DOI: 10.37863/umzh.v73i4.639

UDC 517.9

D. T. Luyen (Hoa Lu Univ., Ninh Nhat, Vietnam)

PICONE'S IDENTITY FOR Δ_γ-LAPLACE OPERATOR AND ITS APPLICATIONS*

ТОТОЖНІСТЬ ПІКОНЕ ДЛЯ Δ_γ-ОПЕРАТОРА ЛАПЛАСА ТА ЇЇ ЗАСТОСУВАННЯ

We prove a nonlinear analogue of Picone's identity for Δ_γ-Laplace operator. As an application, we give a Hardy type inequality and Sturmian comparison principle. We also show the strict monotonicity of the principle eigenvalue and degenerate elliptic system.

Доведено нелінійний аналог тотожності Піконе для Δ_γ-оператора Лапласа. Як застосування наведено нерівність типу Гарді та принцип порівняння Штурма. Також доведено строгу монотонність власного значення принципу та виродженої еліптичної системи.

1. Introduction. It is a well-known fact that in the qualitative theory of elliptic PDEs, Picone's identity plays an important role. The classical Picone's identity says that if u and v are differentiable functions such that $v > 0$ and $u \geq 0$, then

$$|\nabla u|^2 + \frac{u^2}{v^2}|\nabla v|^2 - 2\frac{u}{v}\nabla u \cdot \nabla v = |\nabla u|^2 - \nabla\left(\frac{u^2}{v}\right) \cdot \nabla v \geq 0. \tag{1.1}$$

(1.1) has an enormous applications to second-order elliptic equations and systems (see, for instance, [1 – 3, 22] and the references therein). Nonlinear analogue of (1.1) is established by J. Tyagi [29]. In order to apply (1.1) to p-Laplace equations, (1.1) is extended by W. Allegretto and Y. X. Huang [4]. Nonlinear analogue of Picone's type identity for p-Laplace equations is established by K. Bal [6].

In this article we establish the nonlinear analogue of generalized Picone's identity for Δ_γ-Laplace operator and its applications.

This paper is organized as follows. In Section 2, we recall the definition of the Δ_γ-Laplace operator and the associated functional setting. We further give examples for the class of Δ_γ-Laplace operator. Section 3 deals with nonlinear analogue of Picone's identity. In Section 4, we give several application of Picone's identity to Δ_γ-Laplace equations.

2. The Δ_γ-Laplace operator. The Δ_γ-operator was considered by B. Franchi and E. Lanconelli in [7, 8], and recently reconsidered in [10] under the additional assumption that the operator is homogeneous of degree two with respect to a group dilation in $\mathbb{R}^N$. We consider the operators of the form

$$\Delta_\gamma := \sum_{j=1}^{N} \partial_{x_j}\left(\gamma_j^2 \partial_{x_j}\right), \qquad \partial_{x_j} = \frac{\partial}{\partial x_j}, \qquad j = 1, 2, \ldots, N.$$

Here, the functions $\gamma_j : \mathbb{R}^N \longrightarrow \mathbb{R}$ are assumed to be continuous, different from zero and of class C^1 in $\mathbb{R}^N \backslash \Pi$, where

* This research was supported by Vietnam National Foundation for Science and Technology Development (NAFOSTED) (grant no. 101.02-2020.13).

$$\Pi := \left\{ x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N : \prod_{j=1}^{N} x_j = 0 \right\}.$$

Moreover, we assume the following properties:

i) There exists a group of dilations $\{\delta_t\}_{t>0}$ such that

$$\delta_t : \mathbb{R}^N \longrightarrow \mathbb{R}, \quad \delta_t (x_1, \ldots, x_N) = (t^{\varepsilon_1} x_1, \ldots, t^{\varepsilon_N} x_N), \qquad 1 = \varepsilon_1 \le \varepsilon_2 \le \ldots \le \varepsilon_N,$$

such that γ_j is δ_t-homogeneous of degree $\varepsilon_j - 1$, i.e.,

$$\gamma_j (\delta_t (x)) = t^{\varepsilon_j - 1} \gamma_j (x) \qquad \forall x \in \mathbb{R}^N \quad \forall t > 0, \quad j = 1, \ldots, N.$$

