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ASYMPTOTIC BEHAVIOR OF A CLASS OF PERTURBED DIFFERENTIAL EQUATIONS

АСИМПТОТИЧНА ПОВЕДІНКА ОДНОГО КЛАСУ ЗБУРЕНИХ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ

This paper deals with the problem of stability of nonlinear differential equations with perturbations. Sufficient conditions for global uniform asymptotic stability in terms of Lyapunov-like functions and integral inequality are obtained. The asymptotic behavior is studied in the sense that the trajectories converge to a small ball centered at the origin. Furthermore, an illustrative example in the plane is given to verify the effectiveness of the theoretical results.

Розглядається задача стійкості нелінійних диференціальних рівнянь із збуреннями. Отримано достатні умови глобальної рівномірної асимптотичної стійкості у термінах функцій типу Ляпунова та інтегральних нерівностей. Асимптотична поведінка вивчається в сенсі того, що траєкторії збігаються до малої кулі із центром у початку координат. Крім того, з метою перевірки ефективності отриманих теоретичних результатів наведено відповідний приклад на площині.

1. Introduction. The problem of Lyapunov stability of time-varying differential equations has attracted the attention of serval authors and has produced a large important results [1 – 6]. The two major approaches for Lyapunov stability analysis [11, 15 – 19] are the linearization method and the direct method. Stability of a system can be investigated via the first linearization method, but in general and the most powerful technique is the second direct method. For this method one usually assumes the existence of the so-called Lyapunov function which is a positive definite function with negative derivative along the trajectories of the system. Therefore, because of the existence of some perturbations, stability are defined in terms of the behavior of solutions under other restrictions imposed on the term of perturbation. The natural assumption is to consider some stability property for the unperturbed system with some information on the bound of the perturbed term. Many works are given in this sense [4, 10, 13, 17]. The authors in [7, 8] introduce the concept of exponential rate of convergence and for a specific classes of uncertain systems where they present controllers which guarantee this behavior. In particular, when the origin is not necessarily an equilibrium point, in this case one can study the asymptotic behavior of the solutions in a neighborhood of the origin. This approach is used to study systems whose desired behavior is asymptotic stability about the origin of the state space or a close approximation to this [9, 10, 12 – 14]. In [18], the authors studied the robust stability of nonlinear systems, which admit simultaneously disturbances on the structure of the system and the initial conditions. This kind stability requires that the disturbing function is bounded. However, the system may oscillate close to the state, in such situation for allowable uncertainties and nonlinearities, we can estimate the region of attraction from which all solutions converge to a small ball containing the origin of the state space. The usual techniques is to use the Lyapunov function associated to the nominal system as a Lyapunov candidate for the perturbed system (see [16]). The idea used in [4] is to add in the Lyapunov function associated to the nominal system a special function which is chosen such that the derivative along the trajectories of the system in presence of perturbations is definite negative. Another approach is to use some integral inequalities of Gronwall – Belmann type to deduce the asymptotic stability of systems in presence of perturbations (see [14]). In this paper, under some restrictions on the perturbed term, we show that all state trajectories are bounded and approach a sufficiently small neighborhood of the origin. One also desires that the state approaches the origin (or some sufficiently small neighborhood of it) in a sufficiently fast manner, in this sense, we prove that the solutions of the perturbed systems converge to a small ball centered at the origin. Furthermore, we provide some sufficient conditions for the exponential stability of a class of perturbed systems based on a new nonlinear inequality. Finally, a numerical example is given to show the efficiency and accuracy of the method.

2. Stability analysis. We consider the first unperturbed nonlinear system described by the equation

$$\dot{x} = f(t,x), \quad t \geq 0, \tag{1}$$

where $x \in \mathbb{R}^n$ and $f\colon \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous in (t,x) and locally Lipschitz in x uniformly in t. We suppose that $f(t,0) = 0,\ t \geq 0$, in this case the origin is an equilibrium point of (1). The second one, is a perturbed system given by

$$\dot{y} = f(t,y) + g(t,y), \qquad t \geq 0, \tag{2}$$

where $f,\ g\colon \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous in (t,y) and locally Lipschitz in y uniformly in t.

