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INTEGRAL OPERATORS PRESERVING SUBORDINATION AND SUPERORDINATION FOR MULTIVALENT FUNCTIONS

ІНТЕГРАЛЬНІ ОПЕРАТОРИ, ЯКІ ЗБЕРІГАЮТЬ СУБОРДИНАЦІЮ ТА СУПЕРОРДИНАЦІЮ ДЛЯ БАГАТОЗНАЧНИХ ФУНКЦІЙ

We obtain subordination, superordination and sandwich-preserving new theorems for certain integral operators defined on multivalent functions. The sandwich-type theorem for these integral operators is also derived and our results extend some earlier ones. Combining these new theorems with some previous related results, we give interesting subordination and superordination consequences for a wide class of analytic integral operators.

Отримано нові теореми щодо субординації, суперординації та збереження порядку для деяких інтегральних операторів на багатозначних функціях. Також доведено теореми типу стискання для інтегральних операторів, які узагальнюють деякі відомі результати. Комбінуючи ці нові теореми з деякими відомими відповідними результатами, ми отримуємо цікаві наслідки щодо субординації та суперординації для широкого класу аналітичних інтегральних операторів.

1. Introduction. Let $H(\mathrm{U})$ be the class of functions analytic in the open unit disk $\mathrm{U} = \{z \in \mathbb{C} : |z| < 1\}$, and let denote by $H[a, n]$ the subclass of $H(\mathrm{U})$ consisting of functions of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots, \quad a \in \mathbb{C}, \quad n \in \mathbb{N} = \{1, 2, \ldots\}.$$

Also, let $\mathcal{A}(p)$ denote the subclass of $H(\mathrm{U})$ of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \quad z \in \mathrm{U}, \quad p \in \mathbb{N},$$

and denote $\mathcal{A} := \mathcal{A}(1)$.

If f and F are members of $H(\mathrm{U})$, then the function f is said to be *subordinate* to F, or F is said to be *superordinate* to f, if there exists a function $w \in H(\mathrm{U})$ with $w(0) = 0$ and $|w(z)| < 1$ for $z \in \mathrm{U}$, such that $f(z) = F(w(z))$ for all $z \in \mathrm{U}$, and in such a case we write $f(z) \prec F(z)$.

If F is univalent, then $f(z) \prec F(z)$ if and only if $f(0) = F(0)$ and $f(\mathrm{U}) \subset F(\mathrm{U})$ (see [12, 13]).

Let $\Psi : \mathbb{C}^2 \times \mathrm{U} \to \mathbb{C}$ and h be univalent in U. If $p \in H(\mathrm{U})$ and satisfies the first order differential subordination

$$\Psi\big(p(z), zp'(z); z\big) \prec h(z), \tag{1.1}$$

then p is a *solution of the differential subordination* (1.1). The univalent function q is called a *dominant* of the solutions of the differential subordination (1.1) if $p(z) \prec q(z)$ for all p satisfying (1.1). A univalent dominant $\tilde{q}$ that satisfies $\tilde{q}(z) \prec q(z)$ for all the dominants of (1.1) is called the *best dominant*.

Similarly, if $\Phi\colon \mathbb{C}^2 \times \mathrm{U} \to \mathbb{C}$ such that p and $\Phi\big(p(z), zp'(z); z\big)$ are univalent in U and if p satisfies first order differential superordination

$$h(z) \prec \Phi\big(p(z), zp'(z); z\big), \tag{1.2}$$

then p is called a *solution of the differential superordination* (1.2). An analytic function q is called a *subordinant* of the solutions of the differential superordination (1.2) if $q(z) \prec p(z)$ for all p satisfying (1.2). A univalent subordinant $\tilde{q}$ that satisfies $q(z) \prec \tilde{q}(z)$ for all the subordinants of (1.2) is called the *best subordinant* (see [12, 13]).

For the parameters $\alpha, \beta, \gamma \in \mathbb{C}$ with $\beta \neq 0$ and $p \in \mathbb{N}$, we introduce the integral operators $\mathrm{I}^p_{\alpha,\beta,\gamma}$: $\mathcal{K}^p_{\alpha,\gamma} \to \mathcal{A}(p)$ with $\mathcal{K}^p_{\alpha,\gamma} \subset \mathcal{A}(p)$ defined by

$$\mathrm{I}^p_{\alpha,\beta,\gamma}[f](z) = \left[\frac{\alpha p + \gamma}{z^{(\alpha-\beta)p+\gamma}} \int\limits_0^z f^{\alpha}(t) t^{\gamma-1} dt\right]^{\frac{1}{\beta}}, \tag{1.3}$$

where all powers are principal ones.

***Remark* 1.1.** For $p = 1$ and $\alpha = \beta$ we obtain

$$\mathrm{I}_{\beta,\gamma}[f](z) = \left[\frac{\beta + \gamma}{z^{\gamma}} \int\limits_0^z f^{\beta}(t) t^{\gamma-1} dt\right]^{\frac{1}{\beta}}, \tag{1.4}$$

where $\mathrm{I}_{\beta,\gamma}$ is the integral operator introduced by Miller and Mocanu [12], and studied in [1–3] and more other articles (see [4–6]).

In the present paper we obtain sufficient conditions on the functions g_1, g_2 and on the parameters α, β, γ such that the following *sandwich-type result* holds:

$$z\left(\frac{g_1(z)}{z^p}\right)^{\alpha} \prec z\left(\frac{f(z)}{z^p}\right)^{\alpha} \prec z\left(\frac{g_2(z)}{z^p}\right)^{\alpha}$$

implies

$$z\left(\frac{\mathrm{I}^p_{\alpha,\beta,\gamma}[g_1](z)}{z^p}\right)^{\beta} \prec z\left(\frac{\mathrm{I}^p_{\alpha,\beta,\gamma}[f](z)}{z^p}\right)^{\beta} \prec z\left(\frac{\mathrm{I}^p_{\alpha,\beta,\gamma}[g_2](z)}{z^p}\right)^{\beta}.$$

Moreover, our result is sharp, i.e., the functions $z\left(\frac{\mathrm{I}^p_{\alpha,\beta,\gamma}[g_1](z)}{z^p}\right)^{\beta}$ and $z\left(\frac{\mathrm{I}^p_{\alpha,\beta,\gamma}[g_2](z)}{z^p}\right)^{\beta}$ are, respectively, the best subordinant and the best dominant.