The number

$$\widetilde{N} := \sum_{j=1}^{N} \varepsilon_j$$

is called the homogeneous dimension of $\mathbb{R}^N$ with respect to $\{\delta_t\}_{t>0}$.

ii) $\gamma_1 = 1, \qquad \gamma_j (x) = \gamma_j (x_1, x_2, \ldots, x_{j-1}), \qquad j = 2, \ldots, N.$

iii) There exists a constant $\rho \ge 0$ such that

$$0 \le x_k \partial_{x_k} \gamma_j (x) \le \rho \gamma_j (x) \qquad \forall k \in \{1, 2, \ldots, j-1\} \quad \forall j = 2, \ldots, N,$$

and for every $x \in \overline{\mathbb{R}}^N_+ := \left\{ (x_1, \ldots, x_N) \in \mathbb{R}^N : x_j \ge 0 \ \ \forall j = 1, 2, \ldots, N \right\}.$

iv) Equalities $\gamma_j (x) = \gamma_j (x^*)$, $j = 1, 2, \ldots, N$, are satisfied for every $x \in \mathbb{R}^N$, where

$$x^* = (|x_1|, \ldots, |x_N|) \quad \text{if} \quad x = (x_1, x_2, \ldots, x_N).$$

Many aspects of the theory of degenerate elliptic differential operators are presented in monographs [27, 28] (see also some recent results in [5, 10 – 20, 23 – 26] and the references therein).

Definition 2.1. *By $S^p_\gamma(\Omega)$, $1 \le p < +\infty$, we will denote the set of all functions $u \in L^p(\Omega)$ such that $\gamma_j \partial_{x_j} u \in L^p(\Omega)$ for all $j = 1, \ldots, N$. We define the norm in this space as follows:*

$$\|u\|_{S^p_\gamma(\Omega)} = \left\{ \int_\Omega \left(|u|^p + \sum_{j=1}^{N} \left| \gamma_j \partial_{x_j} u \right|^p \right) dx \right\}^{\frac{1}{p}}.$$

If $p = 2$ we can also define the scalar product in $S^2_\gamma(\Omega)$ as follows:

$$(u, v)_{S^2_\gamma(\Omega)} = (u, v)_{L^2(\Omega)} + \sum_{j=1}^{N} (\gamma_j \partial_{x_j} u, \gamma_j \partial_{x_j} v)_{L^2(\Omega)}.$$

The space $S^p_{\gamma,0}(\Omega)$ is defined as the closure of $C^1_0(\Omega)$ in the space $S^p_\gamma(\Omega)$.

Set

$$\nabla_\gamma u := (\gamma_1 \partial_{x_1} u, \gamma_2 \partial_{x_2} u, \ldots, \gamma_N \partial_{x_N} u), \qquad |\nabla_\gamma u| := \left(\sum_{j=1}^{N} \left| \gamma_j \partial_{x_j} u \right|^2 \right)^{\frac{1}{2}}.$$

We now give some examples of the Δ_γ-Laplace operator. We use the following notations: we split $\mathbb{R}^N$ into

$$\mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \mathbb{R}^{N_3},$$

and write

$$x = \left(x^{(1)}, x^{(2)}, x^{(3)}\right), \qquad x^{(i)} = \left(x_1^{(i)}, x_2^{(i)}, \ldots, x_{N_i}^{(i)}\right) \in \mathbb{R}^{N_i},$$

$$|x^{(i)}|^2 = \sum_{j=1}^{N_i} |x_j^{(i)}|^2, \qquad i = 1, 2, 3.$$

We denote the classical Laplace operator in $\mathbb{R}^{N_i}$ by

$$\Delta_{x^{(i)}} = \sum_{j=1}^{N_i} \partial^2_{x_j^{(i)}}.$$

***Example* 2.1** (see [11, 17]). Let α be a real positive number. The operator

$$\Delta_\gamma = \Delta_{x^{(1)}} + |x^{(1)}|^{2\alpha}(\Delta_{x^{(2)}} + \Delta_{x^{(3)}}),$$

where

$$\gamma = (\underbrace{1, 1, \ldots, 1}_{N_1-\text{times}}, \underbrace{|x^{(1)}|^\alpha, \ldots, |x^{(1)}|^\alpha}_{(N_2+N_3)-\text{times}}),$$

is called the Grushin operator (see [9]).

