DOI: 10.37863/umzh.v73i5.379

UDC 517.9

S. Majumder (Raiganj Univ., West Bengal, India),
S. Saha (Mehendipara Jr. High School, Dist.-Dakshin Dinajpur, West Bengal, India)

A NOTE ON THE UNIQUENESS OF CERTAIN TYPES OF DIFFERENTIAL-DIFFERENCE POLYNOMIALS

ПРО УНІКАЛЬНІСТЬ ДЕЯКИХ ТИПІВ ДИФЕРЕНЦІАЛЬНО-РІЗНИЦЕВИХ ПОЛІНОМІВ

We study the uniqueness problems of certain types of differential-difference polynomials sharing a small function. In this paper, we not only solve the open problem occurred in [A. Banerjee, S. Majumder, *On the uniqueness of certain types of differential-difference polynomials*, Anal. Math., **43**, № 3, 415 – 444 (2017)], but also present our main results in a more generalized way.

Вивчається можливість розв'язання задач єдиності для деяких типів диференціально-різницевих поліномів, які мають спільну малу функцію. У цій роботі не лише наведено розв'язок відкритої задачі з [A. Banerjee, S. Majumder, *On the uniqueness of certain types of differential-difference polynomials*, Anal. Math., **43**, № 3, 415 – 444 (2017)], а й запропоновано більш загальний вигляд отриманого основного результату.

1. Introduction, definitions and results. In this paper by meromorphic functions we shall always mean meromorphic functions in the complex plane. We adopt the standard notations of value distribution theory (see [8]). For a non-constant meromorphic function f, we denote by $T(r, f)$ the Nevanlinna characteristic of f and by $S(r, f)$ any quantity satisfying $S(r, f) = o\{T(r, f)\}$ as $r \to \infty$ possibly outside a set of finite linear measure. A meromorphic function a is called a small function of f, if $T(r, a) = S(r, f)$. We denote by $S(f)$ the set of all small functions of f. Also we denote by $\rho(f)$ the order of f.

Let f and g be two non-constant meromorphic functions. Let $a \in S(f) \cap S(g)$. We say that f and g share a counting multiplicities (CM) if $f(z) - a(z)$ and $g(z) - a(z)$ have the same zeros with the same multiplicities and we say that f and g share a ignoring multiplicities (IM) if we do not consider the multiplicities.

Let f be a transcendental meromorphic function and $n \in \mathbb{N}$. Many authors have investigated the value distributions of $f^n(z)f'(z)$. In 1959, W. K. Hayman (see [7], Corollary of Theorem 9) proved the following theorem.

Theorem A [7]. *Let f be a transcendental meromorphic function and $n \in \mathbb{N}$ such that $n \geq 3$. Then $f^n(z)f'(z) = 1$ has infinitely many solutions.*

The case $n = 2$ was settled by Mues [13] in 1979. Bergweiler and Eremenko [3] showed that $f(z)f'(z) - 1$ has infinitely many zeros. For an analog of the above results Laine and Yang [11] investigated the value distribution of difference products of entire functions in the following manner.

Theorem B [11]. *Let f be a transcendental entire function of finite order and $c \in \mathbb{C} \setminus \{0\}$. Then for $n \in \mathbb{N} \setminus \{1\}$, $f^n(z)f(z + c)$ assumes every $a \in \mathbb{C} \setminus \{0\}$ infinitely often.*

In 2010, Zhang [19] considered zeros of one certain type of difference polynomials that was not studied previously and obtained the following theorem.

Theorem C [19]. *Let f be a transcendental entire function of finite order, $\alpha(\not\equiv 0) \in S(f)$, $c \in \mathbb{C} \setminus \{0\}$ and $n \in \mathbb{N}$. If $n \geq 2$, then $f^n(z)(f(z)-1)f(z+c) - \alpha(z)$ has infinitely many zeros.*

In 2012, Chen and Chen [5] further extended Theorem C as follows.

Theorem D [5]. *Let f be a transcendental entire function of finite order, $\alpha(\not\equiv 0) \in S(f)$, $c_j \in \mathbb{C}$ and $d, m, n, \nu_j \in \mathbb{N}$, where $j = 1, 2, \ldots, d$. If $n \geq 2$, then $f^n(z)(f^m(z)-1)\prod_{j=1}^{d}(f(z+c_j))^{\nu_j} - \alpha(z)$ has infinitely many zeros.*

Chen and Chen [5] also found the uniqueness result corresponding to Theorem D. In 2014, Zhang and Yi [18] treat the above investigations into a different way that was not dealt earlier. They paid their attention to the kth derivative of more generalized difference expression and obtained a series of results as follows.

Theorem E [18]. *Let f be a transcendental entire function of finite order, $\alpha(\not\equiv 0) \in S(f)$, $c_j \in \mathbb{C}$ be distinct and d, m, n, $\nu_j \in \mathbb{N} \cup \{0\}$, $j = 1, 2, \ldots, d$. If $n \geq k+2$, then the differential-difference polynomial $\left(f^n(z)(f^m(z)-1)\prod_{j=1}^{d}(f(z+c_j))^{\nu_j}\right)^{(k)} - \alpha(z)$ has infinitely many zeros.*

Theorem F [18]. *Let f be a transcendental entire function of finite order, $\alpha(\not\equiv 0) \in S(f)$, $c_j \in \mathbb{C}$ be distinct and d, m, n, $\nu_j \in \mathbb{N} \cup \{0\}$, $j = 1, 2, \ldots, d$. If one of the following conditions holds:*

(i) $n \geq k+2$, *when* $m \leq k+1$;

(ii) $n \geq 2k - m + 3$, *when* $m > k+1$,

then the differential-difference polynomial $\left(f^n(z)(f(z)-1)^m\prod_{j=1}^{d}(f(z+c_j))^{\nu_j}\right)^{(k)} - \alpha(z)$ has infinitely many zeros.

Theorem G [18]. *Let f and g be two transcendental entire functions of finite order, $\alpha(\not\equiv 0) \in S(f) \cap S(g)$, $c_j \in \mathbb{C}$ be distinct and d, m, n, $\nu_j \in \mathbb{N} \cup \{0\}$, where $j = 1, 2, \ldots, d$ and $\sigma = \sum_{j=1}^{d}\nu_j$. If $n \geq 2k+m+\sigma+5$ and $\left(f^n(z)(f^m(z)-1)\prod_{j=1}^{d}(f(z+c_j))^{\nu_j}\right)^{(k)}$, $\left(g^n(z)(g^m(z)-1)\prod_{j=1}^{d}(g(z+c_j))^{\nu_j}\right)^{(k)}$ share $\alpha(z)$ CM, then $f \equiv tg$, where $t^m = t^{n+\sigma} = 1$.*

Theorem H [18]. *Under the same situation of Theorem G if $n \geq 4k - m + \sigma + 9$ and $\left(f^n(z)(f(z)-1)^m\prod_{j=1}^{d}(f(z+c_j))^{\nu_j}\right)^{(k)}$, $\left(g^n(z)(g(z)-1)^m\prod_{j=1}^{d}(g(z+c_j))^{\nu_j}\right)^{(k)}$ share $\alpha(z)$ CM, then $f \equiv g$.*

In 2017, with the notion of weighted sharing as introduced in [10], Banerjee and Majumder [1] rectified the errors occurred in Theorems G and H and generalised the results as follows.