Combining these new theorems with some previous related results, we give subordination and superordination consequences for a wide class of analytic integral operators.

2. Preliminaries. The following definitions and lemmas will be required in our present investigation.

Definition 2.1 [12]. *Denote by $\mathcal{Q}$ the set of all functions q that are analytic and injective on $\overline{\mathrm{U}} \setminus E(q)$, where*

$$E(q) = \left\{\zeta \in \partial\mathrm{U} \colon \lim_{z\to\zeta} q(z) = \infty\right\},$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial\mathrm{U} \setminus E(q)$. Further, let denote by $\mathcal{Q}(a)$ the subclass of the functions $q \in \mathcal{Q}$ for which $q(0) = a$.

Definition 2.2. *A function $L(z;t)\colon \mathrm{U} \times [0, +\infty) \to \mathbb{C}$ is called a subordination* (*or a Loewner*) *chain if $L(\cdot;t)$ is analytic and univalent in* U *for all $t \geq 0$ and $L(z;s) \prec L(z;t)$ when $0 \leq s \leq t$.*

The next known lemma gives a sufficient condition so that the $L(z;t)$ function will be a subordination chain.

Lemma 2.1 [14, p. 159]. *Let $L(z;t) = a_1(t)z + a_2(t)z^2 + \ldots$ with $a_1(t) \neq 0$ for all $t \geq 0$ and* $\lim_{t\to+\infty}|a_1(t)| = +\infty$. *Suppose that $L(\cdot;t)$ is analytic in* U *for all $t \geq 0$, $L(z;\cdot)$ is continuously differentiable on $[0, +\infty)$ for all $z \in \mathrm{U}$. If $L(z;t)$ satisfies*

$$\operatorname{Re}\left[z\frac{\partial L(z;t)/\partial z}{\partial L(z;t)/\partial t}\right] > 0, \quad z \in \mathrm{U}, \quad t \geq 0,$$

and

$$|L(z;t)| \leq K_0\, |a_1(t)|\,, \quad |z| < r_0 < 1, \quad t \geq 0,$$

for some positive constants K_0 and r_0, then $L(z;t)$ is a subordination chain.

Lemma 2.2 [8]. *Suppose that the function $H : \mathbb{C}^2 \to \mathbb{C}$ satisfies the condition*

$$\operatorname{Re} H(is,t) \leq 0, \quad s \in \mathbb{R}, \quad t \leq -\frac{n\big(1+s^2\big)}{2},$$

where $n \in \mathbb{N}$. If the function $p(z) = 1 + p_n z^n + p_{n+1}z^{n+1} + \ldots$ is analytic in U *and*

$$\operatorname{Re} H\big(p(z), zp'(z)\big) > 0, \quad z \in \mathrm{U},$$

then $\operatorname{Re} p(z) > 0$ *for* $z \in \mathrm{U}$.

The next result deals with the solutions of the *Briot–Bouquet differential equation* (2.1), and more general forms of the following lemma may be found in [9] (Theorem 1).

Lemma 2.3 [9]. *Let $\lambda, \mu \in \mathbb{C}$ with $\lambda \neq 0$ and $k \in H(\mathrm{U})$ with $k(0) = c$. If* $\operatorname{Re}\big[\lambda k(z)+\mu\big] > 0$, $z \in \mathrm{U}$, *then the solution of the differential equation*

$$q(z) + \frac{zq'(z)}{\lambda q(z) + \mu} = k(z) \tag{2.1}$$

with $q(0) = c$ is analytic in U *and satisfies* $\operatorname{Re}\big[\lambda q(z) + \mu\big] > 0$, $z \in \mathrm{U}$.

Lemma 2.4 [12]. *Let $p \in \mathcal{Q}(a)$ and $q(z) = a + a_n z^n + a_{n\gamma+1}z^{n+1} + \ldots$ be analytic in* U *with $q(z) \not\equiv a$ and $n \geq 1$. If q is not subordinate to p, then there exist two points $z_0 = r_0e^{i\theta} \in \mathrm{U}$ and $\zeta_0 \in \partial\mathrm{U} \setminus E(q)$, and a number $m \geq n$ such that*

$$q(\mathrm{U}_{r_0}) \subset p(\mathrm{U}), \qquad q(z_0) = p(\zeta_0), \qquad \text{and} \qquad z_0p'(z_0) = m\zeta_0 p(\zeta_0),$$

where $\mathrm{U}_{r_0} = \big\{z \in \mathbb{C} \colon |z| < r_0\big\}$.

Lemma 2.5 [14]. *Let* $\Phi:\mathbb{C}^2\to\mathbb{C},\ q\in H[a,1]$ *and set* $\Phi\big(q(z),zq'(z)\big)=h(z)$. *If* $L(z;t)=\Phi\big(q(z),tzq'(z)\big)$ *is a subordination chain and* $q\in H[a,1]\cap\mathcal{Q}(a)$, *then*

$$h(z)\prec\Phi\big(p(z),zp'(z)\big)$$

implies that $q(z)\prec p(z)$. *Furthermore, if* $\Phi\big(q(z),zq'(z)\big)=h(z)$ *has a univalent solution* $q\in\mathcal{Q}(a)$, *then* q *is the best subordinant*.

Let $c\in\mathbb{C}$ with $\operatorname{Re}c>0,\ n\in\mathbb{N}$ and

$$C_n=C_n(c)=\frac{n}{\operatorname{Re}c}\left[|c|\sqrt{1+2\operatorname{Re}\left(\frac{c}{n}\right)}+\operatorname{Im}c\right].$$

If R is the univalent function $R(z)=\dfrac{2C_nz}{1-z^2}$, then the *open door function* $R_{c,n}$ is defined by

$$R_{c,n}(z)=R\left(\frac{z+b}{1+\bar{b}z}\right),\quad z\in\mathrm{U},$$

where $b=R^{-1}(c)$.