***Example* 2.2** (see [11, 17]). Let α, β be nonnegative real numbers. The operator

$$\Delta_\gamma = \Delta_{x^{(1)}} + \Delta_{x^{(2)}} + |x^{(1)}|^{2\alpha}|x^{(2)}|^{2\beta}\Delta_{x^{(3)}},$$

where

$$\gamma = (\underbrace{1, 1, \ldots, 1}_{(N_1+N_2)-\text{times}}, \underbrace{|x^{(1)}|^\alpha|x^{(2)}|^\beta, \ldots, |x^{(1)}|^\alpha|x^{(2)}|^\beta}_{N_3-\text{times}}),$$

is called the strongly degenerate elliptic operators (see [24, 28]).

3. Generalized Picone's inequality.

Theorem 3.1. *Let $v > 0$ and $u \geq 0$ be be two non-constant differentiable functions in Ω. Also assume that $f \in C^1(\mathbb{R}, (0, \infty))$ satisfies $f'(y) \geq 1$ for all $y \in (0, \infty)$. Define*

$$L(u, v) = |\nabla_\gamma u|^2 - \frac{2u\nabla_\gamma u \cdot \nabla_\gamma v}{f(v)} + \frac{u^2 f'(v)\,|\nabla_\gamma v|^2}{f^2(v)},$$

$$R(u, v) = |\nabla_\gamma u|^2 - \nabla_\gamma\left(\frac{u^2}{f(v)}\right) \cdot \nabla_\gamma v.$$

Then $L(u, v) = R(u, v) \geq 0$. Moreover, $L(u, v) = 0$ a.e. in Ω if and only if $\nabla_\gamma\left(\dfrac{u}{v}\right) = 0$ a.e. in Ω, i.e., $u = kv$ for some constant k in each component of Ω.

Proof. Expanding $R(u,v)$ one easily sees that $L(u,v) = R(u,v)$. To show $L(u,v) \geq 0$ we proceed as follows:

$$L(u,v) = |\nabla_\gamma u|^2 - \frac{2u\nabla_\gamma u \cdot \nabla_\gamma v}{f(v)} + \frac{u^2 f'(v)\,|\nabla_\gamma v|^2}{f^2(v)} =$$

$$= |\nabla_\gamma u|^2 + \frac{u^2 f'(v)\,|\nabla_\gamma v|^2}{f^2(v)} - \frac{2u\,|\nabla_\gamma u|\,|\nabla_\gamma v|}{f(v)} +$$

$$+ \frac{2u}{f(v)}\left(|\nabla_\gamma u|\,|\nabla_\gamma v| - \nabla_\gamma u \cdot \nabla_\gamma v\right) =$$

$$= \left(|\nabla_\gamma u|^2 + \frac{u^2\,|\nabla_\gamma v|^2}{f^2(v)}\right) - \frac{u^2\,|\nabla_\gamma v|^2}{f^2(v)} - \frac{2u\,|\nabla_\gamma u|\,|\nabla_\gamma v|}{f(v)} +$$

$$+ \frac{u^2 f'(v)\,|\nabla_\gamma v|^2}{f^2(v)} + \frac{2u}{f(v)}\left(|\nabla_\gamma u|\,|\nabla_\gamma v| - \nabla_\gamma u \cdot \nabla_\gamma v\right).$$

By using Cauchy's inequality, we get

$$|\nabla_\gamma u|^2 + \frac{u^2\,|\nabla_\gamma v|^2}{f^2(v)} \geq \frac{2u\,|\nabla_\gamma u|\,|\nabla_\gamma v|}{f(v)}. \tag{3.1}$$

