DOI: 10.37863/umzh.v73i6.388

UDC 517.9

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A STUDY OF A MORE GENERAL CLASS OF NONLOCAL INTEGRO-MULTIPOINT BOUNDARY-VALUE PROBLEMS OF FRACTIONAL INTEGRO-DIFFERENTIAL INCLUSIONS *

ВИВЧЕННЯ ВІДНОСНО ЗАГАЛЬНОГО КЛАСУ НЕЛОКАЛЬНИХ ІНТЕГРАЛЬНИХ БАГАТОТОЧКОВИХ КРАЙОВИХ ЗАДАЧ ДЛЯ ДРОБОВИХ ІНТЕГРО-ДИФЕРЕНЦІАЛЬНИХ ВКЛЮЧЕНЬ

We develop the existence theory for a more general class of nonlocal integro-multipoint boundary value problems of Caputo type fractional integro-differential inclusions. Our results include the convex and non-convex cases for the given problem and rely on standard fixed point theorems for multivalued maps. The obtained results are illustrated with the aid of examples.

Запропоновано теорію існування для відносно загального класу нелокальних інтегральних багатоточкових крайових задач для дробових інтегро-диференціальних включень типу Капуто. Наші результати охоплюють опуклі та неопуклі випадки даної проблеми і базуються на стандартних теоремах про нерухому точку для багатозначних відображень. Отримані результати проілюстровано відповідними прикладами.

1. Introduction. The tools of fractional calculus revolutionized the mathematical modeling of various phenomena occurring in sciences and engineering by producing fractional-order models for them, which are found to be more informative and realistic than their integer-order counterparts. The interest in this branch of mathematical analysis owes to the nonlocal nature of fractional order operators which are capable to trace the history of processes and materials involved in the phenomenon at hand. Examples include continuum and statistical mechanics [1], protein dynamics [2], chaos and fractional dynamics [3], bio-engineering [4], chaos synchronization [5], viscoelasticity [6], ecology [7], infectious diseases [8, 9], financial economics [10], etc.

Widespread applications of fractional calculus motivated many researchers to develop the theory of initial and boundary-value problems arising in the fractional order models associated with numerous real world phenomena. In particular, boundary-value problems of nonlinear fractional differential equations and inclusions have been extensively studied by several researchers during the past few decades, for instance, see [11 – 18]. Recently, in [19], the authors proved some existence results for fractional differential equations with non-separated type nonlocal multipoint and multistrip boundary conditions.

In this paper, we consider the inclusions (multivalued) case of the problem addressed in [19] and investigate the existence of solutions for the problem at hand. In precise terms we study the following multivalued problem:

$$ {}^cD^q x(t) \in F(t, x(t), {}^cD^\beta x(t), I^\gamma x(t)), \qquad 1 < q \leq 2, \quad 0 < \beta < 1, \quad \gamma > 0,\ t \in [0,1], $$

* This paper was supported by the Deanship of Scientific Research (DSR), King Abdulaziz University, Saudi Arabia under grant no. (KEP-PhD-40-130-41).

$$ax(0) + bx(1) = \sum_{i=1}^{m-2} \alpha_i x(\sigma_i) + \sum_{j=1}^{p-2} r_j \int_{\xi_j}^{\eta_i} x(s)ds,$$

$$cx'(0) + dx'(1) = \sum_{i=1}^{m-2} \delta_i x'(\sigma_i) + \sum_{j=1}^{p-2} \gamma_j \int_{\xi_j}^{\eta_j} x'(s)ds, \tag{1.1}$$

$$0 < \sigma_1 < \sigma_2 < \ldots < \sigma_{m-2} < \ldots < \xi_1 < \eta_1 < \xi_2 < \eta_2 < \ldots < \xi_{p-2} < \eta_{p-2} < 1,$$

where ${}^cD^{(.)}$ denotes the Caputo fractional derivative of order $(.)$, $I^{(.)}$ denotes the Riemann–Liouville integral of fractional order $(.)$, $F\colon [0,1] \times \mathbb{R}^3 \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$, and a, b, c, d are real constants and α_i, δ_i, $i = 1, 2, \ldots, m-2$, r_j, γ_j, $j = 1, 2, \ldots, p-2$, are real constants.