Theorem I [1]. *Let f and g be two transcendental entire functions of finite order, $c_j \in \mathbb{C}$, $j = 1, 2, \ldots, s$, be distinct and let $a(\not\equiv 0, \infty) \in S(f) \cap S(g)$ with finitely many zeros. Let m, n, $\mu_j \in \mathbb{N}$, $j = 1, 2, \ldots, s$, such that $n > 2k + m + \sigma + 4$, where $\sigma = \sum_{j=1}^{s}\mu_j$ and $P(\omega) = \sum_{j=0}^{m} a_j\omega^j$ be a polynomial, where $a_0(\neq 0), a_1, \ldots, a_m(\neq 0) \in \mathbb{C}$. If $\left(f^n(z)P(f(z))\prod_{j=1}^{s}(f(z+c_j))^{\mu_j}\right)^{(k)} - a(z)$ and $\left(g^n(z)P(g(z))\prod_{j=1}^{s}(g(z+c_j))^{\mu_j}\right)^{(k)} - a(z)$ share $(0,2)$, then:*

(I) *when $P(\omega) = \sum_{j=0}^{m} a_j\omega^j$ is a non-zero polynomial, one of the following two cases holds:*

(I_1) *$f \equiv tg$, $t \in \mathbb{C}\setminus\{0\}$ such that $t^d = 1$, where d is the GCD of the elements of J, $J = \{k \in I : a_k \neq 0\}$ and $I = \{n+\sigma, n+\sigma+1, \ldots, n+\sigma+m\}$,* (I$_2$) $f^n(z)P(f(z))\prod_{j=1}^{s}(f(z+c_j))^{\mu_j}\equiv g^n(z)P(g(z))\prod_{j=1}^{s}(g(z+c_j))^{\mu_j}$;

(II) *when* $P(\omega)=\omega^m-1$ *and* $n\geq\sigma+2s+3$, *then* $f\equiv tg$, $t\in\mathbb{C}\backslash\{0\}$ *such that* $t^m=t^{n+\sigma}=1$;

(III) *when* $P(\omega)=(\omega-1)^m(m\geq 2)$, *one of the following two cases holds:*

(III$_1$) $f\equiv g$,

(III$_2$) $f^n(z)(f(z)-1)^m\prod_{j=1}^{s}(f(z+c_j))^{\mu_j}\equiv g^n(z)(g(z)-1)^m\prod_{j=1}^{s}(g(z+c_j))^{\mu_j}$.

In the same paper, Banerjee and Majumder [1] emerged the following question as an open problem.

***Question* 1.** Whether Theorem I can be obtained for any small function $a\in S(f)\cap S(g)$?

Our first objective to write this paper is to solve Question 1. Throughout this paper we use $\mathcal{P}(\omega)$ as follows:

$$\mathcal{P}(\omega)=a_m\omega^m+a_{m-1}\omega^{m-1}+\ldots+a_1\omega+a_0, \tag{1.1}$$

where $a_i\in S(f)\cap S(g)$ for $i=0,1,2,\ldots,m$ such that $a_0\not\equiv 0,\ a_m\not\equiv 0$.

Let $c\in\mathbb{C}$ such that $\mathcal{P}(c)\not\equiv 0$ and let $\omega_1=\omega-c$. Then $\mathcal{P}(\omega)=\mathcal{P}(\omega_1+c)=\mathcal{P}_1(\omega_1)$, say, where $\mathcal{P}_1(\omega_1)$ is of the form

$$\mathcal{P}_1(\omega_1)=b_m\omega_1^m+b_{m-1}\omega_1^{m-1}+\ldots+b_1\omega_1+b_0, \tag{1.2}$$

$b_i\in S(f)\cap S(g)$ for $i=0,1,2,\ldots,m$ such that $b_0\equiv a_mc^m+a_{m-1}c^{m-1}+\ldots+a_1c+a_0\not\equiv 0$, $b_m\equiv a_m\not\equiv 0$. Throughout this paper we use $\mathcal{P}_1(\omega_1)$ defined as in (1.2).

Our second objective to write this paper is to solve the following questions.

***Question* 2.** Is Theorem I hold for $\mathcal{P}(\omega)$ instead of $P(\omega)$?

***Question* 3.** Can one deduce more generalized result in which Theorem I will be included?

In 2017, with the notion of weakly weighted sharing and relaxed weighted sharing as introduced in [12] and [2], respectively, Sahoo and Karmakar [15] obtained the following results.

Theorem J [15]. *Let f and g be two transcendental entire functions of finite order and $\alpha(\not\equiv 0)\in S(f)\cap S(g)$. Suppose that $\eta\in\mathbb{C}\setminus\{0\}$, $k\in\mathbb{N}\cup\{0\}$ and $m,n(>k)\in\mathbb{N}$ satisfying $n\geq 2k+m+6$, when $m\leq k+1$, and $n\geq 4k-m+10$, when $m>k+1$. If $(f^n(z)(f(z)-1)^mf(z+\eta))^{(k)}$ and $(g^n(z)(g(z)-1)^mg(z+\eta))^{(k)}$ share "$(\alpha(z),2)$" and if f, g have no 1-points with multiplicity less than or equal to k/m, when $m\leq k$, then either $f\equiv g$ or f and g satisfy the equation $R(f,g)\equiv 0$, where $R(\omega_1,\omega_2)$ is given by*

$$R(\omega_1,\omega_2)=\omega_1^n(\omega_1-1)^m\omega_1(z+\eta)-\omega_2^n(\omega_2-1)^m\omega_2(z+\eta).$$

Theorem K [15]. *Let f and g be two transcendental entire functions of finite order and $\alpha(\not\equiv 0)\in S(f)\cap S(g)$. Suppose that $\eta\in\mathbb{C}\setminus\{0\}$, $k\in\mathbb{N}\cup\{0\}$ and $m,n(>k)\in\mathbb{N}$ satisfying $n\geq 3k+2m+8$, when $m\leq k+1$, and $n\geq 6k-m+13$, when $m>k+1$. If $(f^n(z)(f(z)-1)^mf(z+\eta))^{(k)}$ and $(g^n(z)(g(z)-1)^mg(z+\eta))^{(k)}$ share $(\alpha(z),2)^*$ and if f, g have no 1-points with multiplicity less than or equal to k/m, when $m\leq k$, then the conclusions of Theorem J hold.*

Now our third objective to write this paper is to solve the following question.

***Question* 4.** Can one remove the condition "f, g have no 1-points with multiplicity less than or equal to k/m, when $m\leq k$" in Theorems J and K?

In this paper, taking the possible answers of the above questions into back ground we obtain main results as follows.

Theorem 1. *Let f and g be two transcendental entire functions of finite order, $c \in \mathbb{C}$, $c_j \in \mathbb{C}$, $j = 1, 2, \ldots, s$, be distinct and let $a(\not\equiv 0, \infty) \in S(f) \cap S(g)$. Let k, $m \in \mathbb{N} \cup \{0\}$, n, $\sigma \in \mathbb{N}$, $\mu_j \in \mathbb{N} \cup \{0\}$, $j = 1, 2, \ldots, s$, such that $n \geq k + 1$ and $\sigma = \sum_{j=1}^{s} \mu_j$. Suppose that*

$$\left((f(z) - c)^n \mathcal{P}(f(z)) \prod_{j=1}^{s} (f(z + c_j) - c)^{\mu_j}\right)^{(k)} - a(z)$$

and

$$\left((g(z) - c)^n \mathcal{P}(g(z)) \prod_{j=1}^{s} (g(z + c_j) - c)^{\mu_j}\right)^{(k)} - a(z)$$

share $(0, 2)$, where $\mathcal{P}(\omega)$ is defined as in (1.1). *Now:*

(I) *when $\mathcal{P}(\omega) \not\equiv (\omega - c)^m - \beta$, $(\omega - c - \beta)^m (m \geq 2)$, where $\beta \in S(f) \cap S(g)$ and $n \geq \geq 2k + m + \sigma + 5$, then one of the following two cases holds:*