The function $g(.,.)$ is the perturbation term which could result from errors in modeling a nonlinear system, aging of parameters or uncertainties and disturbances. In practice, if we know some information on the upper bound of this term and if the associated nominal system (1) has a uniformly asymptotically stable equilibrium point at the origin, what we can say about the stability behavior of the perturbed system (2). Unless otherwise stated, we assume throughout the paper that the function $f(.,.)$ encountered is sufficiently smooth. We often omit arguments of function to simplify notation, $\mathbb{R}^n$ is the n-dimensional Euclidean vector space; $\mathbb{R}^+$ is the set of all nonnegative real numbers; $\|x\|$ is the Euclidean norm of a vector $x \in \mathbb{R}^n$. $B_r = \{x \in \mathbb{R}^n/\|x\| \leq r,\ r > 0\}$ denotes the ball centered at the origin and of radius $r > 0$. For all $x_0 \in \mathbb{R}^n$ and $t_0 \in \mathbb{R}^+$, we will denote by $x(t,t_0,x_0)$, or simply by $x(t)$, the unique solution of (1) (respectively, by $y(t,t_0,y_0)$, or simply by $y(t)$, the unique solution of (2)) at time t_0 starting from the point x_0 (respectively, starting from the point y_0 at time t_0). We recall now some standard comparison functions which are used in stability theory to characterize the stability properties and uniform asymptotic stability (see [11, 16]): $\mathcal{K}$ is the class of functions $\mathbb{R}^+ \to \mathbb{R}^+$ which are zero at the origin, strictly increasing and continuous. $\mathcal{K}_\infty$ is the subset of $\mathcal{K}$ functions that are unbounded. $\mathcal{L}$ is the set of functions $\mathbb{R}^+ \to \mathbb{R}^+$ which are continuous, decreasing and converging to zero as their argument tends to $+\infty$. $\mathcal{KL}$ is the class of functions $\mathbb{R}^+ \longrightarrow \mathbb{R}^+$ which are class $\mathcal{K}$ on the first argument and class $\mathcal{L}$ on the second argument.

We will consider more general case when we do not know that $g(t,0) = 0$ for all $t \geq 0$. The origin may not be an equilibrium point of the perturbed system (2). We can non longer study the stability of the origin as an equilibrium point, nor should we expect the solution of the perturbed system to approach the origin as t goes to infinity. The best we can hope that for a small perturbation term the solution approach to the a small set which contains the origin. The asymptotic behavior of the solutions of (2) can be studied in a neighborhood of the origin, in this case the solutions converge to a certain small ball. Let consider the system (2), in the case where $g(t,0) \neq 0$ for a certain $t \geq 0$. We introduce some basic definitions and preliminary facts which we shall need in the sequel. First, we give the definition of global uniform asymptotic stability of a ball B_r (see [3, 9]).

Definition 1. *Let $r > 0$ be a positive number. The ball B_r is said to be globally uniformly asymptotically stable for* (1), *if there exists a class $\mathcal{KL}$ functions β such that the solution of* (1) *from any initial state $x_0 \in \mathbb{R}^n$ and initial time $t_0 \in \mathbb{R}^+$ satisfies the estimation*

$$\|x(t)\| \leq \beta(\|x_0\|, t - t_0) + r \quad \textit{for all} \quad t \geq t_0 \geq 0.$$

The next definition concerns the global uniform exponential stability.

Definition 2. *B_r is globally uniformly exponentially stable, if there exist $\gamma > 0$ and $k > 0$ such that, for all $t \geq t_0 \geq 0$ and $x_0 \in \mathbb{R}^n$,*

$$\|x(t)\| \leq k\|x_0\|\exp(-\gamma(t - t_0)) + r.$$

Note that, in the above definition, if we take $r = 0$, then one deals with the standard concept of the global exponential stability of the origin viewed as an equilibrium point. Moreover, if $g(t, 0) \neq 0$, we shall study the asymptotic behavior of a small ball centered at the origin for $0 \leq \|x(t)\| - r$ $\forall t \geq t_0 \geq 0$, so that the initial conditions are taken outside the ball B_r.

Lyapunov's direct method allows us to determine the stability of a system without explicitly integrating the differential equation.