Remark that $R_{c,n}$ is univalent in $\mathrm{U},\ R_{c,n}(0)=c$ and $R_{c,n}(\mathrm{U})=R(\mathrm{U})$ is the complex plane slit along the half-lines $\operatorname{Re}w=0,\ \operatorname{Im}w\geq C_n$ and $\operatorname{Re}w=0,\ \operatorname{Im}w\leq -C_n$. Moreover, if $c>0$, then $C_{n+1}>C_n$ and $\lim_{n\to\infty}C_n=\infty$, hence $R_{c,n}(z)\prec R_{c,n+1}(z)$ and $\lim_{n\to\infty}R_{c,n}(\mathrm{U})=\mathbb{C}$. In this paper we will use the notation $R_c:=R_{c,1}$.

3. Main results. Unless otherwise mentioned, we assume throughout this section that $\alpha,\beta,\gamma\in\mathbb{C}$ with $\beta\neq 0,\ \operatorname{Re}(\alpha p+\gamma)\geq 1$ and $p\in\mathbb{N}$.

First we will determine the subset $\mathcal{K}^p_{\alpha,\gamma}\subset\mathcal{A}(p)$ such that the integral operator $\mathrm{I}^p_{\alpha,\beta,\gamma}$ given by (1.3) will be well-defined.

If we denote by $\mathcal{A}^n_p$ the class of functions

$$\mathcal{A}^n_p=\big\{f\in H(\mathrm{U})\colon f(z)=z^p+a_{p+n}z^{p+n}+\ldots\big\},\quad n\in\mathbb{N},$$

then $\mathcal{A}(p)=\mathcal{A}^1_p$.

Lemma 3.1. *Let* $\phi,\Phi\in H[1,n]$ *with* $\phi(z)\cdot\Phi(z)\neq 0$ *for* $z\in\mathrm{U}$. *Let* $\alpha,\beta,\gamma,\delta\in\mathbb{C}$ *with* $\beta\neq 0$, $\alpha p+\delta=\beta p+\gamma$ *and* $\operatorname{Re}(\alpha p+\delta)>0$. *If the function* f *belongs to* $\mathcal{A}^n_p$ *and satisfies*

$$\alpha\frac{zf'(z)}{f(z)}+\frac{z\phi'(z)}{\phi(z)}+\delta\prec R_{\alpha p+\delta,n}(z), \tag{3.1}$$

then

$$F(z)=\left[\frac{\beta p+\gamma}{z^{\gamma}\Phi(z)}\int\limits_0^z f^{\alpha}(t)\phi(t)t^{\delta-1}dt\right]^{\frac{1}{\beta}}=z^p+A_{p+n}z^{p+n}+\ldots\in\mathcal{A}^n_p,$$

$\dfrac{F(z)}{z^p}\neq 0,\ z\in\mathrm{U}$, *and*

$$\operatorname{Re}\left[\beta\frac{zF'(z)}{F(z)}+\frac{z\Phi'(z)}{\Phi(z)}+\gamma\right]>0,\quad z\in\mathrm{U}.$$

(*All powers are principal ones*.)

Proof. The idea of the proof is similar with those of Theorem 2.5c of [12]. Thus, the subordination (3.1) implies that $\dfrac{f(z)}{z^p} \neq 0$ for all $z \in \mathrm{U}$. Since $\mathrm{Re}(\alpha p + \delta) > 0$, like in the proof of Lemma 1.2c of [12] we can easily show that the function

$$p(z) = \frac{1}{z^{\alpha p+\delta}\varphi(z)} \left[\frac{f(z)}{z^p}\right]^{-\alpha} \int\limits_0^z \left[\frac{f(t)}{t^p}\right]^{\alpha} t^{\alpha p+\delta-1}\varphi(t)dt =$$

$$= \frac{1}{z^{\delta} f^{\alpha}(z)\varphi(z)} \int\limits_0^z f^{\alpha}(t)t^{\delta-1}\varphi(t)dt = \frac{1}{\alpha p+\delta} + p_n z^n + \ldots$$

is analytic in U and $p \in H[1/(\alpha p + \delta), n]$. Differentiating the above definition formula of p, it is easy to show that the function p satisfies the differential equation

$$zp'(z) + P(z)p(z) = 1$$

with

$$P(z) = \alpha\frac{zf'(z)}{f(z)} + \frac{z\phi'(z)}{\phi(z)} + \delta.$$

Starting from this point the proof is similar with those of Theorem 2.5c of [12], hence it will omitted.

Lemma 3.1 is proved.

Note that for the special case $p = 1$ the above lemma represents the *Integral Existence Theorem* [12] (Theorem 2.5c) (see also [10, 11]).

Lemma 3.2. *Let* $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ *with* $\beta \neq 0$ *and* $\mathrm{Re}(\alpha p + \gamma) > 0$. *If* f *belongs to* $\mathcal{K}^p_{\alpha,\gamma}$, *where*

$$\mathcal{K}^p_{\alpha,\gamma} = \left\{ f \in \mathcal{A}(p) : \alpha\frac{zf'(z)}{f(z)} + \gamma \prec R_{\alpha p+\gamma}(z) \right\},$$

then $\mathrm{I}^p_{\alpha,\beta,\gamma}[f] \in \mathcal{A}(p)$, $\dfrac{\mathrm{I}^p_{\alpha,\beta,\gamma}[f](z)}{z^p} \neq 0$ *for all* $z \in \mathrm{U}$, *and*

$$\mathrm{Re}\left[\beta\frac{z\left(\mathrm{I}^p_{\alpha,\beta,\gamma}[f](z)\right)'}{\mathrm{I}^p_{\alpha,\beta,\gamma}[f](z)} + \gamma\right] > 0, \quad z \in \mathrm{U},$$

where $\mathrm{I}^p_{\alpha,\beta,\gamma}$ *is the integral operator defined by* (1.3).

Proof. Taking in the Lemma 3.1 the values $n := 1$, $\alpha := \alpha$, $\beta := \beta$, $\gamma := (\alpha-\beta)p+\gamma$, $\delta := \gamma$ and $\varphi(z) = \Phi(z) \equiv 1$, it follows that the assumption of this lemma holds. Also, the subordination condition (3.1) becomes

$$\alpha\frac{zf'(z)}{f(z)} + \gamma \prec R_{\alpha p+\gamma,1}(z),$$

while $F(z) = \mathrm{I}^p_{\alpha,\beta,\gamma}[f](z)$.