(I_1) *$f - c \equiv t(g - c)$, $t \in \mathbb{C} \setminus \{0\}$ such that $t^d = 1$, where d is the GCD of the elements of J, $J = \{k \in I : b_k \neq 0\}$ and $I = \{0, 1, \ldots, m\}$;*

(I_2) $(f(z) - c)^n \mathcal{P}(f(z)) \prod_{j=1}^{s} (f(z + c_j) - c)^{\mu_j} \equiv (g(z) - c)^n \mathcal{P}(g(z)) \prod_{j=1}^{s} (g(z + c_j) - c)^{\mu_j}$;

(II) *when $\mathcal{P}(\omega) = (\omega - c)^m - \beta$, where $\beta \in S(f) \cap S(g)$ and $n \geq \max\{2k + m + \sigma + 5, \sigma + 2s + 3\}$, then $f - c \equiv t(g - c)$, $t \in \mathbb{C} \setminus \{0\}$ such that $t^m = t^{n+\sigma} = 1$;*

(III) *when $\mathcal{P}(\omega) = (\omega - c - \beta)^m$, $m \geq 2$, where $\beta \in S(f) \cap S(g)$ and*

$$n \geq \begin{cases} 2k + m + \sigma + 5, & \text{if} \quad m \leq k + 1, \\ 4k - m + \sigma + 9, & \text{if} \quad m > k + 1, \end{cases}$$

then one of the following two cases holds:

(III_1) $f \equiv g$,

(III_2) $(f(z) - c)^n (f(z) - c - \beta(z))^m \prod_{j=1}^{s} (f(z + c_j) - c)^{\mu_j} \equiv (g(z) - c)^n (g(z) - c - - \beta(z))^m \prod_{j=1}^{s} (g(z + c_j) - c)^{\mu_j}$.

***Corollary* 1.** *Let f and g be two transcendental entire functions of finite order, $c \in \mathbb{C}$, $c_j \in \mathbb{C}$, $j = 1, 2, \ldots, s$, be distinct, $a(\not\equiv 0, \infty) \in S(f) \cap S(g)$ and $k, m \in \mathbb{N} \cup \{0\}$, $n, \sigma \in \mathbb{N}$, $\mu_j \in \mathbb{N} \cup \{0\}$, $j = 1, 2, \ldots, s$, such that $n \geq k + 1$ and $\sigma = \sum_{j=1}^{s} \mu_j$. Suppose that*

$$\left((f(z) - c)^n \mathcal{P}(f(z)) \prod_{j=1}^{s} (f(z + c_j) - c)^{\mu_j}\right)^{(k)} - a(z)$$

and

$$\left((g(z) - c)^n \mathcal{P}(g(z)) \prod_{j=1}^{s} (g(z + c_j) - c)^{\mu_j}\right)^{(k)} - a(z)$$

share "$(0, 2)$", where $\mathcal{P}(\omega) = (\omega - c - \beta)^m$ and $\beta \in S(f) \cap S(g)$. Now when

$$n \geq \begin{cases} 2k+m+\sigma+5, & \text{if} \quad m \leq k+1, \\ 4k-m+\sigma+9, & \text{if} \quad m > k+1, \end{cases}$$

then one of the conclusions (III_1) *and* (III_2) *of Theorem 1 holds.*

***Corollary* 2.** *Let f and g be two transcendental entire functions of finite order, $c \in \mathbb{C}$, $c_j \in \mathbb{C}$, $j = 1, 2, \ldots, s$, be distinct, $a(\not\equiv 0, \infty) \in S(f) \cap S(g)$ and k, $m \in \mathbb{N} \cup \{0\}$, n, $\sigma \in \mathbb{N}$, $\mu_j \in \mathbb{N} \cup \{0\}$, $j = 1, 2, \ldots, s$, such that $n \geq k+1$ and $\sigma = \sum_{j=1}^{s} \mu_j$. Suppose that*

$$\left((f(z)-c)^n \mathcal{P}(f(z)) \prod_{j=1}^{s} (f(z+c_j)-c)^{\mu_j}\right)^{(k)} - a(z)$$

and

$$\left((g(z)-c)^n \mathcal{P}(g(z)) \prod_{j=1}^{s} (g(z+c_j)-c)^{\mu_j}\right)^{(k)} - a(z)$$

share $(0, 2)^*$, where $\mathcal{P}(\omega) = (\omega - c - \beta)^m$ and $\beta \in S(f) \cap S(g)$. Now when

$$n \geq \begin{cases} 3k+2m+2\sigma+6, & \text{if} \quad m \leq k+1, \\ 6k-m+2\sigma+11, & \text{if} \quad m > k+1, \end{cases}$$

then one of the conclusions (III_1) *and* (III_2) *of Theorem* 1 *holds.*

2. Lemmas. Let F and G be two non-constant meromorphic functions. Henceforth we shall denote by H the function

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right). \tag{2.1}$$

Lemma 1 [16]. *Let f be a non-constant meromorphic function and let $a_n(\not\equiv 0), a_{n-1}, \ldots, a_0 \in \in S(f)$. Then $T\left(r, \sum_{i=0}^{n} a_i f^i\right) = nT(r,f) + S(r,f)$.*

Lemma 2 [4]. *Let f be a meromorphic function of finite order ρ and $c \in \mathbb{C} \setminus \{0\}$ be fixed. Then for each $\varepsilon > 0$, we have*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = O\left(r^{\rho-1+\varepsilon}\right).$$

The following lemma has little modifications of the original version (Theorem 2.1 of [4]).

Lemma 3. *Let f be a transcendental meromorphic function of finite order, $c \in \mathbb{C} \setminus \{0\}$ be fixed. Then $T(r, f(z+c)) = T(r,f) + S(r,f)$.*

Lemma 4 [6]. *Let f be a non-constant meromorphic function of finite order and $c \in \mathbb{C}$. Then*

$$N(r, 0; f(z+c)) \leq N(r, 0; f(z)) + S(r,f) \qquad \text{and} \qquad N(r, \infty; f(z+c)) \leq N(r, \infty; f) + S(r,f).$$

Lemma 5 ([8], Lemma 3.5)**.** *Suppose that F is meromorphic in a domain D and set $f = \dfrac{F'}{F}$. Then, for $n \in \mathbb{N}$,*

$$\frac{F^{(n)}}{F} = f^n + \frac{n(n-1)}{2} f^{n-2} f' + A_n f^{n-3} f'' + B_n f^{n-4} (f')^2 + P_{n-3}(f),$$

where $A_n = \dfrac{1}{6} n(n-1)(n-2)$, $B_n = \dfrac{1}{8} n(n-1)(n-2)(n-3)$ and $P_{n-3}(f)$ is a differential polynomial with constant coefficients, which vanishes identically for $n \leq 3$ and has degree $n-3$ when $n > 3$.

Lemma 6. *Let f be a transcendental meromorphic function of finite order such that $N(r,\infty;f) = S(r,f)$ and $c_j \in \mathbb{C}$, $m \in \mathbb{N} \cup \{0\}$, n, $\sigma \in \mathbb{N}$, $\mu_j \in \mathbb{N} \cup \{0\}$, $j = 1, 2, \ldots, s$. Then, for each $\varepsilon > 0$, we have*

$$T\left(r, f^n(z) \mathcal{P}(f(z)) \prod_{j=1}^{s} (f(z+c_j))^{\mu_j}\right) = T\left(r, f^{n+\sigma}(z) \mathcal{P}(f(z))\right) + S(r,f).$$

Proof of lemma follows from the proof of Lemma 6 [1].