Differential inclusions play a key role in studying dynamical systems and stochastic processes. An important application of differential inclusions can be found in the area of sweeping processes, which are modeled by evolution differential inclusions. For a detailed account of this subject and its applications, we refer the reader to the texts [20, 21]. Differential inclusions help to study sweeping process [22], granular systems [23, 24], nonlinear dynamics of wheeled vehicles [25], control problems [26, 27], synchronization process [28], etc.

This paper is organized as follows. In Section 2, we recall some useful preliminaries from multivalued analysis and fractional calculus. Section 3 contains the main results. The first existence result dealing with convex valued maps involved in (1.1) is proved by applying the nonlinear alternative of Leray–Schauder type, while the second result for the problem (1.1) is concerned with the non-convex valued maps and relies on a fixed point theorem for contractive multivalued maps due to Covitz and Nadler. The methods used in our analysis are known, however their exposition in the framework of problem (1.1) facilitates the enhancement of the topic. Examples illustrating the main results are also constructed. Finally, we discuss some special cases of the work presented in this article.

2. Background material. Let $X = \big\{x\colon x \in C([0,1], \mathbb{R}) \text{ and } {}^cD^\beta x \in C([0,1],\ R)\big\}$ denotes the space equipped with the norm $\|x\|_X = \|x\| + \|{}^cD^\beta x\| = \sup_{t\in[0,1]} |x(t)| + \sup_{t\in[0,1]} |{}^cD^\beta x(t)|$. Observe that $(X, \|\cdot\|_X)$ is a Banach space.

In the forthcoming analysis, we need the following spaces: $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X)\colon Y \text{ is closed}\}$, $\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X)\colon Y \text{ is bounded}\}$, $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X)\colon Y \text{ is compact}\}$, and $\mathcal{P}_{cp,c}(X) = \{Y \in \mathcal{P}(X)\colon Y \text{ is compact and convex}\}$.

Next we state some known results related to our proposed work.

Lemma 2.1 ([29], Proposition 1.2)**.** *If $G\colon X \to \mathcal{P}_{cl}(X)$ is u.s.c., then $Gr(G)$ is a closed subset of $X \times Y$, i.e., for every sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ and $\{y_n\}_{n\in\mathbb{N}} \subset Y$, if when $n \to \infty$, $x_n \to x_*$, $y_n \to y_*$ and $y_n \in G(x_n)$, then $y_* \in G(x_*)$. Conversely, if G is completely continuous and has a closed graph, then it is upper semicontinuous.*

Lemma 2.2 [30]**.** *Let X be a Banach space. Let $F\colon [0,1] \times X^3 \to \mathcal{P}_{cp,c}(X)$ be an L^1-Carathéodory multivalued map and let Θ be a linear continuous mapping from $L^1\big([0,1], X\big)$ to $C\big([0,1], X\big)$. Then the operator*

$$\Theta \circ S_{F,x}\colon C\big([0,1], X\big) \to \mathcal{P}_{cp,c}\big(C([0,1], X)\big), \quad x \mapsto (\Theta \circ S_{F,x})(x) = \Theta(S_{F,x})$$

is a closed graph operator in $C\big([0,1],X\big)\times C\big([0,1],X\big)$.

Lemma 2.3 (Nonlinear alternative for Kakutani maps [31])**.** *Let E be a Banach space, C a closed convex subset of E, U an open subset of C and $0\in U$. Suppose that $F:\overline{U}\to\mathcal{P}_{cp,c}(C)$ is a upper semicontinuous compact map. Then or* (i) *F has a fixed point in $\overline{U}$, or* (ii) *there is a $u\in\partial U$ and $\lambda\in(0,1)$ with $u\in\lambda F(u)$.*