Lemma 3.2 is proved.

Theorem 3.1. *Let* $\alpha,\beta,\gamma\in\mathbb{C}$ *with* $\beta\neq 0$ *such that* $\operatorname{Re}(\alpha p+\gamma)\geq 1$. *For* $f,g\in\mathcal{K}^p_{\alpha,\gamma}$, *suppose that the function* ϕ, *defined by*

$$\phi(z)=z\left(\frac{g(z)}{z^p}\right)^{\alpha}, \tag{3.2}$$

satisfies the inequality

$$\operatorname{Re}\left[1+\frac{z\phi''(z)}{\phi'(z)}\right]>-\delta_0,\quad z\in\mathrm{U}, \tag{3.3}$$

where δ_0 *is given by*

$$\delta_0=\begin{cases}\dfrac{1+|\alpha p+\gamma-1|^2-\left|1-(\alpha p+\gamma-1)^2\right|}{4\operatorname{Re}(\alpha p+\gamma-1)}, & \text{if}\quad \operatorname{Re}(\alpha p+\gamma)>1,\\ 0, & \text{if}\quad \operatorname{Re}(\alpha p+\gamma)=1.\end{cases} \tag{3.4}$$

Then the subordination condition

$$z\left(\frac{f(z)}{z^p}\right)^{\alpha}\prec z\left(\frac{g(z)}{z^p}\right)^{\alpha} \tag{3.5}$$

implies that

$$z\left(\frac{\mathrm{I}^p_{\alpha,\beta,\gamma}[f](z)}{z^p}\right)^{\beta}\prec z\left(\frac{\mathrm{I}^p_{\alpha,\beta,\gamma}[g](z)}{z^p}\right)^{\beta},$$

and the function $z\left(\dfrac{\mathrm{I}^p_{\alpha,\beta,\gamma}[g](z)}{z^p}\right)^{\beta}$ *is the best dominant*. (*All powers are principal ones*.)

Proof. Let define the functions F and G by

$$F(z):=z\left(\frac{\mathrm{I}^p_{\alpha,\beta,\gamma}[f](z)}{z^p}\right)^{\beta},\qquad G(z):=z\left(\frac{\mathrm{I}^p_{\alpha,\beta,\gamma}[g](z)}{z^p}\right)^{\beta},\quad z\in\mathrm{U}, \tag{3.6}$$

respectively, where all powers are principal ones. From Lemma 3.2 it follows that F, G belong to $\mathcal{A}$. First we will show that if

$$q(z)=1+\frac{zG''(z)}{G'(z)},\quad z\in\mathrm{U}, \tag{3.7}$$

then

$$\operatorname{Re}q(z)>0,\quad z\in\mathrm{U}. \tag{3.8}$$

From (1.3) and the definitions of the functions G and ϕ, we obtain

$$\phi(z)=\left(1-\frac{1}{\alpha p+\gamma}\right)G(z)+\frac{1}{\alpha p+\gamma}zG'(z). \tag{3.9}$$

Differentiating both side of (3.9) with respect to z, we have

$$\phi'(z)=G'(z)+\frac{zG''(z)}{\alpha p+\gamma}, \tag{3.10}$$

and combining (3.7) and (3.10) we easily get

$$k(z) := 1 + \frac{z\phi''(z)}{\phi'(z)} = q(z) + \frac{zq'(z)}{q(z) + \alpha p + \gamma - 1}. \tag{3.11}$$

According to (3.11), from (3.3) it follows that

$$\operatorname{Re}\big[k(z) + \alpha p + \gamma - 1\big] > -\delta_0 + \operatorname{Re}(\alpha p + \gamma - 1) \geq 0, \quad z \in \mathrm{U},$$

whenever

$$\delta_0 \leq \operatorname{Re}(\alpha p + \gamma - 1). \tag{3.12}$$

Supposing that the inequality (3.12) holds, according to Lemma 2.3 we conclude that the differential equation (3.11) has a solution $q \in H(\mathrm{U})$ with $k(0) = q(0) = 1$.

If we let

$$H(u, v) = u + \frac{v}{u + \alpha p + \gamma - 1} + \delta,$$

from (3.11) and (3.4) we obtain

$$\operatorname{Re} H\big(q(z), zq'(z)\big) > 0, \quad z \in \mathrm{U}.$$

To verify the condition

$$\operatorname{Re} H(is, t) \leq 0 \quad \text{for} \quad s \in \mathbb{R}, \quad t \leq -\frac{1 + s^2}{2}, \tag{3.13}$$

first we see that

$$\operatorname{Re} H(is, t) = \frac{t \operatorname{Re}(\alpha p + \gamma - 1)}{|\alpha p + \gamma - 1 + is|^2} + \delta = \delta \leq \delta_0 = 0$$

for $\operatorname{Re}(\alpha p + \gamma - 1) = 0$. If $\operatorname{Re}(\alpha p + \gamma - 1) > 0$, then

$$\operatorname{Re} H(is, t) = \operatorname{Re}\left[is + \frac{t}{is + \alpha p + \gamma - 1} + \delta_0\right] =$$

$$= \frac{t \operatorname{Re}(\alpha p + \gamma - 1)}{|\alpha p + \gamma - 1 + is|^2} + \delta \leq -\frac{K_{p;\alpha,\gamma;\delta}(s)}{2\,|\alpha p + \gamma - 1 + is|^2} \quad \text{for} \quad t \leq -\frac{1 + s^2}{2},$$

where

$$K_{p;\alpha,\gamma;\delta}(s) := \big[\operatorname{Re}(\alpha p + \gamma - 1) - 2\delta\big]s^2 - 4\delta \operatorname{Im}(\alpha p + \gamma - 1)s -$$

$$- 2\delta|\alpha p + \gamma - 1|^2 + \operatorname{Re}(\alpha p + \gamma - 1).$$

We need to determine the value

$$\delta_0 = \sup\big\{\delta \colon K_{p;\alpha,\gamma;\delta}(s) \geq 0, \;\; s \in \mathbb{R}, \;\; \delta \leq \delta_0\big\}.$$

(i) If $\operatorname{Re}(\alpha p + \gamma - 1) - 2\delta = 0$, then

$$K_{p;\alpha,\gamma;\delta}(s) = \operatorname{Re}(\alpha p + \gamma - 1)\big[-2 \operatorname{Im}(\alpha p + \gamma - 1)\, s + 1 - |\alpha p + \gamma - 1|^2\big] \geq 0$$

for all $s \in \mathbb{R}$ if and only if $(\alpha p + \gamma - 1) \in (0, 1]$, and in this case $\delta_0 = (\alpha p + \gamma - 1)/2$. Thus, it is easy to see that the definition relation (3.4) could be used for this special case.