Lemma 7. *Let f and g be two transcendental entire functions of finite order, c, $c_j \in \mathbb{C}$, and k, $m \in \mathbb{N} \cup \{0\}$, n, $\sigma \in \mathbb{N}$, $\mu_j \in \mathbb{N} \cup \{0\}$, $j = 1, 2, \ldots, s$, such that $n \geq k+1$. Suppose that*

$$F(z) = \frac{\left((f(z)-c)^n \mathcal{P}(f(z)) \prod_{j=1}^{s} (f(z+c_j)-c)^{\mu_j}\right)^{(k)}}{\alpha(z)},$$

$$G(z) = \frac{\left((g(z)-c)^n \mathcal{P}(g(z)) \prod_{j=1}^{s} (g(z+c_j)-c)^{\mu_j}\right)^{(k)}}{\alpha(z)},$$

where $\alpha \in S(f) \cap S(g)$ and $H \equiv 0$. If one of the following conditions holds:

(1) *$\mathcal{P}(\omega) \not\equiv (\omega - c - \beta)^m$, where $\beta \in S(f) \cap S(g)$ and $n \geq 2k + m + \sigma + 5$,*

(2) *$\mathcal{P}(\omega) \equiv (\omega - c - \beta)^m$, where $\beta \in S(f) \cap S(g)$ and $n \geq 2k + m + \sigma + 5$ when $m \leq k+1$ and $n \geq 4k - m + \sigma + 9$ when $m > k+1$,*

then one of the following two cases holds:

(i) $$\left((f(z)-c)^n \mathcal{P}(f(z)) \prod_{j=1}^{s} (f(z+c_j)-c)^{\mu_j}\right)^{(k)} \times$$
$$\times \left((g(z)-c)^n \mathcal{P}(g(z)) \prod_{j=1}^{s} (g(z+c_j)-c)^{\mu_j}\right)^{(k)} \equiv \alpha^2(z),$$

(ii) $(f(z)-c)^n \mathcal{P}(f(z)) \prod_{j=1}^{s} (f(z+c_j)-c)^{\mu_j} \equiv (g(z)-c)^n \mathcal{P}(g(z)) \prod_{j=1}^{s} (g(z+c_j)-c)^{\mu_j}$.

Proof. Note that when $\mathcal{P}(\omega) = (\omega - c - \beta)^m$, where $\beta \in S(f) \cap S(g)$ and $m > k+1$. Then

$$N_{k+1}\left(r, 0; (f(z)-c)^n \mathcal{P}(f(z)) \prod_{j=1}^{s} (f(z+c_j)-c)^{\mu_j}\right) +$$

$$= N_{k+1}\left(r, 0; f_1^n(z)\mathcal{P}_1(f_1(z))\prod_{j=1}^{s}(f_1(z+c_j))^{\mu_j}\right) =$$

$$= N_{k+1}\left(r, 0; f_1^n(z)(f_1(z)-\beta(z))^m\prod_{j=1}^{s}(f_1(z+c_j))^{\mu_j}\right) =$$

$$= N_{k+1}(r, 0; f_1^n) + N_{k+1}(r, 0; (f_1(z)-\beta(z))^m) + N_{k+1}\left(r, 0; \prod_{j=1}^{s}(f_1(z+c_j))^{\mu_j}\right) \leq$$

$$\leq 2(k+1)T(r, f_1) + N\left(r, 0; \prod_{j=1}^{s}(f_1(z+c_j))^{\mu_j}\right) + S(r, f_1),$$

where $f_1 = f - c$. Similar expression holds for $(g(z)-c)^n\mathcal{P}(g(z))\prod_{j=1}^{s}(g(z+c_j)-c)^{\mu_j}$. We omit the detail proof, since proof of lemma follows from the proof of Lemma 7 [1].

Lemma 8 ([1], Lemma 8)**.** *Let f be a transcendental meromorphic function of finite order and $c_j \in \mathbb{C},\ j = 1, 2, \ldots, s$. Suppose that $n, \sigma \in \mathbb{N}$ and $\mu_j \in \mathbb{N} \cup \{0\},\ j = 1, 2, \ldots, s$. Let $\Phi(z) = f^n(z)\prod_{j=1}^{s}(f(z+c_j))^{\mu_j}$. Then $(n-\sigma)\,T(r, f) \leq T(r, \Phi) + S(r, f)$.*

Lemma 9. *Let f and g be transcendental entire functions of finite order, $\alpha\ (\not\equiv 0, \infty) \in S(f) \cap S(g),\ c,\ c_j \in \mathbb{C}$ and $n,\ m,\ s,\ \sigma \in \mathbb{N},\ \ k, \mu_j \in \mathbb{N} \cup \{0\},\ j = 1, 2, \ldots, s$. If $n \geq k+1$, then*

$$\left((f(z)-c)^n\mathcal{P}(f(z))\prod_{j=1}^{s}(f(z+c_j)-c)^{\mu_j}\right)^{(k)}\left((g(z)-c)^n\mathcal{P}(g(z))\prod_{j=1}^{s}(g(z+c_j)-c)^{\mu_j}\right)^{(k)} \not\equiv \alpha^2(z).$$

Proof. Suppose on contrary that

$$\left((f(z)-c)^n\mathcal{P}(f(z))\prod_{j=1}^{s}(f(z+c_j)-c)^{\mu_j}\right)^{(k)}\left((g(z)-c)^n\mathcal{P}(g(z))\prod_{j=1}^{s}(g(z+c_j)-c)^{\mu_j}\right)^{(k)} \equiv \alpha^2(z).$$

Then

$$\left(f_1^n(z)\mathcal{P}_1(f_1(z))\prod_{j=1}^{s}(f_1(z+c_j))^{\mu_j}\right)^{(k)}\left(g_1^n(z)\mathcal{P}_1(g_1(z))\prod_{j=1}^{s}(g_1(z+c_j))^{\mu_j}\right)^{(k)} \equiv \alpha^2(z), \quad (2.2)$$

where $f_1 = f - c$ and $g_1 = g - c$. Note that $S(r, f) = S(r, f_1)$ and $S(r, g) = S(r, g_1)$. Since $n \geq k+1$, from (2.2), we have

$$N(r, 0; f_1) \leq N(r, 0; \alpha^2) = S(r, f_1). \quad (2.3)$$

Since f and g are transcendental entire functions of finite order and so are f_1 and g_1. Therefore we can take $f_1(z) = \gamma(z)e^{\delta(z)}$ and $g_1(z) = \eta(z)e^{\zeta(z)}$, where $\gamma(z)(\not\equiv 0)$, $\eta(z)(\not\equiv 0)$ are entire functions such that $N(r,0;\gamma) = S(r,f_1)$, $N(r,0;\eta) = S(r,f_1)$ and $\delta(z)$, $\zeta(z)$ are non-zero polynomials. We now consider following two cases.

Case 1. Suppose that $k \in \mathbb{N}$. Let

$$F_i(z) = b_i(z)f_1^{n+i}(z)\prod_{j=1}^{s}(f_1(z+c_j))^{\mu_j} =$$

$$= b_i(z)\gamma^{n+i}(z)\prod_{j=1}^{s}(\gamma(z+c_j))^{\mu_j}\, e^{(n+i)\delta(z)+\sum_{j=1}^{s}\mu_j\delta(z+c_j)} = P_{1i}(z)e^{P_{2i}(z)}, \tag{2.4}$$

