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AN EIGENVALUE OF ANISOTROPIC DISCRETE PROBLEM WITH THREE VARIABLE EXPONENTS

ВЛАСНЕ ЗНАЧЕННЯ АНІЗОТРОПНОЇ ДИСКРЕТНОЇ ЗАДАЧІ З ТРЬОМА ЗМІННИМИ ЕКСПОНЕНТАМИ

We study the existence of a continuous spectrum of an anisotropic discrete problem, involving variable exponent. The proposed technical approach is based on the variational methods and critical point theory.

Вивчається проблема існування неперервного спектра анізотропної дискретної задачі із змінною експонентою. Запропонований підхід базується на варіаційних методах та теорії критичних точок.

1. Introduction. Let $T > 2$ be a positive integer and $[1,T]_{\mathbb{Z}} = \{1,2,3,\ldots,T\}$. We consider the discrete anisotropic problem

$$-\Delta\left(|\Delta u(k-1)|^{p(k-1)-2}\Delta u(k-1)\right) + |u(k)|^{p(k)-2}u(k) + |u(k)|^{q(k)-2}u(k) =$$
$$= \lambda|u(k)|^{r(k)-2}u(k) \quad \text{for} \quad k \in [1,T]_{\mathbb{Z}}, \tag{1}$$
$$u(0) = u(T+1) = 0,$$

where Δ denotes the forward difference operator defined by $\Delta u(k) = u(k+1) - u(k)$, $\lambda > 0$ is a real parameter, $p\colon [0,T]_Z \to [2,+\infty)$ and $q,r\colon [1,T]_Z \to [2,+\infty)$ are given functions.

In the last years, the study of boundary-value problems for finite difference equations has captured special attention. This type of problems have an important role in different domains of research, such as control systems, economics, computer science, physics, artificial or biological neural networks, cybernetics, ecology and many others. For example, view the recent results in the references [1 – 5, 17, 18]. The important tools employed to study this kind of problem are critical point theory and variational methods.

However, there is an increasing interest to the existence results to boundary-value problems for difference equations with $p(k)$-Laplacian operator, because of their applications in many fields. To the best of our knowledge, discrete problems involving anisotropic exponents have been discussed for the first time in [13, 16, 20], the authors proved the existence of a continuous spectrum of eigenvalues for the problem

$$-\Delta\left(|\Delta u(k-1)|^{p(k-1)-2}\Delta u(k-1)\right) = \lambda|u(k)|^{q(k)-2}u(k) \quad \text{for} \quad k \in [1,T]_{\mathbb{Z}}, \tag{2}$$
$$u(0) = u(T+1) = 0,$$

In [8 – 11, 19], the authors have studied the existence of at least one solution, multiplicity of solutions and a sequences of solutions for the problem

$$-\Delta\left(|\Delta u(k-1)|^{p(k-1)-2}\Delta u(k-1)\right)=\lambda f(k,u(k)) \quad \text{for} \quad k\in[1,T]_{\mathbb{Z}},$$

$$u(0)=u(T+1)=0,$$

where $f\colon [1,T]_{\mathbb{Z}}\times\mathbb{R}\longmapsto\mathbb{R}$ is a continuous function.

More recently, in [6, 7, 12, 14, 15, 21] the authors have been investigated the existence and multiplicity of solutions for nonlinear discrete boundary-value problems involving $p(.)$-Laplacian operator using variational methods.

Our analysis mainly concern the existence and the nonexistence of a weak solutions to problem (1) more general than (2), with three variable exponents under appropriate assumptions (4) below, between the functions exponents $p(k), q(k)$ and $r(k)$. Our aim is to determine the concrets intervals for the parameter λ for which problem (1) has, or not has, a nontrivial solutions. More precisely, we prove the existence of two positive constants λ_* and λ^* with $\lambda_*\le\lambda^*$ such that for each $\lambda\in[\lambda^*,+\infty)$ the problem (1) has at least one nontrivial solution, while for any $\lambda\in(0,\lambda_*)$ problem (1) has no nontrivial solution. For these results, we use some known tools such as the direct variational methods and the critical point theory.

This paper is organized as follows. The second section is devoted to mathematical preliminaries and statement of main results. In the third section we give the mains results and thier proofs.