Definition 3. *Let $V : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^+$ a Lyapunov function*

$$V(t, 0) = 0 \quad \forall t \geq 0,$$

$$V(t, x) > 0 \quad \forall (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n \setminus \{0\}.$$

(i) *$V(t, x)$ is positive definite, i.e., there exists a continuous, nondecreasing scalar function $\alpha(.)$ such that $\alpha(0) = 0$ and*

$$0 < \alpha(\|x\|) \leq V(t, x) \quad \forall y \neq 0.$$

(ii) *$\dot{V}(t, x)$ is negative definite, that is,*

$$\dot{V}(t, x) \leq -\gamma(\|x\|) < 0,$$

where $\gamma(.)$ is a continuous nondecreasing scalar function such that $\gamma(0) = 0$.

(iii) *$V(t, x) \leq \beta(\|x\|)$, where $\beta(.)$ is a continuous nondecreasing function and $\beta(0) = 0$, i.e., V is decrescent, i.e., the Lyapunov function is upper bounded.*

(iv) *V is radially unbounded, that is, $\alpha(\|x\|) \to \infty$ as $\|x\| \to \infty$.*

The classic Lyapunov direct criterion establishes that, given a nonlinear system, the existence of a smooth positive definite function with the property of having a nonpositive derivative along the solutions implies that the origin of the system is stable. If such derivative happens to be negative, the origin is asymptotically stable. If, in addition, the Lyapunov function of the system, is radially unbounded, then the origin is globally asymptotically stable. Furthermore, converse Lyapunov theorems ensure that the existence of a Lyapunov function is also a necessary condition for stability. The classic Lyapunov result for a nonlinear nonautonomous system can found in [11]. The following theorems give sufficient conditions to ensure global stability of a ball. Their proofs can be deduced from [7 – 9].

Theorem 1. *Consider system* (1) *and suppose that there exist a continuously differentiable function $V : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$, a class $\mathcal{K}_\infty$ functions $\alpha_1(.)$, $\alpha_2(.)$, a class $\mathcal{K}$ function $\alpha_3(.)$ and a small positive real number ϱ such that the inequalities hold, for all $t \geq t_0 \geq 0$ and $x \in \mathbb{R}^n$:*

$$\alpha_1(\|x\|) \leq V(t,x) \leq \alpha_2(\|x\|),$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -\alpha_3(\|x\|) + \varrho.$$

Then the ball

$$B_r = \{x \in \mathbb{R}^n / \|x\| \leq \alpha_1^{-1} \circ \alpha_2 \circ \alpha_3^{-1}(\varrho)\}$$

is globally uniformly asymptotically stable with

$$r = \alpha_1^{-1} \circ \alpha_2 \circ \alpha_3^{-1}(\varrho).$$

Now we state sufficient conditions for global exponential stability (see [3]).

Theorem 2. *Consider system* (1) *and let $V : [0, +\infty[\times \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable Lyapunov function such that*

$$c_1\|x\|^2 \leq V(t,x) \leq c_2\|x\|^2,$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -c_3 V(t,x) + \rho$$

for all $t \geq t_0 \geq 0$ and $x \in \mathbb{R}^n$, where c_1, c_2 and c_3 are positive constants. Then the ball

$$B_r = \{x \in \mathbb{R}^n / \|x\| \leq \sqrt{\rho / c_1 c_2}\}$$

is globally uniformly exponentially stable with

$$r = \sqrt{\rho / c_1 c_2}.$$

3. Stability of perturbed system. We study in this section the asymptotic behavior of perturbed systems via different approaches for certain classes of perturbed differential equations.

3.1. Lyapunov approach. There exist equations of the form (1) such that the origin is globally uniformly asymptotically stable, but there exists bounded perturbations such that (2) becomes unstable. This motivates us to study the problem of uniform stability of perturbed systems by assuming that the nominal associated system is globally uniformly asymptotically stable and some assumptions on the size of perturbations.

We assume that the perturbation term $g(.,.)$ satisfies the uniform bound:

$$\|g(t,y)\| \leq \theta(t)\|y\| + \xi(t) \quad \forall t \geq t_0 \geq 0 \quad \forall y \in \mathbb{R}^n, \tag{3}$$

where $\theta(.)$, $\xi(.) \in C[\mathbb{R}^+, \mathbb{R}^+]$ such that

$$\sup_{t \geq t_0} \theta(t) = \eta < +\infty$$

and the function $\xi(t)$ satisfies

$$\sup_{t \geq t_0} \xi(t) = \kappa < +\infty.$$

For the asymptotic behavior of equation (2), we shall suppose for system (1), that there exists a continuously differentiable function $V(.,.) : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$ which satisfies the inequalities