where $P_{1i}(z) = b_i\gamma^{n+i}(z)\prod_{j=1}^{s}(\gamma(z+c_j))^{\mu_j}$ and $P_{2i}(z) = (n+i)\delta(z) + \sum_{j=1}^{s}\mu_j\delta(z+c_j)$ for $i = 0,1,2,\ldots,m$. Let $J_1 = \{j \in I_1 : b_j(z) \not\equiv 0\}$, where $I_1 = \{0,1,\ldots,m\}$. Note that $N(r,\infty;F_i) = S(r,f_1)$ for $i \in J_1$. Using (2.3) and Lemma 4, we obtain $N(r,0,P_{1i}) = S(r,f_1)$ and $N(r,\infty,P_{1i}) = S(r,f_1)$ for $i \in J_1$. By Lemmas 1 and 6, we have $T(r,F_i) = (n+i+\sigma)T(r,f_1) + S(r,f_1)$ and so $S(r,F_i) = S(r,f_1)$ for $i \in J_1$. Note that $\gamma(z) \not\equiv 0$ and so $\prod_{j=1}^{s}(\gamma(z+c_j))^{\mu_j} \not\equiv 0$. Therefore, we have $P_{1i}(z) \not\equiv 0$ for $i \in J_1$. Let

$$h_i = \frac{F_i'}{F_i} = \frac{P_{1i}'}{P_{1i}} + P_{2i}'$$

for $i \in J_1$. Clearly,

$$T(r,h_i) = N\left(r,\frac{F_i'}{F_i}\right) + m\left(r,\frac{F_i'}{F_i}\right) = \overline{N}(r,\infty;F_i) + \overline{N}(r,0;F_i) + S(r,F_i) =$$

$$= S(r,f_1) + S(r,F_i) = S(r,f_1) \tag{2.5}$$

for $i \in J_1$. By using (2.5), we obtain

$$T\big(r,h_i^{(p)}\big) \le (p+1)T(r,h_i) + S(r,h_i) = S(r,f_1), \tag{2.6}$$

where $p \in \mathbb{N}\cup\{0\}$ and $i \in J_1$. From (2.6) and Lemma 1, we get

$$T\big(r,\big(h_i^{(p)}\big)^q\big) = q\,T\big(r,h_i^{(p)}\big) + S(r,h_i) = S(r,f_1), \tag{2.7}$$

where $q \in \mathbb{N}\cup\{0\}$ and $i \in J_1$. By Lemma 5, we have $F_i^{(k)} = Q_iF_i$, i.e.,

$$\big(F_i(z)\big)^{(k)} = Q_i(z)P_{1i}(z)e^{P_{2i}(z)},$$

where $Q_i = h_i^k + \dfrac{k(k-1)}{2}h_i^{k-2}h_i' + A_kh_i^{k-3}h_i'' + B_kh_i^{k-4}(h_i')^2 + P_{k-3}(h_i)$ and $i \in J_1$. Since $f(z)$ is a transcendental entire function, it follows that $F_i(z)$ is also a transcendental entire function for $i \in J_1$. Consequently, $Q_i(z) \not\equiv 0$ for $i \in J_1$. Then, from (2.5) and (2.7), it follows that

$$T(r,Q_i) = T\left(r,h_i^k + \frac{k(k-1)}{2}h_i^{k-2}h_i' + A_kh_i^{k-3}h_i'' + B_kh_i^{k-4}(h_i')^2 + P_{k-3}(h_i)\right) \le$$

$$\leq T(r,h_i^k)+T(r,h_i^{k-2})+T(r,h_i^{'})+T(r,h_i^{k-3})+T(r,h_i^{''})+$$

$$+T(r,h_i^{k-4})+T(r,(h_i^{'})^2))+T(r,P_{k-3}(h_i))+S(r,f_1)=S(r,f_1)$$

for $i \in J_1$. Note that, for $i \in J_1$,

$$\left(f_1^n(z)\mathcal{P}_1(f_1(z))\prod_{j=1}^{s}(f_1(z+c_j))^{\mu_j}\right)^{(k)}=$$

$$=\sum_{i=0}^{m}\left(b_if_1^{n+i}(z)\prod_{j=1}^{s}(f_1(z+c_j))^{\mu_j}\right)^{(k)}=$$

$$=\sum_{i=0}^{m}Q_i(z)P_{1i}(z)e^{P_{2i}(z)}=$$

$$=\gamma^n(z)\prod_{j=1}^{s}(\gamma(z+c_j))^{\mu_j}\,e^{n\delta(z)+\sum_{j=1}^{s}\mu_j\delta(z+c_j)}\sum_{i=0}^{m}b_iQ_i(z)\gamma^i(z)e^{i\delta(z)}=$$

$$=\gamma^n(z)\prod_{j=1}^{s}(\gamma(z+c_j))^{\mu_j}\,e^{n\delta(z)+\sum_{j=1}^{s}\mu_j\delta(z+c_j)}\sum_{i=0}^{m}b_iQ_i(z)f_1^i(z). \tag{2.8}$$

Also, from (2.2), we have $\left(f_1^n(z)\mathcal{P}_1(f_1(z))\prod_{j=1}^{s}(f_1(z+c_j))^{\mu_j}\right)^{(k)}\not\equiv 0$ and so

$$\overline{N}\left(r,0;\left(f_1^n(z)\mathcal{P}_1(f_1(z))\prod_{j=1}^{s}(f_1(z+c_j))^{\mu_j}\right)^{(k)}\right)\leq N(r,0;\alpha^2)\leq S(r,f_1).$$

From (2.8), we obtain

$$\overline{N}\left(r,0;\sum_{i=0}^{m}b_iQ_if_1^i\right)=S(r,f_1). \tag{2.9}$$

Since $b_i,Q_i\in S(f_1)$, from Lemma 1, we get $m\,T(r,f_1)=T\Big(r,\sum_{i=0}^{m}b_iQ_if_1^i\Big)+S(r,f_1)$. This shows that $S(r,f_1)=S\Big(r,\sum_{i=0}^{m}b_iQ_if_1^i\Big)$. Similarly, we have $S(r,f_1)=S\Big(r,\sum_{i=1}^{m}b_iQ_if_1^i\Big)$. Now we claim that $\sum_{i=1}^{m}b_iQ_if_1^i$ is not a rational function. If possible suppose that $\sum_{i=1}^{m}b_iQ_if_1^i$ is a rational function. Since $b_i,Q_i\in S(f_1)$, we obtain

$$m\,T(r,f_1)=T\left(r,\sum_{i=1}^{m}b_iQ_if_1^i\right)+S(r,f_1)=O(\log r)+S(r,f_1)=S(r,f_1),$$

which is not possible. Hence, $\sum_{i=1}^{m}b_iQ_if_1^i$ is a transcendental meromorphic function such that

$$N\left(r,\infty,\sum_{i=1}^{m}b_iQ_if_1^i\right)=S(r,f_1). \tag{2.10}$$

Note that $T(r, b_0Q_0) = S(r, f_1) = S\Big(r, \sum_{i=1}^{m} b_iQ_if_1^i\Big)$, which shows that b_0Q_0 is a small function of $\sum_{i=1}^{m} b_iQ_if_1^i$. Then, using (2.3), (2.9), (2.10) and Lemma 1, we get from the second fundamental theorem for small functions (see [17]) that

$$m\,T(r, f_1) = T\left(r, \sum_{i=1}^{m} b_iQ_if_1^i\right) + S(r, f_1) \leq$$

$$\leq \overline{N}\left(r, 0; \sum_{i=1}^{m} b_iQ_if_1^i\right) + \overline{N}\left(r, \infty; \sum_{i=1}^{m} b_iQ_if_1^i\right) + \overline{N}\left(r, 0; \sum_{i=0}^{m} b_iQ_if_1^i\right) + S(r, f_1) \leq$$

$$\leq \overline{N}(r, 0; f_1) + \overline{N}\left(r, 0; \sum_{i=1}^{m} b_iQ_if_1^{i-1}\right) + S(r, f_1) \leq$$

$$\leq T\left(r, \sum_{i=1}^{m} b_iQ_if_1^{i-1}\right) + S(r, f_1) = (m-1)\,T(r, f_1) + S(r, f_1),$$

which is not possible.