3. Existence results.

Definition 3.1. *A function $x\in C^2([0,1],\mathbb{R})$ is said to be a solution of the boundary-value problem* (1.1) *if* $ax(0)+bx(1)=\sum_{i=1}^{m-2}\alpha_i x(\sigma_i)+\sum_{j=1}^{p-2}r_j\int_{\xi_j}^{\eta_i}x(s)ds,\ cx'(0)+dx'(1)=$ $=\sum_{i=1}^{m-2}\delta_i x'(\sigma_i)+\sum_{j=1}^{p-2}\gamma_j\int_{\xi_j}^{\eta_j}x'(s)ds$, *and there exists function $v\in L^1\big([0,1],\mathbb{R}\big)$ such that $v(t)\in F\big(t,x(t),{}^cD^\beta x(t),I^\gamma x(t)\big)$ a.e. on $[0,1]$ and*

$$x(t)=\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}v(s)ds+\frac{\Delta_2 t-\Delta_3}{\Delta_1\Delta_2}\left[\sum_{i=1}^{m-2}\delta_i\int_0^{\sigma_i}\frac{(\sigma_i-s)^{q-2}}{\Gamma(q-1)}v(s)ds+\right.$$

$$\left.+\sum_{j=1}^{p-2}\gamma_j\int_{\xi_j}^{\eta_j}\left(\int_0^s\frac{(s-u)^{q-2}}{\Gamma(q-1)}v(u)du\right)ds-d\int_0^1\frac{(1-s)^{q-2}}{\Gamma(q-1)}v(s)ds\right]+$$

$$+\frac{1}{\Delta_2}\left[\sum_{i=1}^{m-2}\alpha_i\int_0^{\sigma_i}\frac{(\sigma_i-s)^{q-1}}{\Gamma(q)}v(s)ds+\right.$$

$$\left.+\sum_{j=1}^{p-2}r_j\int_{\xi_j}^{\eta_j}\left(\int_0^s\frac{(s-u)^{q-1}}{\Gamma(q)}v(u)du\right)ds-b\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}v(s)ds\right],$$

where

$$\Delta_1=c+d-\mu_1\neq 0,\qquad \Delta_2=a+b-\mu_2\neq 0,\qquad \Delta_3=b-\mu_3,$$

$$\mu_1=\sum_{i=1}^{m-2}\delta_i-\sum_{j=1}^{p-2}\gamma_j(\eta_j-\xi_j),\qquad \mu_2=\sum_{i=1}^{m-2}\alpha_i-\sum_{j=1}^{p-2}r_j(\eta_j-\xi_j),\tag{3.1}$$

$$\mu_3=\sum_{i=1}^{m-2}\alpha_i\sigma_i-\sum_{j=1}^{p-2}r_j\frac{\eta_j^2-\xi_j^2}{2}.$$

In the above definition, we have used Lemma 2.5 derived in [19]. For the sake of convenience, we set

$$\Lambda=\frac{1}{\Gamma(q+1)}+\max_{t\in[0,1]}\frac{|\Delta_2 t-\Delta_3|}{|\Delta_1\Delta_2|}\left[\sum_{i=1}^{m-2}|\delta_i|\frac{\sigma_i^{q-1}}{\Gamma(q)}+\sum_{j=1}^{p-2}\frac{|\gamma_j|}{\Gamma(q+1)}|\eta_j^q-\xi_j^q|+\frac{|d|}{\Gamma(q)}\right]+$$

$$+\frac{1}{|\Delta_2|}\left[\sum_{i=1}^{m-2}|\alpha_i|\frac{\sigma_i^q}{\Gamma(q+1)}+\sum_{j=1}^{p-2}\frac{|r_j|}{\Gamma(q+2)}|\eta_j^{q+1}-\xi_j^{q+1}|+\frac{|b|}{\Gamma(q+1)}\right], \tag{3.2}$$

$$\Lambda_1=\frac{1}{\Gamma(q)}+\frac{1}{|\Delta_1|}\left[\sum_{i=1}^{m-2}|\delta_i|\frac{\sigma_i^{q-1}}{\Gamma(q)}+\sum_{j=1}^{p-2}\frac{|\gamma_j|}{\Gamma(q+1)}|\eta_j^{q}-\xi_j^{q}|+\frac{|d|}{\Gamma(q)}\right], \tag{3.3}$$

$$L_1=1+\frac{1}{\Gamma(\gamma+1)}. \tag{3.4}$$

Our first existence result is concerned with the case that the multivalued map F has convex values (upper semicontinuous case) and its proof is based on Lemma 2.3.