Which is possible since both u and f are non negative. Equality holds when

$$|\nabla_\gamma u| = \frac{u}{f(v)}\,|\nabla_\gamma v|. \tag{3.2}$$

Again using the fact that $f'(y) \geq 1$, we have

$$\frac{u^2 f'(v)\,|\nabla_\gamma v|^2}{f^2(v)} \geq \frac{u^2\,|\nabla_\gamma v|^2}{f^2(v)}. \tag{3.3}$$

Equality holds when

$$f'(v) = 1. \tag{3.4}$$

Combining (3.1) and (3.3), we obtain $L(u,v) \geq 0$. Equality holds when (3.2) and (3.4) together with $|\nabla_\gamma u|\,|\nabla_\gamma v| = \nabla_\gamma u \cdot \nabla_\gamma v$ holds simultaneously.

Solving for (3.4) one obtains $f(v) = v$. So, if $L(u,v)(x_0) = 0$ and $u(x_0) \neq 0$, then (3.1) together with $f(v) = v$ and $|\nabla_\gamma u|\,|\nabla_\gamma v| = \nabla_\gamma u \cdot \nabla_\gamma v$ yields, i.e., $\nabla_\gamma u = (u/v)\nabla_\gamma v$ or $\nabla_\gamma (u/v)(x_0) = 0$. On the other hand, if $\Lambda = \{x \in \Omega, u(x) = 0\}$, then $\nabla_\gamma u = 0$ a.e. in Λ (see [17]), and thus $\nabla_\gamma (u/v) = 0$ a.e. in Ω. We conclude that $\nabla_\gamma (u/v) = 0$ a.e. in Ω and consequently $u = kv$ for some constant k.

***Remark* 3.1.** If $\gamma = (\underbrace{1, 1, \ldots, 1}_{N-\text{times}})$ and $f(y) = y$, we get the classical Picone's identity (1.1) for Laplacian operator.

4. Applications. In this section, we will give some applications of nonlinear Picone's identity following the spirit of [4].

4.1. Hardy type result. We start with establishing a Hardy type inequality for Δ_γ-Laplace operator.

Theorem 4.1. *Assume that there is a* $v \in C^1(\Omega)$ *satisfying*

$$-\Delta_\gamma v \geq \lambda g f(v), \qquad v > 0 \quad in \quad \Omega,$$

for some $\lambda > 0$ *and nonnegative continuous function* g. *Then, for any* $u \in C_0^\infty(\Omega),\ u \geq 0,$ *it holds that*

$$\int\limits_\Omega |\nabla_\gamma u|^2\, dx \geq \lambda \int\limits_\Omega g u^2 dx, \tag{4.1}$$

where $f \in C^1(\mathbb{R}, (0,\infty))$ *satisfies* $f'(y) \geq 1$ *for all* $y \in (0,\infty)$.

Proof. Take $\phi \in C_0^\infty(\Omega),\ \phi > 0$. By Theorem 3.1, we have

$$0 \leq \int\limits_\Omega L(\phi, v) dx =$$

$$= \int\limits_\Omega R(\phi, v) dx = \int\limits_\Omega \left(|\nabla_\gamma \phi|^2 - \nabla_\gamma \left(\frac{\phi^2}{f(v)} \right) \cdot \nabla_\gamma v \right) dx =$$

$$= \int\limits_\Omega \left(|\nabla_\gamma \phi|^2 + \frac{\phi^2}{f(v)} \Delta_\gamma v \right) dx \leq$$

$$\leq \int\limits_\Omega \left(|\nabla_\gamma \phi|^2 - \lambda \phi^2 g \right) dx.$$

Letting $\phi \to u$, we get (4.1).

4.2. Strumium comparison principle. Comparison principles play vital role in study of partial differential equations. Here, we establish nonlinear version of Sturmian comparison principle for Δ_γ-Laplace operator.