Case 2. Suppose that $k = 0$. Then, from (2.2), we get

$$f_1^n(z)\mathcal{P}_1(f_1(z))\prod_{j=1}^{s}(f_1(z+c_j))^{\mu_j} g_1^n(z)\mathcal{P}_1(g_1(z))\prod_{j=1}^{s}(g_1(z+c_j))^{\mu_j} \equiv \alpha^2(z). \tag{2.11}$$

Now, from (2.11), we have

$$N(r, 0; \mathcal{P}_1(f_1)) \leq N(r, 0; \alpha^2) = S(r, f_1), \text{ i.e., } N\left(r, 0; \sum_{i=0}^{m} b_if_1^i\right) = S(r, f_1). \tag{2.12}$$

One can easily prove that $\sum_{i=1}^{m} b_if_1^i$ is a transcendental meromorphic function such that

$$N\left(r, \infty; \sum_{i=0}^{m} b_if_1^i\right) = S(r, f_1) \tag{2.13}$$

and b_0 is a small function of $\sum_{i=1}^{m} b_if_1^i$. Now, by using (2.12), (2.13) and Lemma 1, we get, from the second fundamental theorem for small functions (see [17]),

$$m\,T(r, f_1) = T\left(r, \sum_{i=1}^{m} b_if_1^i\right) + S(r, f_1) \leq$$

$$\leq \overline{N}\left(r, 0; \sum_{i=1}^{m} b_if_1^i\right) + \overline{N}\left(r, \infty; \sum_{i=1}^{m} b_if_1^i\right) + \overline{N}\left(r, 0; \sum_{i=0}^{m} b_if_1^i\right) + S(r, f_1) \leq$$

$$\leq \overline{N}(r, 0; f_1) + \overline{N}\left(r, 0; \sum_{i=1}^{m} b_if_1^{i-1}\right) + S(r, f_1) \leq$$

$$\leq T\left(r, \sum_{i=1}^{m} b_i f_1^{i-1}\right) + S(r, f_1) = (m-1)\, T(r, f_1) + S(r, f_1),$$

which is not possible.

Lemma 9 is proved.

Lemma 10. *Let f and g be two transcendental entire functions of finite order*, $c,\ c_j \in \mathbb{C}$, *and* $m \in \mathbb{N} \cup \{0\},\ n, \sigma \in \mathbb{N},\ \mu_j \in \mathbb{N} \cup \{0\},\ j = 1, 2, \ldots, s$. *Suppose that*

$$(f(z)-c)^n \mathcal{P}(f(z)) \prod_{j=1}^{s} (f(z+c_j)-c)^{\mu_j} \equiv (g(z)-c)^n \mathcal{P}(g(z)) \prod_{j=1}^{s} (g(z+c_j)-c)^{\mu_j}.$$

Now:

(I) *when* $\mathcal{P}(\omega) \not\equiv (\omega - c)^m - \beta,\ (\omega - c - \beta)^m,\ m \geq 2$, *where* $\beta \in S(f) \cap S(g)$, *then one of the following two cases holds:*

(I_1) $f - c \equiv t(g - c),\ t \in \mathbb{C} \setminus \{0\}$ *such that* $t^d = 1$, *where* d *is the GCD of the elements of* J, $J = \{k \in I : b_k \neq 0\}$ *and* $I = \{0, 1, \ldots, m\}$;

(I_2) $(f(z)-c)^n \mathcal{P}(f(z)) \prod_{j=1}^{s} (f(z+c_j)-c)^{\mu_j} \equiv (g(z)-c)^n \mathcal{P}(g(z)) \prod_{j=1}^{s} (g(z+c_j)-c)^{\mu_j}$;

(II) *when* $\mathcal{P}(\omega) = (\omega - c)^m - \beta$, *where* $\beta \in S(f) \cap S(g)$ *and* $n > \sigma + 2s + 2$, *then* $f - c \equiv t(g - c)$, $t \in \mathbb{C} \setminus \{0\}$ *such that* $t^m = t^{n+\sigma} = 1$;

(III) *when* $\mathcal{P}(\omega) = (\omega - c - \beta)^m,\ m \geq 2$, *where* $\beta \in S(f) \cap S(g)$, *then one of the following two cases holds:*

(III_1) $f \equiv g$,

(III_2) $(f(z) - c)^n (f(z) - c - \beta(z))^m \prod_{j=1}^{s} (f(z + c_j) - c)^{\mu_j} \equiv (g(z) - c)^n (g(z) - c - \beta(z))^m \prod_{j=1}^{s} (g(z+c_j) - c)^{\mu_j}$.

Proof. Suppose that

$$(f(z)-c)^n \mathcal{P}(f(z)) \prod_{j=1}^{s} (f(z+c_j)-c)^{\mu_j} \equiv (g(z)-c)^n \mathcal{P}(g(z)) \prod_{j=1}^{s} (g(z+c_j)-c)^{\mu_j}, \qquad (2.14)$$

i.e.,

$$f_1^n(z) \mathcal{P}_1(f_1(z)) \prod_{j=1}^{s} (f_1(z+c_j))^{\mu_j} \equiv g_1^n(z) \mathcal{P}_1(g_1(z)) \prod_{j=1}^{s} (g_1(z+c_j))^{\mu_j}, \qquad (2.15)$$

where $f_1(z) = f(z) - c$ and $g_1(z) = g(z) - c$. Now, from (2.15), Lemmas 1 and 6, we have $T(r, f_1) + S(r, f_1) = T(r, g_1) + S(r, g_1)$ and so $S(r, f_1) = S(r, g_1)$. We consider the following cases.

Case 1. Suppose $\mathcal{P}(\omega) \not\equiv (\omega - c)^m - \beta$ or $(\omega - c - \beta)^m,\ m \geq 2$, where $\beta \in S(f) \cap S(g)$.

Let $h = \dfrac{f_1}{g_1}$. If h is a constant, by putting $f_1 = h g_1$ in (2.15), we get

$$b_m g_1^m (h^{n+m+\sigma} - 1) + b_{m-1} g_1^{m-1} (h^{n+m+\sigma-1} - 1) + \ldots + b_1 g_1 (h^{n+\sigma+1} - 1) + b_0 (h^{n+\sigma} - 1) \equiv 0,$$

which implies that $h^d = 1$, where d is the GCD of the elements of $J,\ J = \{k \in I : b_k \neq 0\}$ and $I = \{0, 1, \ldots, m\}$. Otherwise, by Lemma 1, we have $T(r, g_1) = S(r, g_1)$, which is impossible.

Thus, $f_1 \equiv tg_1$, i.e., $f - c = t(g - c)$, $t \in \mathbb{C} \setminus \{0\}$ such that $t^d = 1$, where d is the GCD of the elements of J, $J = \{k \in I : b_k \neq 0\}$ and $I = \{0, 1, \ldots, m\}$.