2. Framework and preliminary results. Solutions to boundary-value problem (1) will be investigated in the space

$$E=\{u\colon [0,T+1]_{\mathbb{Z}}\to\mathbb{R}, \quad u(0)=u(T+1)=0\},$$

which is a T-dimensional Hilbert space [1], with the inner product

$$(u,v)=\sum_{k=0}^{T}\Delta u(k)\Delta v(k) \qquad \forall u,v\in E.$$

The associated norm is defined by

$$\|u\|=\left(\sum_{k=0}^{T}|\Delta u(k)|^2\right)^{\frac{1}{2}}.$$

Moreover, it is useful to introduce other norm on E:

$$|u|_m=\left(\sum_{k=1}^{T}|u(k)|^m\right)^{\frac{1}{m}} \qquad \text{for} \quad m\ge 2. \tag{3}$$

For any function $h\colon [0,T]_Z\to[2,+\infty)$, we use the following notations:

$$h^-=\min_{k\in[0,T]_{\mathbb{Z}}}h(k) \qquad \text{and} \qquad h^+=\max_{k\in[0,T]_{\mathbb{Z}}}h(k).$$

In this paper, we study the boundary-value problem (1) assuming that the functions p, q and r satisfy the following assumptions:

$$2\le p^-\le p^+<r^-\le r^+<q^-\le q^+. \tag{4}$$

We start with the following auxillary result, will be are used later.

Lemma 2.1 [20]. (a) *For any* $m \geq 2$ *there exists a positive constant* C_m *such that*

$$\sum_{k=1}^{T} |u(k)|^m \leq C_m \sum_{k=1}^{T+1} |\Delta u(k-1)|^m \quad \forall u \in E.$$

(b) *There exist two positive constants* C_1 *and* C_2 *such that*

$$\sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)} \geq C_1 \|u\|^{p^-} - C_2 \quad \forall u \in E \quad \textit{with} \quad \|u\| > 1.$$

(c) *There exists a positive constant* C_3 *such that*

$$\sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)} \geq C_3 \|u\|^{p^+} \quad \forall u \in E \quad \textit{with} \quad \|u\| < 1.$$

(d) $\sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)} \leq (T+1)\left(\|u\|^{p^+} + 1\right) \quad \forall u \in E.$

Definition 2.1. *We say that* $\lambda > 0$ *is an eigenvalue of problem* (1) *if there exists* $u \in E$ *such that* $u \neq 0$ *and*

$$\sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)-2} \Delta u(k-1) \Delta v(k-1) + \sum_{k=1}^{T} |u(k)|^{p(k)-2} u(k) v(k) +$$

$$+ \sum_{k=1}^{T} |u(k)|^{q(k)-2} u(k) v(k) = \lambda \sum_{k=1}^{T} |u(k)|^{r(k)-2} u(k) v(k)$$

for any $v \in E$.

If $\lambda > 0$ is an eigenvalue of problem (1), then the corresponding eigenfunction $u_\lambda \in E$ is a weak solution for the problem (1).

To study the boundary-value problem (1), we define the following functionals, for $u \in E$:

$$\varphi_0(u) = \sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)} + \sum_{k=1}^{T} |u(k)|^{p(k)} + \sum_{k=1}^{T} |u(k)|^{q(k)}, \tag{5}$$

$$\psi_0(u) = \sum_{k=1}^{T} |u(k)|^{r(k)}, \tag{6}$$

$$\varphi_1(u) = \sum_{k=1}^{T+1} \frac{|\Delta u(k-1)|^{p(k-1)}}{p(k-1)} + \sum_{k=1}^{T} \frac{|u(k)|^{p(k)}}{p(k)} + \sum_{k=1}^{T} \frac{|u(k)|^{q(k)}}{q(k)}, \tag{7}$$

$$\psi_1(u) = \sum_{k=1}^{T} \frac{|u(k)|^{r(k)}}{r(k)}, \tag{8}$$

and, for any $\lambda > 0$ and $u \in E$, we define the functional I_λ as follows:

$$I_\lambda(u) = \varphi_1(u) - \lambda\psi_1(u). \tag{9}$$

With any fixed $\lambda > 0$ the functionals I_λ is differentiable [11, 20], and its derivatives at u reads

$$\left(I'_\lambda(u), v\right) = \left(\varphi'_1(u), v\right) - \lambda\left(\psi'_1(u), v\right), \tag{10}$$

for any $v \in E$, where

$$\left(\varphi'_1(u), v\right) = \sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)-2}\Delta u(k-1)\Delta v(k-1) +$$

$$+ \sum_{k=1}^{T}\left(|u(k)|^{p(k)-2} + |u(k)|^{q(k)-2}\right)u(k)v(k) \tag{11}$$

and

$$\left(\psi'_1(u), v\right) = \sum_{k=1}^{T} |u(k)|^{r(k)-2}u(k)v(k). \tag{12}$$