Theorem 4.2. *Let* f_1 *and* f_2 *are two weight functions such that* $f_1(\xi) < f_2(\xi)$ *for all* $\xi \in \Omega$ *and* $f \in C^1(\mathbb{R}, (0,\infty))$ *satisfies* $f'(y) \geq 1$ *for all* $y \in (0,\infty)$. *If there is a positive solution* u *satisfying*

$$-\Delta_\gamma u = f_1(x) u \quad in\ \Omega, \qquad u = 0 \quad on\ \partial\Omega,$$

then any nontrivial solution v *of*

$$-\Delta_\gamma v = f_2(x) f(v) \quad in\ \Omega, \qquad v = 0 \quad on\ \partial\Omega, \tag{4.2}$$

must change sign.

Proof. Let us assume that there exists a solution $v > 0$ of (4.2) in Ω. Then by Picone's identity, we have

$$0 \leq \int\limits_\Omega L(u, v) dx = \int\limits_\Omega R(u, v) dx =$$

$$= \int\limits_{\Omega} \left(|\nabla_\gamma u|^2 - \nabla_\gamma \left(\frac{u^2}{f(v)} \right) \cdot \nabla_\gamma v \right) dx =$$

$$= \int\limits_{\Omega} \left(f_1(x)u^2 - f_2(x)u^2 \right) dx = \int\limits_{\Omega} \left(f_1(x) - f_2(x) \right) u^2 dx < 0,$$

which is a contradiction. Hence, v changes sign in Ω.

4.3. Strict monotonicity of principle eigenvalue in domain. Consider the indefinite eigenvalue problem

$$-\Delta_\gamma u = \lambda g(x) u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{4.3}$$

where $g(x)$ is indefinite weight function.

Theorem 4.3. *Let $\lambda_1^+(\Omega) > 0$ be the principle eigenvalue of* (4.3), *then suppose $\Omega_1 \subset \Omega_2$ and $\Omega_1 \neq \Omega_2$. Then $\lambda_1^+(\Omega_1) > \lambda_1^+(\Omega_2)$, if both exist.*

Proof. Let u_i be a positive eigenfunction associated with $\lambda_1^+(\Omega_i)$, $i = 1, 2$. Evidently, for $\phi \in C_0^\infty(\Omega_1)$, we obtain

$$0 \le \int\limits_{\Omega_1} L(\phi, u_2) dx = \int\limits_{\Omega} R(\phi, u_2) dx =$$

$$= \int\limits_{\Omega_1} \left(|\nabla_\gamma \phi|^2 - \nabla_\gamma \left(\frac{\phi^2}{f(u_2)} \right) \cdot \nabla_\gamma u_2 \right) dx =$$

$$= \int\limits_{\Omega_1} |\nabla_\gamma \phi|^2 \, dx + \int\limits_{\Omega_1} \frac{\phi^2}{f(u_2)} \Delta_\gamma u_2 dx =$$

$$= \int\limits_{\Omega_1} |\nabla_\gamma \phi|^2 \, dx - \lambda_1^+(\Omega_2) \int\limits_{\Omega_1} \frac{\phi^2}{f(u_2)} g(x) u_2 dx.$$

Letting $\phi \to u_1$ and $f(y) = y$, we get

$$0 \le \int\limits_{\Omega_1} L(u_1, u_2) dx = \left(\lambda_1^+(\Omega_1) - \lambda_1^+(\Omega_2) \right) \int\limits_{\Omega_1} g(x) u_1^2 dx.$$

This gives $\lambda_1^+(\Omega_1) > \lambda_1^+(\Omega_2)$, as if $\lambda_1^+(\Omega_1) = \lambda_1^+(\Omega_2)$. We conclude that $u_1 = k u_2$ which is not possible as $\Omega_1 \subset \Omega_2$ and $\Omega_1 \neq \Omega_2$.

Remark **4.1.** When $g(x) = 1$, we have $\lambda_1(\Omega_1) > \lambda_1(\Omega_2)$ if $\Omega_1 \subset \Omega_2$ and $\Omega_1 \neq \Omega_2$.

