\alpha = \left\{y_\beta^\lambda : \alpha + 1 \leq \beta < \rho\right\}$$

for every $\alpha < \rho$ and $\lambda < \kappa$ such that $g(\alpha) < \lambda < g(\alpha + 1)$. Then $X \setminus T_\lambda^\alpha \notin \mathrm{Bound}(\mathbb{B})$ for every λ and α (compare with Case 1), so $T_\lambda^\alpha \in \mathrm{Thick}(\mathbb{B}(B, \phi))$. Of course

$$\mathcal{A} = \{T_\lambda^\alpha : g(\alpha) < \lambda < g(\alpha + 1) \wedge \alpha < \rho\}$$

is a disjoint family and thanks to choice of g, $\mathrm{card}(\mathcal{A}) = \kappa$.

We conclude also that Theorem 3.3 of [1] implies the following statement.

***Corollary* 3.** *Let $\mathbb{B} = (X, P, B)$ be a ballean,* $\mathrm{card}(X) = \kappa$, $\mathrm{card}(P) \leq \kappa$. *Then, for every free filter ϕ on X, there exists a disjoint family $\mathcal{F} \subseteq \mathrm{Thick}(\mathbb{B}(B, \phi))$ such that $\mathrm{card}(\mathcal{F}) = \kappa$ and provided that one of the following conditions is satisfied:*

(i) *there exists $\kappa' < \kappa$ such that $\mathrm{card}(B(x, \alpha)) \leq \kappa'$ for all $x \in X$ and $\alpha \in P$;*

(ii) *$\mathrm{card}(B(x, \alpha)) < \kappa$ for all $x \in X$ and $\alpha \in P$ and κ is regular.*

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