***Remark* 2.1.** According to equalities (10)–(12) and the Definition 2.1, it follows that λ is an eigenvalue of problem (1) if and only if there exists $u_\lambda \in E$ such that $u_\lambda \neq 0$ is a critical point of the funtional I_λ.

3. Main results and thier proof. In this paper, we study the boundary-value problem (1) assuming that the functions p, q and r satisfy the hypothesis given in (4).

Theorem 3.1. *Assume that the hypothesis* (4) *holds, then there exists a positive constant* $\lambda_\star$ *such that any* $\lambda \in (0, \lambda_\star)$ *is not an eigenvalue of the problem* (1).

Proof. Put

$$\lambda_\star = \inf_{u \in E - \{0\}} \frac{\varphi_0(u)}{\psi_0(u)}, \tag{13}$$

where φ_0 and ψ_0 are given by (5) and (6).

Firstly, we show that $\lambda_\star > 0$. From (4) we infer that, for all $k \in [1, T]_{\mathbb{Z}}$,

$$p(k) < r(k) < q(k),$$

then, for any $u \in E$ and $k \in [1, T]_{\mathbb{Z}}$, we have

$$|u(k)|^{r(k)} \leq |u(k)|^{p(k)} + |u(k)|^{q(k)}. \tag{14}$$

Then

$$\sum_{k=1}^{T}\left(|u(k)|^{p(k)} + |u(k)|^{q(k)}\right) \geq \sum_{k=1}^{T} |u(k)|^{r(k)},$$

and we deduce that

$$\varphi_0(u) \geq \psi_0(u) \quad \forall u \in E.$$

Therefore,

$$\lambda_\star \geq 1 > 0.$$

Secondly, we show that any $\lambda \in (0, \lambda_\star)$ is not an eigenvalue of the boundary-value problem (1). To do this, assuming by contradiction that there is $\lambda \in (0, \lambda_\star)$ an eigenvalue of problem (1), then by Remark 2.1, we deduce that there exists $u_\lambda \in E$ such that $u_\lambda \neq 0$ and $I'_\lambda(u_\lambda) = 0$. So,

$$\left(\varphi'_1(u_\lambda), v\right) = \lambda \left(\psi'_1(u_\lambda), v\right) \quad \forall v \in E.$$

In particular, for $v = u_\lambda$, we get

$$\varphi_0(u_\lambda) = \lambda \psi_0(u_\lambda).$$

Since $u_\lambda \neq 0$, it follows that $\varphi_0(u_\lambda) > 0$ and $\psi_0(u_\lambda) > 0$. Then from (13) and the fact that $\lambda < \lambda_\star$, we deduce that

$$\varphi_0(u_\lambda) \geq \lambda_\star \psi_0(u_\lambda) > \lambda \psi_0(u_\lambda) = \varphi_0(u_\lambda).$$

This inequality is absurd, then the proof is completed.

Theorem 3.2. *Assume that the hypothesis* (4) *holds*, *then there exists a positive constant* $\lambda^\star$ *such that* $\lambda_\star \leq \lambda^\star$ *and each* $\lambda \in [\lambda^\star, +\infty)$ *is an eigenvalue of the problem* (1).

We need to prove the following lemmas which will be used to show the Theorem 3.2.