$$\alpha_1(\|x\|) \leq V(t,x) \leq \alpha_2(\|x\|), \tag{4}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -\alpha_3(\|x\|) + \varrho(t), \tag{5}$$

$$\left\| \frac{\partial V}{\partial x} \right\| \leq \alpha_4(\|x\|) \tag{6}$$

in $[0,\infty) \times \mathbb{R}^n$, where $\alpha_i(.),\ i = 1,2,3,4$, are class $\mathcal{K}_\infty$ functions and $\varrho(.) \in C[\mathbb{R}^+, \mathbb{R}^+]$ such that

$$\sup_{t \geq t_0} \varrho(t) = \tilde{\varrho} < +\infty.$$

Theorem 3. *Suppose that the conditions* (3), (4)–(6) *hold and*

$$\big(\kappa\alpha_4 + \eta\alpha_4.\alpha_1^{-1} \circ \alpha_2\big) < l\alpha_3, \quad 0 < l < 1. \tag{7}$$

Then the ball

$$B_r = \{x \in \mathbb{R}^n / \|x\| \leq \alpha_1^{-1} \circ \alpha_2 \circ \alpha_0^{-1}(\tilde{\varrho})\}$$

is globally uniformly asymptotically stable respecting the system (2).

Proof. We use $V(.,.)$ as a Lyapunov function candidate for the perturbed system (2). The derivative of $V(t,y)$ along the trajectories of (2) gives

$$\dot{V}(t,y) \leq -\alpha_3(\|y\|) + \rho(t) + \left\| \frac{\partial V}{\partial y} \right\| \|g(t,y)\| \leq$$

$$\leq -\alpha_3(\|y\|) + \alpha_4(\|y\|)\big(\eta\|y\| + \kappa\big) + \varrho(t).$$

Thus,

$$\dot{V}(t,y) \leq -\alpha_3(\|y\|) + \big(\kappa\alpha_4 + \eta\alpha_4.\alpha_1^{-1} \circ \alpha_2\big)(\|y\|) + \tilde{\varrho}.$$

The last expression in conjunction with the fact that the functions $\alpha_i,\ i = 1,\ldots,4$, satisfy (7), yields

$$\dot{V}(t,y) \leq -(1-l)\alpha_3(\|y\|) + \tilde{\varrho}.$$

Let $\alpha_0(r) = (1-l)\alpha_3(r),\ 0 < l < 1$. One has $\alpha_0 \in \mathcal{K}_\infty$, and by using Theorem 1, we deduce that the ball B_r with

$$r = \alpha_1^{-1} \circ \alpha_2 \circ \alpha_0^{-1}(\tilde{\varrho})$$

is globally uniformly asymptotically stable.

***Remark* 1.** If l approaches to zero in the inequality (7), then the radius r of the above ball decreases.

For the exponential behavior of equation (2), we shall suppose for system (1) that there exists a continuously differentiable function $V : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$ which satisfies the inequalities

$$c_1\|x\|^2 \leq V(t,x) \leq c_2\|x\|^2, \tag{8}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -c_3 V(t,x), \tag{9}$$

$$\left\| \frac{\partial V}{\partial y} \right\| \leq \alpha_4(\|x\|) \tag{10}$$

for all $t \geq 0$ and $x \in \mathbb{R}^n$, where $c_1,\ c_2,\ c_3$ and c_4 are positive constants.

Theorem 4. *Suppose that the conditions* (3), (8) – (10) *hold and*

$$\eta < \frac{c_3 c_1}{c_4}.$$

Then the ball

$$B_r = \{x \in \mathbb{R}^n / \|x\| \leq \alpha_1^{-1} \circ \alpha_2 \circ \alpha_0^{-1}(\tilde{\varrho})\}$$

is globally uniformly exponentially stable respecting the system (2).