(ii) If $\operatorname{Re}(\alpha p + \gamma - 1) - 2\delta \neq 0$, then $K_{p;\alpha,\gamma;\delta}(s) \geq 0$ for all $s \in \mathbb{R}$ if and only if

$$\operatorname{Re}(\alpha p + \gamma - 1) - 2\delta \geq 0 \tag{3.14}$$

and

$$4\delta^2 \operatorname{Im}^2(\alpha p + \gamma - 1) -$$
$$- \left[\operatorname{Re}(\alpha p + \gamma - 1) - 2\delta\right]\left[\operatorname{Re}(\alpha p + \gamma - 1) - 2\delta|\alpha p + \gamma - 1|^2\right] \leq 0. \tag{3.15}$$

Using the fact that the inequality (3.15) is equivalent to

$$\chi(\delta) := -4 \operatorname{Re}(\alpha p + \gamma - 1)\, \delta^2 +$$
$$+ 2\left(1 + |\alpha p + \gamma - 1|^2\right)\delta - \operatorname{Re}(\alpha p + \gamma - 1) \leq 0,$$

a simple computation shows that the function χ has the positive zeros $0 < \delta_0 \leq \delta_1$, where δ_0 is given by (3.4). Since $\chi(\delta) \leq 0$ for all $\delta \leq \delta_0$ and

$$\chi\left(\frac{\operatorname{Re}(\alpha p + \gamma - 1)}{2}\right) = \operatorname{Re}(\alpha p + \gamma - 1) \operatorname{Im}^2(\alpha p + \gamma - 1) \geq 0,$$

it follows that $\delta_0 \leq \dfrac{\operatorname{Re}(\alpha p + \gamma - 1)}{2}$, i.e., the condition (3.14) holds for $\delta = \delta_0$.

Moreover, because

$$\delta_0 \leq \frac{\operatorname{Re}(\alpha p + \gamma - 1)}{2} < \operatorname{Re}(\alpha p + \gamma - 1) \quad \text{if} \quad \operatorname{Re}(\alpha p + \gamma - 1) > 0,$$

we conclude that the inequality (3.12) holds whenever $\operatorname{Re}(\alpha p + \gamma - 1) > 0$. Obviously, it holds also for $\operatorname{Re}(\alpha p + \gamma - 1) = 0$, since in this case $\delta_0 = 0$.

In conclusion, for the assumed value of δ_0 given by (3.4), we proved that $K_{p;\alpha,\gamma;\delta}(s) \geq 0$ for all $s \in \mathbb{R}$ which implies that (3.13) holds.

By using Lemma 2.2, we conclude that the inequality (3.8) holds, and from the definition relation (3.7) it follows that G is convex. Hence, G is a univalent function in U.

Next, we will prove that the subordination condition (3.5) implies that $F(z) \prec G(z)$, where the functions F and G are defined by (3.6). For this purpose, let define the function $L(z;t)$ by

$$L(z;t) = \left(1 - \frac{1}{\alpha p + \gamma}\right) G(z) + \frac{1 + t}{\alpha p + \gamma}\, zG'(z), \quad z \in \mathrm{U}, \quad t \geq 0. \tag{3.16}$$

If we denote $L(z;t) = a_1(t)z + \ldots$, then

$$a_1(t) = \frac{\partial L(0;t)}{\partial z} = \left(1 + \frac{t}{\alpha p + \gamma}\right) G'(0) = 1 + \frac{t}{\alpha p + \gamma},$$

hence $\lim_{t \to +\infty} |a_1(t)| = +\infty$. By using the fact that $\operatorname{Re}(\alpha p + \gamma) > 0$, we obtain that $a_1(t) \neq 0$ for all $t \geq 0$.

Since $\operatorname{Re} q(z) > 0,\ z \in \mathrm{U}$, and $\operatorname{Re}(\alpha p + \gamma) \geq 1$, we deduce that

$$\operatorname{Re}\left[z \frac{\partial L(z;t)/\partial z}{\partial L(z;t)/\partial t}\right] = \operatorname{Re}\big[\alpha p + \gamma - 1 + (1+t)q(z)\big] > 0, \quad z \in \mathrm{U}, \quad t \geq 0.$$

From the definition (3.16), since $\operatorname{Re}(\alpha p + \gamma) \geq 1$, for all $t \geq 0$, we have that

$$\frac{|L(z;t)|}{|a_1(t)|} = \frac{\big|(\alpha p + \gamma - 1)G(z) + (1+t)zG'(z)\big|}{|\alpha p + \gamma + t|} \leq$$

$$\leq \frac{|\alpha p + \gamma - 1||G(z)| + (1+t)|zG'(z)|}{|\alpha p + \gamma + t|}. \tag{3.17}$$

Since G is convex, the following known growth and distortion sharp inequalities are true (see [7]):

$$\frac{r}{1+r} \leq |G(z)| \leq \frac{r}{1-r}, \quad \text{if} \quad |z| \leq r,$$
$$\frac{1}{(1+r)^2} \leq |G'(z)| \leq \frac{1}{(1-r)^2}, \quad \text{if} \quad |z| \leq r. \tag{3.18}$$

By using the right-hand sides of these inequalities in (3.17), we obtain

$$\frac{|L(z;t)|}{|a_1(t)|} \leq \frac{r}{(1-r)^2} \frac{t + 1 + |\alpha p + \gamma - 1|(1-r)}{|\alpha p + \gamma + t|}, \quad |z| \leq r, \quad t \geq 0. \tag{3.19}$$

The assumption $\operatorname{Re}(\alpha p + \gamma) \geq 1$ implies

$$|t + \alpha p + \gamma| \geq |\alpha p + \gamma|, \qquad |t + \alpha p + \gamma| \geq |t + 1|, \quad t \geq 0,$$

and from (3.19) we conclude that

$$\frac{|L(z;t)|}{|a_1(t)|} \leq \frac{r}{(1-r)^2}\left[1 + \frac{|\alpha p + \gamma - 1|(1-r)}{|\alpha p + \gamma|}\right], \quad |z| \leq r, \quad t \geq 0.$$

Thus, the second assumption of Lemma 2.1 holds, and according to this lemma we obtain that the function $L(z;t)$ is a subordination chain.