4.4. Quasilinear system with singular nonlinearity. We will use Picone's identity to establish a linear relationship between solutions of a quasilinear system with singular nonlinearity. Consider the singular degenerate elliptic system equations

$$\begin{aligned} -\Delta_\gamma u &= f(v) \quad \text{in } \Omega, \\ -\Delta_\gamma v &= \frac{f^2(v)}{u} \quad \text{in } \Omega, \\ u > 0, &\quad v > 0 \quad \text{in } \Omega, \\ u = 0, &\quad v = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{4.4}$$

where $f \in C^1(\mathbb{R}, (0,\infty))$ satisfies $f'(y) \geq 1$ for all $y \in (0,\infty)$. We have the following result.

Theorem 4.4. *Let (u,v) be a weak solution of* (4.4). *Then $u = kv$, where k is a constant.*

Proof. Let (u,v) be the weak solution of (4.4). Now for any ϕ_1 and ϕ_2 in $S^2_{\gamma,0}(\Omega)$, we have

$$\int\limits_\Omega \nabla_\gamma u \cdot \nabla_\gamma \phi_1 dx = \int\limits_\Omega f(v)\phi_1 dx, \tag{4.5}$$

$$\int\limits_\Omega \nabla_\gamma v \cdot \nabla_\gamma \phi_2 dx = \int\limits_\Omega \frac{f^2(v)}{u}\phi_2 dx. \tag{4.6}$$

Choosing $\phi_1 = u$ and $\phi_2 = u^2/f(v)$ in (4.5) and (4.6), we obtain

$$\int\limits_\Omega |\nabla_\gamma u|^2\, dx = \int\limits_\Omega f(v)u dx = \int\limits_\Omega \nabla_\gamma v \cdot \nabla_\gamma \left(\frac{u^2}{f(v)}\right) dx.$$

Hence, we get

$$\int\limits_\Omega R(u,v)dx = \int\limits_\Omega \left(|\nabla_\gamma u|^2 - \nabla_\gamma v \cdot \nabla_\gamma \left(\frac{u^2}{f(v)}\right)\right) dx = 0,$$

this gives $R(u,v) = 0$, which in turn implies that $u = kv$.

Acknowledgments. This paper was done while the author was staying at the Vietnam Institute of Advanced Study in Mathematics (VIASM) as a research fellow. He would like to thank VIASM for its hospitality and support.

References

1. W. Allegretto, *Positive solutions and spectral properties of weakly coupled elliptic systems*, J. Math. Anal. and Appl., **120**, № 2, 723–729 (1986).
2. W. Allegretto, *On the principal eigenvalues of indefinite elliptic problems*, Math. Z., **195**, № 1, 29–35 (1987).
3. W. Allegretto, *Sturmian theorems for second order systems*, Proc. Amer. Math. Soc., **94**, № 2, 291–296 (1985).
4. W. Allegretto, Y. X. Huang, *A Picone's identity for the p-Laplacian and applications*, Nonlinear Anal., **32**, № 7, 819–830 (1998).
5. C. T. Anh, B. K. My, *Existence of solutions to Δ_λ-Laplace equations without the Ambrosetti–Rabinowitz condition*, Complex Var. and Elliptic Equat., **61**, № 1, 137–150 (2016).
6. K. Bal, *Generalized Picone's identity and its applications*, Electron. J. Different. Equat., № 243 (2013), 6 p.
7. B. Franchi, E. Lanconelli, *A metric associated with a class of degenerate elliptic operators*, Conf. Linear Partial and Pseudodifferential Operators (Torino, 1982), Rend. Sem. Mat. Univ. Politec. Torino, 1983, Special Issue, 105–114 (1984).
8. B. Franchi, E. Lanconelli, *An embedding theorem for Sobolev spaces related to nonsmooth vector fields and Harnack inequality*, Commun. Part. Different. Equat., **9**, № 13, 1237–1264 (1984).
9. V. V. Grushin, *A certain class of hypoelliptic operators*, Mat. Sb. (N.S.), **83**, № 125, 456–473 (1970) (in Russian).
10. A. E. Kogoj, E. Lanconelli, *On semilinear Δ_λ-Laplace equation*, Nonlinear Anal., **75**, № 12, 4637–4649 (2012).
11. A. E. Kogoj, S. Sonner, *Attractors for a class of semi-linear degenerate parabolic equations*, J. Evol. Equat., **13**, № 3, 675–691 (2013).
12. D. T. Luyen, D. T. Huong, L. T. H. Hanh, *Existence of infinitely many solutions for Δ_γ-Laplace problems*, Math. Notes, **103**, № 5, 724–736 (2018).
13. D. T. Luyen, *Two nontrivial solutions of boundary-value problems for semilinear Δ_γ-differential equations*, Math. Notes, **101**, № 5, 815–823 (2017).