Proof. We use $V(.,.)$ as a Lyapunov function candidate for the perturbed system (2). The derivative of $V(t,y)$ along the trajectories of (2) gives

$$\dot{V}(t,y) \leq -c_3 V(t,y) + \left\| \frac{\partial V}{\partial y} \right\| \|g(t,y)\| \leq$$

$$\leq -c_3 V(t,y) + c_4(\|y\|)\big(\eta\|y\| + \kappa\big).$$

Thus,

$$\dot{V}(t,y) \leq -\left(c_3 - \frac{c_4\eta}{c_1}\right) V(t,y) + \frac{c_4\kappa}{\sqrt{c_1}}\sqrt{V(t,y)}.$$

By assumption, the term $c_3 - \dfrac{c_4\eta}{c_1} > 0$. Let

$$v(t) = \sqrt{V(t,y)}.$$

The derivative with respect time gives

$$\dot{v}(t) \leq -\frac{1}{2}\left(c_3 - \frac{c_4\eta}{c_1}\right) v(t) + \frac{1}{2}\frac{c_4\kappa}{\sqrt{c_1}}.$$

It implies that

$$v(t) \leq v(t_0) \exp -\frac{1}{2}\left(c_3 - \frac{c_4\eta}{c_1}\right)(t - t_0) + \frac{\dfrac{c_4\kappa}{\sqrt{c_1}}}{c_3 - \dfrac{c_4\eta}{c_1}}.$$

The last expression in conjunction with (8) yields

$$\|y(t)\| \leq \frac{\sqrt{c_2}}{\sqrt{c_1}}\|y(0)\| \exp -\frac{1}{2}\left(c_3 - \frac{c_4\eta}{c_1}\right)(t - t_0) + \frac{c_4\kappa}{c_1 c_3 - c_4\eta}.$$

Hence, the ball B_r with

$$r = \frac{c_4\kappa}{c_1 c_3 - c_4\eta}$$

is globally uniformly exponentially stable.

Next, we will prove a new nonlinear generalized Gronwall – Bellman-type integral inequality, and, as application, we give a new class of time-varying perturbed systems which is globally uniformly exponentially stable in the sense that the trajectories converge to a small ball centered at the origin. Moreover, we give an example to illustrate the applicability of the result.

3.2. Integral inequality approach. We consider now, the nonlinear perturbed differential equation

$$\dot{y} = A(t)y + g(t, y). \tag{11}$$

Here, the nominal system is supposed linear, where $A(.)$ is continuous bounded matrix defined on $\mathbb{R}^+$ and $g\colon \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous in (t, y) and locally Lipschitz with respect y uniformly in t.

The natural assumption is to consider some stability property for the unperturbed system with some information on the bound of the perturbed term where the nominal system is supposed to be globally uniformly asmptotically stable.

Suppose that, for

$$\dot{x} = A(t)x,$$

$x = 0$ is globally uniformly exponentially stable equilibrium point, this is equivalent to say that

$$\|\Phi(t, t_0)\| \leq k \exp(-\gamma(t - t_0)), \qquad t \geq t_0 \geq 0, \tag{12}$$

where $k > 0, \gamma > 0,$ $\Phi(t, t_0)$ is the state transition matrix of the matrix $A(t)$.

We assume that the perturbation term $g(.,.)$ satisfies the uniform bound

$$\|g(t, y)\| \leq \theta(t)\|y\| + \xi(t) \quad \forall t \geq t_0 \geq 0 \quad \forall y \in \mathbb{R}^n, \tag{13}$$

where $\theta(.),\ \xi(.) \in C[\mathbb{R}^+, \mathbb{R}^+]$.

The solution of the perturbed equation which starts at (t_0, y_0) is given by

$$y(t) = \Phi(t, t_0)y(t_0) + \int_{t_0}^{t} \Phi(t, s)g(s, y(s))ds, \qquad t \geq t_0 \geq 0.$$

By using (12), we obtain

$$\|y(t)\| \leq ke^{-\gamma(t-t_0)}\|y(t_0)\| + \int_{t_0}^{t} ke^{-\gamma(t-s)}\|g(s, y(s))\|ds.$$

Thus,

$$e^{\gamma t}\|y(t)\| \leq ke^{\gamma t_0}\|y(t_0)\| + \int_{t_0}^{t} ke^{\gamma s}\|g(s, y(s))\|ds.$$

By using inequality (13) imposed on the term of perturbation, we have

$$e^{\gamma t}\|y(t)\| \leq ke^{\gamma t_0}\|y(t_0)\| + \int_{t_0}^{t} ke^{\gamma s}\big(\theta(s)\|y(s)\| + \xi(s)\big)ds.$$

It follows that

$$e^{\gamma t}\|y(t)\| \leq ke^{\gamma t_0}\|y(t_0)\| + \int\limits_{t_0}^{t} k\theta(s)e^{\gamma s}\|y(s)\| + \big(ke^{\gamma s}\xi(s)\big)\big)ds, \qquad t \geq t_0 \geq 0. \tag{14}$$

To obtain an estimation on the solutions of the perturbed system (11), we need the following lemma which a generalization of Gronwall type integral inequality.