By using Lemma 2.4, we will show that $F(z) \prec G(z)$. Without loss of generality, we can assume that ϕ and G are analytic and univalent in $\overline{\mathrm{U}}$ and $G'(\zeta) \neq 0$ for $|\zeta| = 1$. If not, then we could replace ϕ with $\phi_\rho(z) = \phi(\rho z)$ and G with $G_\rho(z) = G(\rho z)$, where $\rho \in (0,1)$. These new functions have the desired properties on $\overline{\mathrm{U}}$ and we can use them in our proof. Therefore, the results would follow by letting $\rho \to 1$.

From the definition of the subordination chain it follows

$$\phi(z) = \left(1 - \frac{1}{\alpha p + \gamma}\right)G(z) + \frac{1}{\alpha p + \gamma} zG'(z) = L(z;0)$$

and

$$L(z;0) \prec L(z;t), \quad t \geq 0,$$

which implies

$$L(\zeta;t)\notin L(\mathrm{U};0)=\phi(\mathrm{U}),\quad \zeta\in\partial\mathrm{U},\quad t\geq 0. \tag{3.20}$$

According to Lemma 2.4, if $F(z)\not\prec G(z)$ it follows that there exist two points $z_0\in\mathrm{U}$, $\zeta_0\in\partial\mathrm{U}$ and a number $m=1+t_0\geq 1$ such that

$$F(z_0)=G(\zeta_0)\qquad\text{and}\qquad z_0F'(z_0)=(1+t_0)\,\zeta_0G'(\zeta_0),\quad t_0\geq 0. \tag{3.21}$$

Hence, by virtue of (3.21), we have

$$L(\zeta_0;t_0)=\left(1-\frac{1}{\alpha p+\gamma}\right)G(\zeta_0)+\frac{1+t_0}{\alpha p+\gamma}\,\zeta_0G'(\zeta_0)=$$

$$=\left(1-\frac{1}{\alpha p+\gamma}\right)F(z_0)+\frac{1}{\alpha p+\gamma}\,z_0F'(z_0)\in\phi(\mathrm{U}),$$

which contradicts the above remark (3.20), i.e., $L(\zeta_0;t_0)\notin\phi(\mathrm{U})$. Consequently, the subordination condition (3.5) implies that $F(z)\prec G(z)$, and considering $F=G$ we conclude that the function G is the best dominant.

Theorem 3.1 is proved.

***Remark* 3.1.** (i) Taking $p=1$ and $\alpha=\beta$ in Theorem 3.1, we obtain a subordination results for the class integral operators studied in [1, 2].

(ii) Note that in [1] (Theorem 1) the author supposed that $0<\beta+\gamma\leq 1$, in [2] (Theorem 3.1) the assumption was extended to $0<\beta+\gamma\leq 2$, while the above theorem extends the range of these parameters to $\mathrm{Re}(\beta+\gamma)\geq 1$.

According to this last remark, for the special case $p=1$ and $\beta+\gamma>0$, combining Theorem 3.1 with Theorem 3.1 in [2], we obtain the following result.

***Corollary* 3.1.** *Let $\beta,\gamma\in\mathbb{C}$ with $\beta\neq 0$ such that $\beta+\gamma>0$. For $f,g\in\mathcal{K}^1_{\alpha,\gamma}$, suppose that the function ϕ, defined by*

$$\phi(z)=z\left[\frac{g(z)}{z}\right]^{\beta},$$

satisfies the inequality

$$\mathrm{Re}\left[1+\frac{z\phi''(z)}{\phi'(z)}\right]>\widetilde{\delta},\quad z\in\mathrm{U},$$

where $\widetilde{\delta}$ is given by

$$\widetilde{\delta}=\begin{cases}1-(\beta+\gamma), & \textit{if}\quad 0<\beta+\gamma\leq 1,\\ \dfrac{1-(\beta+\gamma)}{2}, & \textit{if}\quad 1\leq\beta+\gamma\leq 2,\\ -\dfrac{1}{2(\beta+\gamma-1)}, & \textit{if}\quad \beta+\gamma\geq 2.\end{cases}$$

Then the subordination condition

$$z\left[\frac{f(z)}{z}\right]^{\beta}\prec z\left[\frac{g(z)}{z}\right]^{\beta}\quad\textit{implies}\quad z\left[\frac{\mathrm{I}_{\beta,\gamma}[f](z)}{z}\right]^{\beta}\prec z\left[\frac{\mathrm{I}_{\beta,\gamma}[g](z)}{z}\right]^{\beta},$$

where the integral operator $\mathrm{I}_{\beta,\gamma}$ *is given by* (1.4). *Moreover, the function* $z\left[\dfrac{\mathrm{I}_{\beta,\gamma}[g](z)}{z}\right]^{\beta}$ *is the best dominant.*

We now derive the following superordination result.

Theorem 3.2. *Let* $\alpha,\beta,\gamma\in\mathbb{C}$ *with* $\beta\neq 0$ *such that* $\mathrm{Re}(\alpha p+\gamma)>1$. *For* $f,g\in\mathcal{K}^p_{\alpha,\gamma}$, *suppose that the function* ϕ *defined by* (3.2) *satisfies the condition* (3.3), *where* δ_0 *is given by* (3.4).