Theorem 3.1. *Assume that*:

(H_1) $F\colon[0,1]\times\mathbb{R}^3\to\mathcal{P}(\mathbb{R})$ *is L^1-Carathéodory and has nonempty compact and convex values*;

(H_2) *there exist a function $\phi\in C([0,1],\mathbb{R}^+)$ and a nondecreasing, subhomogeneous (that is, $\Omega(\mu x)\le\mu\Omega(x)$ for all $\mu\ge 1$ and $x\in\mathbb{R}^+$) function $\Omega\colon\mathbb{R}^+\to\mathbb{R}^+$ such that* $\|F(t,x)\|_{\mathcal{P}}:=$ $:=\sup\big\{|w|: w\in F(t,x,y,z)\big\}\le\phi(t)\Omega\big(\|x\|+\|y\|+\|z\|\big)$ *for each* $(t,x,y,z)\in[0,1]\times\mathbb{R}^3$;

(H_3) *there exists a constant $M>0$ such that*

$$\frac{M}{\left(\Lambda+\dfrac{\Lambda_1}{\Gamma(2-\beta)}\right)\|\phi\|L_1\Omega(M)}>1,$$

where Λ, Λ_1 and L_1 are defined by (3.2)–(3.4).

Then the boundary-value problem (1.1) *has at least one solution on* $[0,1]$.

Proof. Define an operator $\Omega_F: C\big([0,1],\mathbb{R}\big)\to\mathcal{P}\big(C\big([0,1],\mathbb{R}\big)\big)$ by

$$\Omega_F(x)=\left\{\begin{array}{c} h\in C\big([0,1],\mathbb{R}\big):\\ h(t)=\left\{\begin{array}{l}\displaystyle\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}v(s)ds+\frac{\Delta_2t-\Delta_3}{\Delta_1\Delta_2}\left[\sum\nolimits_{i=1}^{m-2}\delta_i\int_0^{\sigma_i}\frac{(\sigma_i-s)^{q-2}}{\Gamma(q-1)}v(s)ds+\right.\\ \displaystyle\left.+\sum\nolimits_{j=1}^{p-2}\gamma_j\int_{\xi_j}^{\eta_j}\left(\int_0^s\frac{(s-u)^{q-2}}{\Gamma(q-1)}v(u)du\right)ds-d\int_0^1\frac{(1-s)^{q-2}}{\Gamma(q-1)}v(s)ds\right]+\\ \displaystyle+\frac{1}{\Delta_2}\left[\sum\nolimits_{i=1}^{m-2}\alpha_i\int_0^{\sigma_i}\frac{(\sigma_i-s)^{q-1}}{\Gamma(q)}v(s)ds+\right.\\ \displaystyle\left.+\sum\nolimits_{j=1}^{p-2}r_j\int_{\xi_j}^{\eta_j}\left(\int_0^s\frac{(s-u)^{q-1}}{\Gamma(q)}v(u)du\right)ds-b\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}v(s)ds\right]\end{array}\right.\end{array}\right\}$$

for $v\in S_{F,x}$, where $S_{F,x}:=\big\{v\in L^1([0,1],\mathbb{R})\colon v(t)\in F(t,x(t),{}^cD^\beta x(t),I^\gamma x(t))$ for a.e. $t\in$ $\in[0,1]\big\}$ denotes the set of selections of F. We split the proof into several steps to show that the operator Ω_F satisfies the assumptions of Lemma 2.3. As a first step, we show that Ω_F *is convex for each* $x\in C\big([0,1],\mathbb{R}\big)$. This step is obvious since $S_{F,x}$ is convex (F has convex values), and therefore we omit the proof.

In the second step, we show that Ω_F *maps bounded sets* (*balls*) *into bounded sets in* $C\big([0,1],\mathbb{R}\big)$. For a positive number ρ, let $B_\rho=\big\{x\in C\big([0,1],\mathbb{R}\big):\|x\|\le\rho\big\}$ be a bounded ball in $C\big([0,1],\mathbb{R}\big)$.