Lemma 1. *Let $u(.)$, $\vartheta(.)$ and $\zeta(.)$ be continuous functions on $[0,+\infty)$ such that $\vartheta(.) \geq 0$. We suppose that the inequality*

$$u(t) \leq c + \int\limits_{a}^{t} (\vartheta(s)u(s) + \zeta(s))ds \qquad \textit{for all} \quad t \geq a \tag{15}$$

holds, where a and c are arbitrary positive constants. Then

$$u(t) \leq ce^{\int_a^t \vartheta(\tau)d\tau} + \int\limits_{a}^{t} \zeta(s)e^{\int_s^t \vartheta(\tau)d\tau}ds \qquad \textit{for all} \quad t \geq a.$$

Proof. For $t \geq a$, we put

$$F(t) = \int\limits_{a}^{t} (\vartheta(s)u(s) + \zeta(s))ds,$$

then F is differentiable on $[a,+\infty)$ and verifies

$$F'(t) = \vartheta(t)u(t) + \zeta(t).$$

By using the inequality (15), we obtain

$$F'(t) \geq c\vartheta(t) + \vartheta(t)F(t) + \zeta(t).$$

Let y the maximal solution of the linear equation

$$y'(t) = \vartheta(t)y(t) + c\vartheta(t) + \zeta(t)$$

with $y(a) = 0$. It is clear that

$$y(t) = ce^{\int_a^t \vartheta(s)ds} - c + \int\limits_{a}^{t} \zeta(s)e^{\int_s^t \vartheta(\tau)d\tau}.$$

By comparison, we deduce that

$$F(t) \leq y(t).$$

Finally, we obtain

$$u(t) \leq ce^{\int_a^t \vartheta(\tau)d\tau} + \int\limits_{a}^{t} \zeta(s)e^{\int_s^t \vartheta(\tau)d\tau}ds \qquad \text{for all} \quad t \geq a.$$

Lemma 2. *Let $\theta \in L^p(\mathbb{R}_+, \mathbb{R}_+)$, where $p \in [1, +\infty) \cup \{+\infty\}$. We denote by $\|\theta\|_p$ the p-norm of θ. Then, for all $t \geq t_0 \geq 0$,*

$$\int\limits_{t_0}^{t} \theta(u)du \leq \frac{\|\theta\|_p}{p} + \left[\left(1 - \frac{1}{p}\right)\|\theta\|_p\right](t - t_0).$$

Proof. We first consider the case $p \in (1, +\infty)$. By using Hölder inequality to the function θ, we obtain, for all $t \geq t_0$,

$$\int\limits_{t_0}^{t} \theta(\sigma)d\sigma \leq \left(\int\limits_{t_0}^{t} \theta^p(\sigma)d\sigma\right)^{\frac{1}{p}} \left(\int\limits_{t_0}^{t} d\sigma\right)^{\frac{p-1}{p}} \leq$$

$$\leq (t - t_0)^{\frac{p-1}{p}} \left(\int\limits_{t_0}^{+\infty} \theta^p(\sigma)d\sigma\right)^{\frac{1}{p}}.$$

We put

$$f(x) = \frac{1}{p} + \frac{p-1}{p}x - x^{\frac{p-1}{p}} \quad \forall x \geq 0,$$

then f is continuous on $[0, +\infty)$ and differentiable on $(0, +\infty)$ and verifying

$$f'(x) = \frac{p-1}{p}\left(1 - x^{-\frac{1}{p}}\right).$$

Hence, f is decreasing on $[0, 1]$ and increasing on $[1, +\infty)$. Since $f(1) = 0$, we conclude that f is positive on $[0, +\infty)$ which means that

$$x^{\frac{p-1}{p}} \leq \frac{1}{p} + \frac{p-1}{p}x \quad \forall x \geq 0.$$

Consequently, we have

$$(t - t_0)^{\frac{p-1}{p}} \leq \frac{1}{p} + \frac{p-1}{p}(t - t_0) \quad \forall t \geq t_0.$$

Then

$$\int\limits_{t_0}^{t} \theta(u)du \leq \frac{\|\theta\|_p}{p} + \left[\left(1 - \frac{1}{p}\right)\|\theta\|_p\right](t - t_0).$$

This inequality holds also for $p \in \{1, +\infty\}$.