If the function $z\left(\dfrac{f(z)}{z^p}\right)^{\alpha}$ *is univalent in* U *and* $z\left(\dfrac{\mathrm{I}^p_{\alpha,\beta,\gamma}[f](z)}{z^p}\right)^{\beta}\in\mathcal{Q}(0)$, *then the superordination condition*

$$z\left(\frac{g(z)}{z^p}\right)^{\alpha}\prec z\left(\frac{f(z)}{z^p}\right)^{\alpha} \tag{3.22}$$

implies that

$$z\left(\frac{\mathrm{I}^p_{\alpha,\beta,\gamma}[g](z)}{z^p}\right)^{\beta}\prec z\left(\frac{\mathrm{I}^p_{\alpha,\beta,\gamma}[f](z)}{z^p}\right)^{\beta},$$

and the function $z\left(\dfrac{\mathrm{I}^p_{\alpha,\beta,\gamma}[g](z)}{z^p}\right)^{\beta}$ *is the best subordinant.*

Proof. Like in the proof of Theorem 3.1, suppose that the functions F, G and q are defined by (3.6) and (3.7), respectively. Applying a similar method as in the proof of Theorem 3.1, we get that the inequality (3.8) holds, and from the definition (3.7) it follows that G is convex. Hence, G is a univalent function in U.

Next, we will prove that the superordination condition (3.22) implies that $G(z)\prec F(z)$. For this, we define the function $L(z;t)$ by

$$L(z;t)=\left(1-\frac{1}{\alpha p+\gamma}\right)G(z)+\frac{t}{\alpha p+\gamma}\,zG'(z),\quad z\in\mathrm{U},\quad t\geq 0. \tag{3.23}$$

If we denote $L(z;t)=a_1(t)z+\ldots$, then

$$a_1(t)=\frac{\partial L(0;t)}{\partial z}=\left(1+\frac{t-1}{\alpha p+\gamma}\right)G'(0)=1+\frac{t-1}{\alpha p+\gamma}.$$

Hence, $\lim_{t\to+\infty}|a_1(t)|=+\infty$, and, using the assumption $\mathrm{Re}(\alpha p+\gamma)>1$, we obtain $a_1(t)\neq 0$ for all $t\geq 0$.

Using the facts that $\mathrm{Re}\,q(z)>0$, $z\in\mathrm{U}$, and $\mathrm{Re}(\alpha p+\gamma)>1$, we get

$$\mathrm{Re}\left[z\frac{\partial L(z;t)/\partial z}{\partial L(z;t)/\partial t}\right]=\mathrm{Re}\big[\alpha p+\gamma-1+tq(z)\big]>0,\quad z\in\mathrm{U},\quad t\geq 0.$$

From the definition (3.23), since $\mathrm{Re}(\alpha p+\gamma)>1$, for all $t\geq 0$, we have

$$\frac{|L(z;t)|}{|a_1(t)|}=\frac{|(\alpha p+\gamma-1)G(z)+tzG'(z)|}{|\alpha p+\gamma+t-1|}\leq$$

$$\leq\frac{|\alpha p+\gamma-1||G(z)|+t|zG'(z)|}{|\alpha p+\gamma+t-1|}.$$

Since G is convex, using in the above relation the right-hand sides of the inequalities (3.18), we obtain

$$\frac{|L(z;t)|}{|a_1(t)|} \le \frac{r}{(1-r)^2}\,\frac{t+|\alpha p+\gamma-1|(1-r)}{|\alpha p+\gamma+t-1|}, \quad |z|\le r, \quad t\ge 0. \tag{3.24}$$

The assumption $\operatorname{Re}(\alpha p+\gamma)>1$ implies

$$|t-1+\alpha p+\gamma| \ge |\alpha p+\gamma-1|, \quad |t-1+\alpha p+\gamma|>|t|, \quad t\ge 0,$$

and, from (3.24), we conclude that

$$\frac{|L(z;t)|}{|a_1(t)|} < \frac{r(2-r)}{(1-r)^2}, \quad |z|\le r, \quad t\ge 0.$$

Hence, all the assumptions of Lemma 2.1 hold and we conclude that the function $L(z;t)$ is a subordination chain.

According to Lemma 2.5, the supeordination condition (3.22) implies that $G(z)\prec F(z)$, and since the differential equation

$$\phi(z)=\left(1-\frac{1}{\alpha p+\gamma}\right)G(z)+\frac{1}{\alpha p+\gamma}\,zG'(z)=\Phi\big(G(z),zG'(z)\big)$$

has a univalent solution G, the function G is the best subordinant.

Theorem 3.2 is proved.

***Remark* 3.2.** Taking $p=1$ and $\alpha=\beta$ in Theorem 3.2, we obtain a superordination result that generalizes the result from Theorem 3.1 in [3], where a similar implication was obtained for $1<\beta+\gamma\le 2$. In the present paper this result was extended by assuming that $\operatorname{Re}(\beta+\gamma)>1$.

Combining the above-mentioned subordination and superordination results involving the operator $\mathrm{I}^p_{\alpha,\beta,\gamma}$, we have the following *sandwich-type result*.

Theorem 3.3. *Let* $\alpha,\beta,\gamma\in\mathbb{C}$ *with* $\beta\ne 0$ *such that* $\operatorname{Re}(\alpha p+\gamma)>1$. *For* $f_k,g_k\in\mathcal{K}^p_{\alpha,\gamma}$, $k=1,2$, *suppose that the functions* ϕ_k, *defined by*

$$\phi_k(z)=z\left(\frac{g_k(z)}{z^p}\right)^{\alpha},$$

satisfy the inequalities

$$\operatorname{Re}\left[1+\frac{z\phi_k''(z)}{\phi_k'(z)}\right]>-\delta_0, \quad z\in\mathrm{U}, \quad k=1,2,$$

where δ_0 *is given by* (3.4).