Lemma 3.1. *If the condition* (4) *is true*, *then*

$$\lim_{\|u\| \to 0} \frac{\varphi_0(u)}{\psi_0(u)} = +\infty.$$

Proof. For any $k \in [1, T]_{\mathbb{Z}}$, we have $r^- \leq r(k) \leq r^+$. Then, for any $u \in E$, we get

$$|u(k)|^{r(k)} \leq |u(k)|^{r^-} + |u(k)|^{r^+}.$$

Summing for k from 1 to T, we obtain, for any $u \in E$,

$$\psi_0(u) \leq \left(\sum_{k=1}^{T} |u(k)|^{r^-} + \sum_{k=1}^{T} |u(k)|^{r^+}\right).$$

By using Lemma 2.1(a), we infer that

$$\psi_0(u) \leq \left(C_{r^-} \sum_{k=1}^{T+1} |\Delta u(k-1)|^{r^-} + C_{r^+} \sum_{k=1}^{T+1} |\Delta u(k-1)|^{r^+}\right).$$

Again by Lemma 2.1(d), we deduce that

$$\psi_0(u) \leq (1+T)\left(C_{r^-}(1 + \|u\|^{r^-}) + C_{r^+}(1 + \|u\|^{r^+})\right). \tag{15}$$

Next, for any $u \in E$, with $\|u\| < 1$, from (5) and Lemma 2.1(c), we have

$$\varphi_0(u) \geq C_3 \|u\|^{p^+}. \tag{16}$$

Then, for any $u \in E$ with $\|u\| < 1$, small enough, from the inequalities (15) and (16), we get

$$\frac{\varphi_0(u)}{\psi_0(u)} \geq \frac{C_3}{(1+T)} \frac{\|u\|^{p^+}}{C_{r^-}(1+\|u\|^{r^-}) + C_{r^+}(1+\|u\|^{r^+})}.$$

Since $r^+ \geq r^- > p^+$, passing to the limit as $\|u\| \to 0$, in the above inequality we prove that $\lim_{\|u\|\to 0} \dfrac{\varphi_0(u)}{\psi_0(u)} = +\infty$.

Lemma 3.1 is proved.

Lemma 3.2. *If the condition* (4) *is true, then, for any* $\lambda > 0$, I_λ *is coercive, i.e.,*

$$\lim_{\|u\|\to\infty} (\varphi_1(u) - \lambda\psi_1(u)) = +\infty.$$

Proof. For any $u \in E$, from (7) we have

$$\varphi_1(u) = \sum_{k=1}^{T+1} \frac{|\Delta u(k-1)|^{p(k-1)}}{p(k-1)} + \sum_{k=1}^{T} \frac{|u(k)|^{p(k)}}{2p(k)} + \sum_{k=1}^{T} \left(\frac{|u(k)|^{p(k)}}{2p(k)} + \frac{|u(k)|^{q(k)}}{q(k)} \right) \geq$$

$$\geq \frac{1}{p^+} \sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)} + \frac{1}{\max(2p^+, q^+)} \sum_{k=1}^{T} \left(|u(k)|^{p(k)} + |u(k)|^{q(k)} \right). \tag{17}$$

Let s fix such that $r^+ < s < q^-$, then, for any $u \in E$ and $k \in [1,T]_{\mathbb{Z}}$, we get

$$|u(k)|^{p(k)} + |u(k)|^{q(k)} \geq |u(k)|^s,$$

and, by (17), we obtain

$$\varphi_1(u) \geq \frac{1}{p^+} \sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)} + \frac{1}{\max(2p^+, q^+)} |u|_s^s. \tag{18}$$

Next, since $|u(k)|^{r(k)} \leq (|u(k)|^{r^-} + |u(k)|^{r^+})$, then, from (8), we have

$$\psi_1(u) \leq \frac{1}{r^-} \left(\sum_{k=1}^{T} |u(k)|^{r^-} + \sum_{k=1}^{T} |u(k)|^{r^+} \right). \tag{19}$$

By using Hölder's inequality, we prove that, for any $u \in E$,

$$\sum_{k=1}^{T} |u(k)|^{r^-} \leq T^{\frac{s-r^-}{s}} \left(\sum_{k=1}^{T} \left(|u(k)|^{r^-} \right)^{\frac{s}{r^-}} \right)^{\frac{r^-}{s}} = A|u|_s^{r^-} \tag{20}$$

and

$$\sum_{k=1}^{T} |u(k)|^{r^+} \leq T^{\frac{s-r^+}{s}} \left(\sum_{k=1}^{T} \left(|u(k)|^{r^+} \right)^{\frac{s}{r^+}} \right)^{\frac{r^+}{s}} = B|u|_s^{r^+}, \tag{21}$$

where

$$A = T^{\frac{s-r^-}{s}} > 0 \quad \text{and} \quad B = T^{\frac{s-r^+}{s}} > 0. \tag{22}$$