Theorem 5. *Suppose that the conditions* (12), (13) *hold and $\xi \in L^\infty(\mathbb{R}_+, \mathbb{R}_+)$, $\theta \in L^p(\mathbb{R}_+, \mathbb{R}_+)$ with*

$$\|\theta\|_p < \left(1 + \frac{1}{p-1}\right)\frac{\gamma}{k}.$$

Then the ball

$$B_r = \left\{ x \in \mathbb{R}^n / \|x\| \leq \frac{k\|\xi\|_\infty e^{\frac{k\|\theta\|_p}{p}}}{\gamma - k\left(1 - \dfrac{1}{p}\right)\|\theta\|_p} \right\}$$

is globally uniformly exponentially stable respecting the system (11).

Proof. Since the conditions (12) and (13) are satisfied, then one gets an estimation as in (14),

$$e^{\gamma t}\|y(t)\| \leq ke^{\gamma t_0}\|y(t_0)\| + \int\limits_{t_0}^{t} (k\theta(s)e^{\gamma s}\|y(s)\| + ke^{\gamma s}\xi(s))ds.$$

Let

$$u(t) = e^{\gamma t}\|y(t)\|.$$

Then, by applying Lemma 1, we obtain

$$u(t) \leq ku(t_0)e^{k\int_{t_0}^{t}\theta(s)ds} + \int\limits_{t_0}^{t} ke^{\gamma s}\xi(s)e^{k\int_s^t \theta(\tau)d\tau}ds.$$

Since

$$\|y(t)\| = e^{-\gamma t}u(t),$$

we obtain the estimation

$$\|y(t)\| \leq k\|y(t_0)\|e^{\int_{t_0}^{t} k\theta(s)ds - \gamma(t-t_0)} + \int\limits_{t_0}^{t} k\xi(s)e^{\int_s^t k\theta(\tau)d\tau - \gamma(t-s)}ds.$$

Then

$$\|y(t)\| \leq k\|y(t_0)\|e^{k\int_{t_0}^{t}\theta(s)ds - \gamma(t-t_0)} + k\|\xi(t)\|_\infty \int\limits_{t_0}^{t} e^{k\int_s^t \theta(\tau)d\tau + \gamma(s-t)}ds.$$

Now, by using Lemma 2, we get

$$\int\limits_{t_0}^{t} \theta(u)du \leq \frac{\|\theta\|_p}{p} + \left[\left(1 - \frac{1}{p}\right)\|\theta\|_p\right](t - t_0).$$

Consequently,

$$\|y(t)\| \leq k\|y(t_0)\|e^{k\frac{\|\theta\|_p}{p}}e^{-\left(\gamma - k\left(1-\frac{1}{p}\right)\|\theta\|_p\right)(t-t_0)} + k\|\xi\|_\infty e^{k\frac{\|\theta\|_p}{p}} \int\limits_{t_0}^{t} e^{\left(\gamma - k\left(1-\frac{1}{p}\right)\|\theta\|_p\right)(s-t)}ds.$$

Using the fact

$$\|\theta\|_p < \left(1 + \frac{1}{p-1}\right)\frac{\gamma}{k},$$

which implies $\gamma - k\left(1 - \dfrac{1}{p}\right)\|\theta\|_p > 0$. Finally, we obtain the estimation

$$\|y(t)\| \leq ke^{k\frac{\|\theta\|_p}{p}}\|y(t_0)\|e^{-\left(\gamma - k\left(1-\frac{1}{p}\right)\|\theta\|_p\right)(t-t_0)} + \frac{k\|\xi\|_\infty e^{k\frac{\|\theta\|_p}{p}}}{\gamma - k\left(1 - \dfrac{1}{p}\right)\|\theta\|_p}.$$

Hence, the ball B_r is globally uniformly exponentially stable with

$$r = \frac{k\|\xi\|_\infty e^{\frac{k\|\theta\|_p}{p}}}{\gamma - k\left(1 - \dfrac{1}{p}\right)\|\theta\|_p}.$$