14. D. T. Luyen, *Existence of nontrivial solution for fourth-order semilinear Δ_γ-Laplace equation in $\mathbb{R}^N$*, Electron. J. Qual. Theory Different. Equat., **78**, 1 – 12 (2019).
15. D. T. Luyen, *Multiple solutions for semilinear Δ_γ-differential equations in $\mathbb{R}^N$ with sign-changing potential*, Commun. Math. Anal., **22**, № 1, 61 – 75 (2019).
16. D. T. Luyen, L. T. H. Hanh, *Three nontrivial solutions of boundary-value problems for semilinear Δ_γ-Laplace equation*, Bol. Soc. Parana. Mat. doi:10.5269/bspm.45841 (2019).
17. D. T. Luyen, N. M. Tri, *Existence of solutions to boundary-value problems for semilinear Δ_γ-differential equations*, Math. Notes, **97**, № 1, 73 – 84 (2015).
18. D. T. Luyen, N. M. Tri, *Large-time behavior of solutions to damped hyperbolic equation involving strongly degenerate elliptic differential operators*, Siberian Math. J., **57**, № 4, 632 – 649 (2016).
19. D. T. Luyen, N. M. Tri, *Global attractor of the Cauchy problem for a semilinear degenerate damped hyperbolic equation involving the Grushin operator*, Ann. Polon. Math., **117**, № 2, 141 – 162 (2016).
20. D. T. Luyen, N. M. Tri, *Existence of infinitely many solutions for semilinear degenerate Schrödinger equations*, J. Math. Anal. and Appl., **461**, № 2, 1271 – 1286 (2018).
21. D. T. Luyen, N. M. Tri, *On the existence of multiple solutions to boundary value problems for semilinear elliptic degenerate operators*, Complex Var. and Elliptic Equat., **64**, № 6, 1050 – 1066 (2019).
22. A. Manes, A. M. Micheletti, *Un'estensione della teoria variazionale classica degli autovalori per operatori ellittici del secondo ordine*, Boll. Unione Mat. Ital., **7**, № 4, 285 – 301 (1973) (in Italian).
23. B. Rahal, M. K. Hamdani, *Infinitely many solutions for Δ_α-Laplace equations with sign-changing potential*, J. Fixed Point Theory and Appl., **20**, № 4 (2018).
24. P. T. Thuy, N. M. Tri, *Nontrivial solutions to boundary value problems for semilinear strongly degenerate elliptic differential equations*, Nonlinear Different. Equat. and Appl., **19**, № 3, 279 – 298 (2012).
25. P. T. Thuy, N. M. Tri, *Long time behavior of solutions to semilinear parabolic equations involving strongly degenerate elliptic differential operators*, Nonlinear Different. Equat. and Appl., **20**, № 3, 1213 – 1224 (2013).
26. N. M. Tri, *Critical Sobolev exponent for hypoelliptic operators*, Acta Math. Vietnam, **23**, № 1, 83 – 94 (1998).
27. N. M. Tri, *Semilinear degenerate elliptic differential equations, local and global theories*, Lambert Acad. Publ. (2010).
28. N. M. Tri, *Recent progress in the theory of semilinear equations involving degenerate elliptic differential operators*, Publ. House Sci. and Technology Vietnam Acad. Sci. and Technology (2014).
29. J. Tyagi, *A nonlinear Picone's identity and its applications*, Appl. Math. Lett., **26**, № 6, 624 – 626 (2013).

Received 13.03.17,
after revision — 29.09.20