Therefore, for any $u \in E$ with $\|u\| > 1,$ from (11), inequalities (18)–(21) and Lemma 2.1(b), we deduce that, for any $\lambda > 0,$

$$I_\lambda(u) \geq \frac{1}{p^+}\left(C_1\|u\|^{p^-} - C_2\right) + \frac{1}{\max(2p^+, q^+)}|u|_s^s - \lambda\frac{1}{r^-}\left(A|u|_s^{r^-} + B|u|_s^{r^+}\right) \geq$$

$$\geq \frac{C_1\|u\|^{p^-} - C_2}{p^+} + \frac{|u|_s^s}{2\max(2p^+, q^+)} - \lambda\frac{A|u|_s^{r^-}}{r^-} +$$

$$+ \frac{|u|_s^s}{2\max(2p^+, q^+)} - \lambda\frac{B|u|_s^{r^+}}{r^-},$$

so,

$$I_\lambda(u) \geq \frac{C_1\|u\|^{p^-} - C_2}{p^+} - \left(\alpha|u|_s^{r^-} - \beta|u|_s^s\right) - (\gamma|u|_s^{r^+} - \beta|u|_s^s), \tag{23}$$

where $\alpha = \dfrac{A\lambda}{r^-} > 0,$ $\gamma = \dfrac{B\lambda}{r^-} > 0$ and $\beta = \dfrac{1}{2\max(2p^+, q^+)} > 0.$

Let $h_1,\ h_2 \colon]0, +\infty[\to \mathbb{R}$ two real functions, given by

$$h_1(t) = \alpha t^{r^-} - \beta t^s \qquad \text{and} \qquad h_2(t) = \gamma t^{r^+} - \beta t^s \qquad \forall t > 0.$$

It is easy to show that h_1 and h_2 achieves its positive global maximums $M_1 = h_1(t_1)$ and $M_2 = h_2(t_2),$ where

$$t_1 = \left(\frac{\alpha r^-}{\beta s}\right)^{\frac{1}{s-r^-}} > 0 \quad \text{and} \quad t_2 = \left(\frac{\gamma r^+}{\beta s}\right)^{\frac{1}{s-r^+}} > 0.$$

Then we infer that $h_1(t) \leq M_1$ and $h_2(t) \leq M_2$ $\forall t > 0.$

Therefore, for any $u \in E$ with $\|u\| > 1$ and $\lambda > 0,$ from (23), we get that

$$I_\lambda(u) \geq \frac{C_1\|u\|^{p^-} - C_2}{p^+} - M_1 - M_2. \tag{24}$$

Passing to the limit as $\|u\| \to \infty$ in (24), we complete the proof of Lemma 3.2.

***Proof of Theorem* 3.2.** Put

$$\lambda^\star = \inf_{u \in E - \{0\}} \frac{\varphi_1(u)}{\psi_1(u)}. \tag{25}$$

Step 1. We show that $\lambda^\star > 0.$

By (14) and from (4), we infer that, for any $u \in E,$

$$\frac{|u(k)|^{p(k)}}{p(k)} + \frac{|u(k)|^{q(k)}}{q(k)} \geq \frac{|u(k)|^{r(k)}}{q(k)} \geq \frac{|u(k)|^{r(k)}}{q^+}.$$

Then

$$\sum_{k=1}^{T}\frac{|u(k)|^{p(k)}}{p(k)} + \sum_{k=1}^{T}\frac{|u(k)|^{q(k)}}{q(k)} \geq \frac{r^-}{q^+}\sum_{k=1}^{T}\frac{|u(k)|^{r(k)}}{r(k)}$$

and

$$\varphi_1(u) \geq \frac{r^-}{q^+}\psi_1(u) \quad \forall u \in E.$$

So,

$$\lambda^\star \geq \frac{r^-}{q^+} > 0.$$

Thus, step 1 is verified.

Step 2. We show that each $\lambda \in (\lambda^\star, +\infty)$ is an eigenvalue of the problem (1).