3.3. Example. Consider the system

$$\begin{aligned} \dot{x}_1 &= -x_1 - tx_2 + \frac{1}{(1+t^2)^2}\frac{x_1^2}{1+\sqrt{x_1^2+x_2^2}} + \frac{e^{-t}}{1+t^2)(1+x_1^2)}, \\ \dot{x}_2 &= tx_1 - x_2 + \frac{t}{(1+t^2)^2}\frac{x_2^2}{1+\sqrt{x_1^2+x_2^2}} \end{aligned} \tag{16}$$

which can be writing as

$$\dot{x} = A(t)x + h(t,x),$$

where

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

$$A(t) = \begin{pmatrix} -1 - t \\ t - 1 \end{pmatrix}$$

and

$$h(t,x) = \begin{pmatrix} h_1(t,x) \\ h_2(t,x) \end{pmatrix}$$

with

$$\begin{aligned} h_1(t,x) &= \frac{1}{(1+t^2)^2}\frac{x_1^2}{1+\sqrt{x_1^2+x_2^2}} + \frac{e^{-t}}{(1+t^2)(1+x_1^2)}, \\ h_2(t,x) &= \frac{t}{(1+t^2)^2}\frac{x_2^2}{1+\sqrt{x_1^2+x_2^2}}. \end{aligned}$$

It is clear that the system

$$\dot{x} = A(t)x$$

is globally uniformly asymptotically stable. Indeed, the transition matrix $R(t,t_0)$ satisfies

$$R(t,t_0) = e^{-(t-t_0)}\begin{pmatrix} \cos t - \sin t \\ \sin t \cos t \end{pmatrix},$$

thus, we obtain

$$\|R(t,t_0)\| = ke^{-\gamma(t-t_0)}$$

with $\gamma = k = 1$.

On the other hand,

$$\|h(t,x)\|^2 = h_1^2(t,x) + h_2^2(t,x) \leq$$

$$\leq \frac{1}{(1+t^2)^2}(x_1^2 + x_2^2) + \frac{2e^{-t}}{(1+t^2)^2}.$$

By using the classic inequality

$$\sqrt{a^2+b^2} \leq a + b \quad \forall a, b \geq 0,$$

we get

$$\|h(t,x)\| \leq \phi(t)\|x(t)\| + \varepsilon(t) \quad \forall t \geq 0,$$

where

$$\phi(t) = \frac{1}{1+t^2}$$

and

$$\varepsilon(t) = \frac{\sqrt{2}e^{-\frac{t}{2}}}{1+t^2}.$$

It is easy to verify that ϕ and ε are continuous, positive and bounded on $[0, +\infty)$. In particular,

$$\|\varepsilon\|_\infty = \sqrt{2}, \quad \phi \in L^p(\mathbb{R}_+, \mathbb{R}_+)$$

and

$$\|\phi\|_p = \frac{(2(p-1))!}{2^{2(p-1)}((p-1)!)^2}\frac{\pi}{2} \quad \forall p \in [1, +\infty).$$

Then

$$\|\phi\|_p < 1 + \frac{1}{p-1} \quad \forall p \geq 1,$$

and we can apply Theorem 5 to prove that the ball B_r, where

$$r = \frac{\sqrt{2}e^{\frac{\frac{(2(p-1))!}{2^{2(p-1)}((p-1)!)^2}\frac{\pi}{2}}{p}}}{1 - \left(1 - \frac{1}{p}\right)\frac{(2(p-1))!}{2^{2(p-1)}((p-1)!)^2}\frac{\pi}{2}},$$

is globally uniformly exponentially stable respecting the system (16).

***Remark* 2.** The above example show that the trajectories of the system converge exponentially to a small ball centered at the origin under some sufficient conditions on the perturbed term. This fact motivated to study systems whose desired behavior is asymptotic stability about the origin of the state space or a close approximation to this, e.g., all state trajectories are bounded and approach a sufficiently small neighborhood of the origin. Quite often, one also desires that the state approaches the origin (or some sufficiently small neighborhood of it) in a sufficiently fast manner. So, one can introduce a small parameter $\epsilon > 0$ on the perturbations so that when $\epsilon \to 0$ the solutions of the system tend to zero when t goes to infinity.

4. Conclusion. Sufficient conditions for global uniform exponential stability of perturbed systems are obtained using Lyapunov techniques or integral inequalities approach. It is shown that the trajectories converge to a small ball centered at the origin where a new nonlinear generalized Gronwall – Bellman-type integral inequality is proved. As an application, we give a new class of time-varying perturbed systems which is globally uniformly exponentially stable. Moreover, an example is given to illustrate the applicability of the main result.

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