Then, for each $h \in \Omega_F(x), x \in B_\rho$, there exists $v \in S_{F,x}$ such that

$$h(t) = \int\limits_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} v(s)ds + \frac{\Delta_2 t - \Delta_3}{\Delta_1 \Delta_2} \left[\sum_{i=1}^{m-2} \delta_i \int\limits_0^{\sigma_i} \frac{(\sigma_i - s)^{q-2}}{\Gamma(q-1)} v(s)ds + \right.$$

$$\left. + \sum_{j=1}^{p-2} \gamma_j \int\limits_{\xi_j}^{\eta_j} \left(\int\limits_0^s \frac{(s-u)^{q-2}}{\Gamma(q-1)} v(u)du \right) ds - d \int\limits_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} v(s)ds \right] +$$

$$+ \frac{1}{\Delta_2} \left[\sum_{i=1}^{m-2} \alpha_i \int\limits_0^{\sigma_i} \frac{(\sigma_i - s)^{q-1}}{\Gamma(q)} v(s)ds + \right.$$

$$\left. + \sum_{j=1}^{p-2} r_j \int\limits_{\xi_j}^{\eta_j} \left(\int\limits_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} v(u)du \right) ds - b \int\limits_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} v(s)ds \right].$$

Then, for $t \in [0,1]$, we have

$$|h(t)| \leq \int\limits_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} |v(s)|ds + \max_{t\in[0,1]} \frac{|\Delta_2 t - \Delta_3|}{|\Delta_1 \Delta_2|} \left[\sum_{i=1}^{m-2} |\delta_i| \int\limits_0^{\sigma_i} \frac{(\sigma_i - s)^{q-2}}{\Gamma(q-1)} |v(s)|ds + \right.$$

$$\left. + \sum_{j=1}^{p-2} |\gamma_j| \int\limits_{\xi_j}^{\eta_j} \left(\int\limits_0^s \frac{(s-u)^{q-2}}{\Gamma(q-1)} |v(u)|du \right) ds + |d| \int\limits_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} |v(s)|ds \right] +$$

$$+ \frac{1}{\Delta_2} \left[\sum_{i=1}^{m-2} |\alpha_i| \int\limits_0^{\sigma_i} \frac{(\sigma_i - s)^{q-1}}{\Gamma(q)} |v(s)|ds + \right.$$

$$\left. + \sum_{j=1}^{p-2} |r_j| \int\limits_{\xi_j}^{\eta_j} \left(\int\limits_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} |v(u)|du \right) ds + |b| \int\limits_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} |v(s)|ds \right] \leq$$

$$\leq \left(\frac{1}{\Gamma(q+1)} + \max_{t\in[0,1]} \frac{|\Delta_2 t - \Delta_3|}{|\Delta_1 \Delta_2|} \left[\sum_{i=1}^{m-2} |\delta_i| \frac{\sigma_i^{q-1}}{\Gamma(q)} + \sum_{j=1}^{p-2} \frac{|\gamma_j|}{\Gamma(q+1)} |\eta_j^q - \xi_j^q| + \frac{|d|}{\Gamma(q)} \right] + \right.$$

$$\left. + \frac{1}{|\Delta_2|} \left[\sum_{i=1}^{m-2} |\alpha_i| \frac{\sigma_i^q}{\Gamma(q+1)} + \sum_{j=1}^{p-2} \frac{|r_j|}{\Gamma(q+2)} |\eta_j^{q+1} - \xi_j^{q+1}| + \frac{|b|}{\Gamma(q+1)} \right] \right) \|\phi\| \Omega(L_1 \|x\|_X) \leq$$

$$\leq \Lambda \|\phi\| L_1 \Omega(\|x\|_X),$$

which, on taking the norm for $t \in [0,1]$ yields $\|h\| \le \Lambda\|\phi\|L_1\Omega(\|x\|_X)$. Also, we get

$$|h'(t)| \le \int\limits_0^t \frac{(t-s)^{q-2}}{\Gamma(q-1)}|v(s)|ds + \frac{1}{|\Delta_1|}\left[\sum_{i=1}^{m-2}|\delta_i|\int\limits_0^{\sigma_i}\frac{(\sigma_i-s)^{q-2}}{\Gamma(q-1)}|v(s)|ds + \right.$$