If the function $z\left(\frac{f(z)}{z^p}\right)^{\alpha}$ *is univalent in* U *and* $z\left(\frac{\mathrm{I}^p_{\alpha,\beta,\gamma}[f](z)}{z^p}\right)^{\beta}\in\mathcal{Q}(0)$, *then the condition*

$$z\left(\frac{g_1(z)}{z^p}\right)^{\alpha}\prec z\left(\frac{f(z)}{z^p}\right)^{\alpha}\prec z\left(\frac{g_2(z)}{z^p}\right)^{\alpha}$$

implies that

$$z\left(\frac{\mathrm{I}^p_{\alpha,\beta,\gamma}[g_1](z)}{z^p}\right)^\beta \prec z\left(\frac{\mathrm{I}^p_{\alpha,\beta,\gamma}[f](z)}{z^p}\right)^\beta \prec z\left(\frac{\mathrm{I}^p_{\alpha,\beta,\gamma}[g_2](z)}{z^p}\right)^\beta,$$

and the functions $z\left(\frac{\mathrm{I}^p_{\alpha,\beta,\gamma}[g_1](z)}{z^p}\right)^\beta$ *and* $z\left(\frac{\mathrm{I}^p_{\alpha,\beta,\gamma}[g_2](z)}{z^p}\right)^\beta$ *are, respectively, the best subordinant and the best dominant.*

Remark **3.3.** (i) Taking $p = 1$ and $\alpha = \beta$ in Theorem 3.3, we obtain the sandwich superordination result that generalizes the result from Theorem 3.2 in [3].

(ii) While in this previously mentioned article the assumption for the parameters $\beta, \gamma \in \mathbb{C}$ was $1 < \beta + \gamma \leq 2$, we proved now that the implication holds for $\operatorname{Re}(\beta + \gamma) > 1$.

Thus, for the special case $p = 1$ and $\beta + \gamma > 1$ we deduce the following *sandwich-type result*.

Corollary **3.2.** *Let* $\beta, \gamma \in \mathbb{C}$ *with* $\beta \neq 0$ *such that* $\beta + \gamma > 1$. *For* $f, g_1, g_2 \in \mathcal{K}^1_{\alpha,\gamma}$, *suppose that the functions* ϕ_k, $k = 1, 2$, *defined by*

$$\phi_k(z) = z\left[\frac{g_k(z)}{z}\right]^\beta, \quad k = 1, 2,$$

satisfy the inequality

$$\operatorname{Re}\left[1 + \frac{z\phi_k''(z)}{\phi_k'(z)}\right] > \widehat{\delta}, \quad z \in \mathrm{U}, \quad k = 1, 2,$$

where $\widehat{\delta}$ *is given by*

$$\widehat{\delta} = \begin{cases} \dfrac{1 - (\beta + \gamma)}{2}, & \text{if} \quad 1 < \beta + \gamma \leq 2, \\ -\dfrac{1}{2(\beta + \gamma - 1)}, & \text{if} \quad \beta + \gamma \geq 2. \end{cases}$$

If the function $z\left[\frac{f(z)}{z}\right]^\beta$ *is univalent in* U *and* $z\left[\frac{\mathrm{I}_{\beta,\gamma}[f](z)}{z}\right]^\beta \in \mathcal{Q}(0)$, *then the condition*

$$z\left[\frac{g_1(z)}{z}\right]^\beta \prec z\left[\frac{f(z)}{z}\right]^\beta \prec z\left[\frac{g_2(z)}{z}\right]^\beta$$

implies that

$$z\left[\frac{\mathrm{I}_{\beta,\gamma}[g_1](z)}{z}\right]^\beta z\left[\frac{\mathrm{I}_{\beta,\gamma}[f](z)}{z}\right]^\beta \prec z\left[\frac{\mathrm{I}_{\beta,\gamma}[g_2](z)}{z}\right]^\beta,$$

where the integral operator $\mathrm{I}_{\beta,\gamma}$ *is given by* (1.4). *Moreover, the functions* $z\left[\frac{\mathrm{I}_{\beta,\gamma}[g_1](z)}{z}\right]^\beta$ *and* $z\left[\frac{\mathrm{I}_{\beta,\gamma}[g_2](z)}{z}\right]^\beta$ *are, respectively, the best subordinant and the best dominant.*

References

1. T. Bulboacă, *Integral operators that preserve the subordination*, Bull. Korean Math. Soc., **32**, 627 – 636 (1997).
2. T. Bulboacă, *On a class of integral operators that preserve the subordination*, Pure Math. and Appl. (PU.M.A.), **13**, 87 – 96 (2002).
3. T. Bulboacă, *A class of superordination-preserving integral operators*, Indag. Math (N.S.), **13**, 301 – 311 (2002).
4. T. Bulboacă, *Sandwich-type theorems for a class of integral operators*, Bull. Belg. Math. Soc. Simon Stevin, **13**, № 3, 537 – 550 (2006).
5. T. Bulboacă, *Sandwich-type results for a class of convex integral operators*, Acta Math. Sci. Ser. B (Engl. Ed.), **32**, № 3, 989 – 1001 (2012).
6. N. E. Cho, T. Bulboacă, *Subordination and superordination properties for a class of integral operators*, Acta Math. Sin. (Engl. Ser.), **26**, № 3, 515 – 522 (2010).
7. T. H. Gronwall, *Some remarks on conformal representation*, Ann. Math., **16**, 72 – 76 (1914 – 1915).
8. S. S. Miller, P. T. Mocanu, *Differential subordinations and univalent functions*, Michigan Math. J., **28**, № 2, 157 – 172 (1981).
9. S. S. Miller, P. T. Mocanu, *Univalent solutions of Briot – Bouquet differential equations*, J. Different. Equat., **56**, № 3, 297 – 309 (1985).
10. S. S. Miller, P. T. Mocanu, *Integral operators on certain classes of analytic functions, Univalent Functions, Fractional Calculus and their Applications*, Halstead Press, J. Wiley & Sons, New York (1989), p. 153 – 166.
11. S. S. Miller, P. T. Mocanu, *Classes of univalent integral operators*, J. Math. Anal. and Appl., **157**, № 1, 147 – 165 (1991).
12. S. S. Miller, P. T. Mocanu, *Differential subordinations: theory and applications, series on monographs and textbooks in pure and applied mathematics*, **225**, Marcel Dekker, New York, Basel (2000).
13. S. S. Miller, P. T. Mocanu, *Subordinants of differential superordinations*, Complex Var., Theory and Appl., **48**, № 10, 815 – 826 (2003).
14. Ch. Pommerenke, *Univalent functions*, Vandenhoeck and Ruprecht, Göttingen (1975).

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