We fix $\lambda \in (\lambda^\star, +\infty)$. According to Lemma 3.2, we have I_λ is coercive and is weakly lower semicontinuous. Applying Theorem 1.2 in [22] in order to prove that there exists $u_\lambda \in E$ as a global minimum point of I_λ and, thus, as a critical point of I_λ.

In order to finish the proof of step 2, it is enough to prove that u_λ is nontrivial. Indeed, since $\lambda > \lambda^\star$ and from (13) there exists $v_\lambda \in E$ such that

$$\varphi_1(v_\lambda) < \lambda\psi_1(v_\lambda),$$

that is,

$$I_\lambda(v_\lambda) < 0,$$

Then $u_\lambda \neq 0_E$, and we conclude that there exists $u_\lambda \in E$ with $u_\lambda \neq 0_E$, which is a critical point of I_λ or λ is an eigenvalue of the problem (1). Thus, step 2 is true.

Step 3. We show that $\lambda^\star$ is an eigenvalue of problem (1). For this we will prove that there exists $u^\star \in E$ such that $u^\star \neq 0$ and $I'_{\lambda^\star}(u^\star) = 0$.

Let $\lambda_n > 0$ be a minimizing sequence for $\lambda^\star$ (i.e., $\lambda_{\mathrm{n}} > \lambda^\star$). From step 2, we deduce that for each n there exists a sequence $\{u_n\} \in E$ such that $u_n \neq 0$ and $I'_{\lambda_n}(u_n) = 0$. So,

$$\left(\varphi_1'(u_n), v\right) = \lambda_n \left(\psi_1'(u_n), v\right) \quad \forall v \in E. \tag{26}$$

For $v = u_n$, we find that

$$\varphi_0(u_n) - \lambda_n\psi_0(u_n) = 0, \tag{27}$$

and passing to the limit as $n \to +\infty$ in relation (27), we have

$$\lim_{n\to+\infty} (\varphi_0(u_n) - \lambda_n\psi_0(u_n)) = 0. \tag{28}$$

On the other hand, a similar argument as those used in proof of Lemma 3.2, we show that

$$\lim_{\|u_n\|\to+\infty} (\varphi_0(u_n) - \lambda_n\psi_0(u_n)) = +\infty. \tag{29}$$

Then, from (28) and (29) we show that the sequence $\{u_n\}$ is bounded in E. Since E is a finite dimensional Hilbert space, then there exists a subsequence, still denoted by $\{u_n\}$ and $u^\star \in E$, such that $u_n \to u^\star$ as $n \to +\infty$.

Therefore, passing to the limit as $n \to +\infty$ in relation (26), we get that

$$\left(\varphi_1'(u^\star), v\right) = \lambda^\star \left(\psi_1'(u^\star), v\right) \qquad \forall v \in E$$

or

$$\left(I'_{\lambda^\star}(u^\star), v\right) = 0 \quad \forall v \in E.$$

So, $u^\star$ is a critical point of $I_{\lambda^\star}$.

It remains to show that $u^\star$ is nontrivial. In fact, if not we have $u_n \to 0$ in E as $n \to +\infty$ or $\|u_n\| \to 0$, then Lemma 3.1 implies that

$$\lim_{n\to+\infty}\left(\frac{\varphi_0(u_n)}{\psi_0(u_n)}\right) = +\infty.$$

From the equality (27), we deduce that

$$\lim_{n\to+\infty}\left(\frac{\varphi_0(u_n)}{\psi_0(u_n)}\right) = \lambda^\star,$$

which is a contradiction. Consequently, $u^\star \neq 0$ and, thus, $\lambda^\star$ is an eigenvalue of the problem (1).

Step 4. We prove that $\lambda_\star \leq \lambda^\star$. Since $\lambda^\star$ is an eigenvalue of the problem (1), so Theorem 3.1 implies that

$$\lambda^\star \quad \notin \quad]0; \lambda_\star[.$$

Since $0 < \lambda^\star$, therefore, $\lambda_\star \leq \lambda^\star$.

Theorem 3.2 is proved.

***Remark* 3.1.** We are not able deduce whether $\lambda_\star = \lambda^\star$ or $\lambda_\star < \lambda^\star$. In the latter case, an intersting open problem consern the existence of eigenvalue of problem (1) in the interval $[\lambda_\star < \lambda^\star)$.

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