If h is not a constant, then we know by (2.14) that

$$(f(z) - c)^n \mathcal{P}(f(z)) \prod_{j=1}^{s} (f(z + c_j) - c)^{\mu_j} \equiv (g(z) - c)^n \mathcal{P}(g(z)) \prod_{j=1}^{s} (g(z + c_j) - c)^{\mu_j}.$$

Case 2. Suppose $\mathcal{P}(\omega) = (\omega - c)^m - \beta$, where $\beta \in S(f) \cap S(g)$. Clearly $\beta \in S(f_1) \cap S(g_1)$. Then, from (2.15), we have

$$f_1^n(z)(f_1^m(z) - \beta(z)) \prod_{j=1}^{s} (f_1(z + c_j))^{\mu_j} \equiv g_1^n(z)(g_1^m(z) - \beta(z)) \prod_{j=1}^{s} (g_1(z + c_j))^{\mu_j}. \tag{2.16}$$

Let $h = \dfrac{f_1}{g_1}$. Clearly, from (2.16), we get

$$g_1^m(z) \left(h^{n+m}(z) \prod_{j=1}^{s} (h(z + c_j))^{\mu_j} - 1 \right) \equiv \beta(z) \left(h^n(z) \prod_{j=1}^{s} (h(z + c_j))^{\mu_j} - 1 \right). \tag{2.17}$$

First we suppose that h is non-constant. We assert that both $h^{n+m}(z) \prod_{j=1}^{s} (h(z + c_j))^{\mu_j} (\not\equiv 0)$ and $h^n(z) \prod_{j=1}^{s} (h(z + c_j))^{\mu_j} (\not\equiv 0)$ are non-constant. If not, let $h^{n+m}(z) \prod_{j=1}^{s} (h(z + c_j))^{\mu_j} \equiv d_1 \in$ $\in \mathbb{C} \setminus \{0\}$. Then we have

$$h^{n+m}(z) \equiv \frac{d_1}{\prod_{j=1}^{s} (h(z + c_j))^{\mu_j}}.$$

Now, by Lemmas 1, 2 and 4, we get

$$(n + m)\, T(r, h) = T(r, h^{n+m}) + S(r, h) = T\left(r, \frac{d_1}{\prod_{j=1}^{s} (h(z + c_j))^{\mu_j}} \right) + S(r, h) \leq$$

$$\leq \sum_{j=1}^{s} \mu_j N(r, 0; h(z + c_j)) + \sum_{j=1}^{s} \mu_j\, m\left(r, \frac{1}{h(z + c_j)} \right) + S(r, h) \leq$$

$$\leq \sum_{j=1}^{s} \mu_j\, N(r, 0; h(z)) + \sum_{j=1}^{s} \mu_j\, m\left(r, \frac{1}{h(z)} \right) + S(r, h) \leq$$

$$\leq \sigma\, T(r, h) + S(r, h),$$

which is a contradiction. Similarly we can prove that $h^n(z) \prod_{j=1}^{s} (h(z + c_j))^{\mu_j}$ is non-constant.

Thus, from (2.17), we have

$$f_1^m(z) \equiv \beta(z) h^m(z) \frac{h^n(z) \prod_{j=1}^{s} (h(z + c_j))^{\mu_j} - 1}{h^{n+m}(z) \prod_{j=1}^{s} (h(z + c_j))^{\mu_j} - 1}$$

and

$$g_1^m(z) \equiv \beta(z) \frac{h^n(z) \prod_{j=1}^{s} (h(z+c_j))^{\mu_j} - 1}{h^{n+m}(z) \prod_{j=1}^{s} (h(z+c_j))^{\mu_j} - 1}. \tag{2.18}$$

First we claim that h is a transcendental meromorphic function. If not, suppose that h is a rational function. Then, from (2.18) and Lemma 1, we have $m\,T(r, g_1) = S(r, g_1)$, which is impossible. Hence, h is a transcendental meromorphic function. Note that $T(r, h) \le T(r, f_1) + T(r, g_1) = 2\,T(r, f_1) + S(r, f_1)$ and so $T(r, h) = O(T(r, f_1))$. By Lemmas 1 and 3, we obtain

$$T\left(r, h^n(z) \prod_{j=1}^{s} (h(z+c_j))^{\mu_j} - 1\right) \le T\left(r, h^n(z) \prod_{j=1}^{s} (h(z+c_j))^{\mu_j}\right) + O(1) \le$$

$$\le T(r, h^n) + T\left(r, \prod_{j=1}^{s} (h(z+c_j))^{\mu_j}\right) + O(1) \le$$

$$\le T(r, h^n) + \sum_{j=1}^{s} \mu_j\, T(r, h(z+c_j)) + O(1) =$$

$$= T(r, h^n) + \sum_{j=1}^{s} \mu_j\, T(r, h) + S(r, h) =$$

$$= (n + \sigma)\, T(r, h) + S(r, h).$$

Similarly, we have $T(r, h^{n+m}(z) \prod_{j=1}^{s} (h(z+c_j))^{\mu_j} - 1) \le (n + m + \sigma)\, T(r, h) + S(r, h)$. Now, from (2.18) and Lemma 1, we get $m\,T(r, g_1) \le (2n + m + 2\sigma) T(r, h) + S(r, h) + S(r, g_1)$, i.e., $T(r, g_1) = O(T(r, h))$. This shows that $S(r, g_1) = S(r, h)$ and so $\beta \in S(h)$. Let z_0 be a zero of $h^{n+m}(z) \prod_{j=1}^{s} (h(z+c_j))^{\mu_j} - 1$ such that $\beta(z_0) \ne 0, \infty$. Since g_1 is an entire function, it follows that z_0 is also a zero of $h^n(z) \prod_{j=1}^{s} (h(z+c_j))^{\mu_j} - 1$. Then clearly $h^m(z_0) - 1 = 0$ and so

$$\overline{N}\Big(r, 1; h^{n+m} \prod_{j=1}^{s} (h(z+c_j))^{\mu_j}\Big) \le \overline{N}(r, 1; h^m) \le m\,T(r, h) + S(r, h).$$

So, in view of Lemmas 1, 4, and 8 and the second fundamental theorem, we get

$$(n + m - \sigma)\, T(r, h) =$$

$$= T\left(r, h^{n+m}(z) \prod_{j=1}^{s} (h(z+c_j))^{\mu_j}\right) + S(r, h) \le$$

$$\le \overline{N}\left(r, 0; h^{n+m} \prod_{j=1}^{s} (h(z+c_j))^{\mu_j}\right) + \overline{N}\left(r, \infty; h^{n+m} \prod_{j=1}^{s} (h(z+c_j))^{\mu_j}\right) +$$

$$+\overline{N}\left(r,1;h^{n+m}\prod_{j=1}^{s}(h(z+c_j))^{\mu_j}\right)+S(r,h)\leq$$

$$\leq\overline{N}(r,0;h)+\sum_{j=1}^{s}\overline{N}(r,0;h(z+c_j))+\overline{N}(r,\infty;h)+$$

$$+\sum_{j=1}^{s}\overline{N}(r,\infty;h(z+c_j))+m\,T(r,h)+S(r,h)\leq$$

$$\leq N(r,0;h)+\sum_{j=1}^{s}N(r,0;h(z))+N(r,\infty;h)+\sum_{j=1}^{s}N(r,\infty;h(z))+m\,T(r,h)+S(r,h)\leq$$

$$\leq(m+2s+2)\,T(r,h)+S(r,h),$$

which contradicts with $n>\sigma+2s+2$. Hence, h is a constant. Since g_1 is transcendental entire function, from (2.17), we have

$$h^{n+m}(z)\prod_{j=1}^{s}(h(z+c_j))^{\mu_j}-1\equiv 0\Longleftrightarrow h^{n}(z)\prod_{j=1}^{s}(h(z+c_j))^{\mu_j}-1\equiv 0$$

and so $h^m(z)=1$ and $h^{n+\sigma}=1$. Thus, $f\equiv tg$, $t\in\mathbb{C}\setminus\{0\}$ such that $t^m=t^{n+\sigma}=1$.

Case 3. Suppose $\mathcal{P}(\omega)=(\omega-c-\beta)^m$, $m\geq 2$, where $\beta\in S(f)\cap S(g)$. Then, from (2.14), we have

$$(f(z)-c)^n(f(z)-c-\beta(z))^m\prod_{j}^{s}(f(z+c_j)-c)^{\mu_j}\equiv$$

$$\equiv(g(z)-c)^n(g(z)-c-\beta(z))^m\prod_{j=1}^{s}(g(z+c_j)-c)^{\mu_j}, \tag{2.19}$$

i.e.,

$$f_1^n(z)(f_1(z)-\beta(z))^m\prod_{j}^{s}(f_1(z+c_j))^{\mu_j}\equiv g_1^n(z)(g_1(z)-\beta(z))^m\prod_{j=1}^{s}(g_1(z+c_j))^{\mu_j}. \tag{2.20}$$

Let $h=\dfrac{f_1}{g_1}$. First we suppose that h is non-constant. Then we know from (2.19), that

$$(f(z)-c)^n(f(z)-c-\beta(z))^m\prod_{j}^{s}(f(z+c_j)-c)^{\mu_j}\equiv$$

$$\equiv(g(z)-c)^n(g(z)-c-\beta(z))^m\prod_{j=1}^{s}(g(z+c_j)-c)^{\mu_j}.$$

Next we suppose that h is constant. Then, from (2.20), we get

$$f_1^n(z)\prod_{j=1}^{s}(f_1(z+c_j))^{\mu_j}\ \sum_{i=0}^{m}{}^mC_{m-i}\ f_1^{m-i}(z)\beta^i(z)\equiv$$

$$\equiv g_1^n(z)\prod_{j=1}^{s}(g_1(z+c_j))^{\mu_j}\ \sum_{i=0}^{m}{}^mC_{m-i}g_1^{m-i}(z)\beta^i(z). \tag{2.21}$$

Now substituting $f_1 = hg_1$ in (2.21), we get

$$\sum_{i=0}^{m}{}^mC_{m-i}\ \beta^i g_1^{m-i}(z)(h^{n+m+\sigma-i}(z)-1)\equiv 0,$$

which implies that $h=1$. Hence, $f_1\equiv g_1$, i.e., $f\equiv g$.

Lemma 10 is proved.

3. Proof of Theorem 1. Let

$$F(z)=\frac{\Big((f(z)-c)^n\mathcal{P}(f(z))\prod_{j=1}^{s}(f(z+c_j)-c)^{\mu_j}\Big)^{(k)}}{a(z)}=$$

$$=\frac{\Big(f_1^n(z)\mathcal{P}_1(f_1(z))\prod_{j=1}^{s}(f_1(z+c_j))^{\mu_j}\Big)^{(k)}}{a(z)}$$

and

$$G(z)=\frac{\Big((g(z)-c)^n\mathcal{P}(g(z))\prod_{j=1}^{s}(g(z+c_j)-c)^{\mu_j}\Big)^{(k)}}{a(z)}=$$

$$=\frac{\Big(g_1^n(z)\mathcal{P}_1(g_1(z))\prod_{j=1}^{s}(g_1(z+c_j))^{\mu_j}\Big)^{(k)}}{a(z)},$$

where $f_1 = f-c$ and $g_1 = g-c$. Clearly F and G share $(1,2)$ except for the zeros and poles of a. Note that when $\mathcal{P}(\omega)=(\omega-c-\beta)^m$, where $\beta\in S(f)\cap S(g)$ and $m>k+1$, then

$$N_{k+2}\left(r,0;(f(z)-c)^n\mathcal{P}(f(z))\prod_{j=1}^{s}(f(z+c_j)-c)^{\mu_j}\right)=$$

$$=N_{k+2}\left(r,0;f_1^n(z)\mathcal{P}_1(f_1(z))\prod_{j=1}^{s}(f_1(z+c_j))^{\mu_j}\right)=$$

$$=N_{k+2}\left(r,0;f_1^n(z)(f_1(z)-\beta(z))^m\prod_{j=1}^{s}(f_1(z+c_j))^{\mu_j}\right)\le$$

$$\le 2(k+2)T(r,f_1)+N\left(r,0;\prod_{j=1}^{s}(f_1(z+c_j))^{\mu_j}\right)+S(r,f_1).$$

Similar expression holds for $(g(z)-c)^n\mathcal{P}(g(z))\prod_{j=1}^{s}(g(z+c_j)-c)^{\mu_j}$. So we omit the detail proof, since when $H\not\equiv 0$ we follow the proof of Theorem 2 [1] while for $H\equiv 0$ we follow Lemmas 7, 9 and 10.

Theorem 1 is proved.

***Proof of Corollary* 1.** When $H\not\equiv 0$ we follow the proof of Theorem 1 [14], while for $H\equiv 0$ we follow Lemmas 7, 9 and 10. So we omit the detail proof.

***Proof of Corollary* 2.** When $H\not\equiv 0$ we follow the proof of Theorem 2 [14], while for $H\equiv 0$ we follow Lemmas 7, 9 and 10. So we omit the detail proof.

References

1. A. Banerjee, S. Majumder, *On the uniqueness of certain types of differential-difference polynomials*, Anal. Math., **43**, № 3, 415 – 444 (2017).
2. A. Banerjee, S. Mukherjee, *Uniqueness of meromorphic functions concerning differential monomials sharing the same value*, Bull. Math. Soc. Sci. Math. Roumanie (N.S.), **50**, 191 – 206 (2007).
3. W. Bergweiler, A. Eremenko, *On the singularities of the inverse to a meromorphic function of finite order*, Rev. Mat. Iberoam, **11**, 355 – 373 (1995).
4. Y. M. Chiang, S. J. Feng, *On the Nevanlinna characteristic $f(z+\eta)$ and difference equations in complex plane*, Ramanujan J., **16**, 105 – 129 (2008).
5. M. R. Chen, Z. X. Chen, *Properties of difference polynomials of entire functions with finite order*, Chinese Ann. Math. Ser. A, **33**, 359 – 374 (2012).
6. J. Heittokangas, R. Korhonen, I. Laine, J. Rieppo, J. L. Zhang, *Value sharing results for shifts of meromorphic function, and sufficient conditions for periodicity*, J. Math. Anal. and Appl., **355**, 352 – 363 (2009).
7. W. K. Hayman, *Picard values of meromorphic functions and their derivatives*, Ann. Math., **70**, 9 – 42 (1959).
8. W. K. Hayman, *Meromorphic functions*, The Clarendon Press, Oxford (1964).
9. I. Lahiri, *Value distribution of certain differential polynomials*, Int. J. Math. and Math. Sci., **28**, 83 – 91 (2001).
10. I. Lahiri, *Weighted value sharing and uniqueness of meromorphic functions*, Complex Var., Theory and Appl., **46**, 241 – 253 (2001).
11. I. Laine, C. C. Yang, *Value distribution of difference polynomials*, Proc. Japan Acad. Ser. A, **83**, 148 – 151 (2007).
12. S. Lin, W. C. Lin, *Uniqueness of meromorphic functions concerning weakly weighted sharing*, Kodai Math. J., **29**, 269 – 280 (2006).
13. E. Mues, *Über ein Problem von Hayman*, Math. Z., **164**, 239 – 259 (1979).
14. P. Sahoo, H. Karmakar, *Results on uniqueness of entire functions whose certain difference polynomials share a small function*, Anal. Math., **41**, 257 – 272 (2015).
15. P. Sahoo, H. Karmakar, *Erratum to: Results on uniqueness of entire functions whose certain difference polynomials share a small function*, Anal. Math., **43**, № 4, 629 – 630 (2017).
16. C. C. Yang, *On deficiencies of differential polynomials II*, Math. Z., **125**, 107 – 112 (1972).
17. K. Yamanoi, *The second main theorem for small functions and related problems*, Acta Math., **192**, 225 – 294 (2004).
18. K. Zhang, H. X. Yi, *The value distribution and uniqueness of one certain type of differential-difference polynomials*, Acta Math. Sci. Ser. B, **34**, № 3, 719 – 728 (2014).
19. J. L. Zhang, *Value distribution and shared sets of differences of meromorphic functions*, J. Math. Anal. and Appl., **367**, 401 – 408 (2010).

Received 30.09.18,
after revision — 17.03.19