$$\left. + \sum_{j=1}^{p-2}|\gamma_j|\int\limits_{\xi_j}^{\eta_j}\left(\int\limits_0^s \frac{(s-u)^{q-2}}{\Gamma(q-1)}|v(u)|du\right)ds + |d|\int\limits_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)}|v(s)|ds\right] \le$$

$$\le \left\{\frac{1}{\Gamma(q)} + \frac{1}{|\Delta_1|}\left[\sum_{i=1}^{m-2}|\delta_i|\frac{\sigma_i^{q-1}}{\Gamma(q)} + \sum_{j=1}^{p-2}\frac{|\gamma_j|}{\Gamma(q+1)}|\eta_j^q - \xi_j^q| + \frac{|d|}{\Gamma(q)}\right]\right\}\|\phi\|\Omega(L_1\|x\|_X) \le$$

$$\le \Lambda_1\|\phi\|L_1\Omega(\|x\|_X),$$

which, in view of the definition of Caputo fractional derivative with $0 < \beta < 1$, implies that

$$\left|{}^cD^\beta h(t)\right| \le \int\limits_0^t \frac{(t-s)^{-\beta}}{\Gamma(1-\beta)}\left|h'(s)\right|ds \le \Lambda_1\|\phi\|L_1\Omega\big(\|x\|_X\big)\int\limits_0^t \frac{(t-s)^{-\beta}}{\Gamma(1-\beta)}ds \le$$

$$\le \frac{1}{\Gamma(2-\beta)}\Lambda_1\|\phi\|L_1\Omega\big(\|x\|_X\big).$$

Hence,

$$\|h\|_X = \|h\| + \|{}^cD^\beta h\| \le \left(\Lambda + \frac{\Lambda_1}{\Gamma(2-\beta)}\right)\|\phi\|L_1\Omega(r). \tag{3.5}$$

Now we show that Ω_F *maps bounded sets into equicontinuous sets of* $C\big([0,1],\mathbb{R}\big)$. Let $t_1, t_2 \in$ $\in [0,1]$ with $t_1 < t_2$ and $x \in B_\rho$. For each $h \in \Omega_F(x)$, we obtain

$$\left|h(t_2) - h(t_1)\right| \le$$

$$\le \left|\frac{1}{\Gamma(\alpha)}\int\limits_0^{t_1}[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}]|v(s)|ds + \frac{1}{\Gamma(\alpha)}\int\limits_{t_1}^{t_2}(t_2-s)^{\alpha-1}|v(s)|ds\right| +$$

$$+\frac{|t_2-t_1|}{|\Delta_1|}\left[\sum_{i=1}^{m-2}|\delta_i|\int\limits_0^{\sigma_i}\frac{(\sigma_i-s)^{q-2}}{\Gamma(q-1)}|v(s)|ds\ + \right.$$

$$\left. + \sum_{j=1}^{p-2}|\gamma_j|\int\limits_{\xi_j}^{\eta_j}\left(\int\limits_0^s \frac{(s-u)^{q-2}}{\Gamma(q-1)}|v(u)|du\right)ds + |d|\int\limits_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)}|v(s)|ds\right] \le$$

$$\le \frac{\|\phi\|L_1\Omega(r)}{\Gamma(\alpha+1)}|2(t_2-t_1)^\alpha + t_1^\alpha - t_2^\alpha| +$$

$$+\frac{\|\phi\|L_1\Omega(r)|t_2-t_1|}{|\Delta_1|}\left[\sum_{i=1}^{m-2}|\delta_i|\frac{\sigma_i^{q-1}}{\Gamma(q)}+\sum_{j=1}^{p-2}\frac{|\gamma_j|}{\Gamma(q+1)}|\eta_j^q-\xi_j^q|+\frac{|d|}{\Gamma(q)}\right]$$

and

$$\left|{}^cD^\beta h(t_2)-{}^cD^\beta h(t_1)\right|\le$$

$$\le\frac{1}{\Gamma(1-\beta)}\left\{\int\limits_0^{t_1}\frac{\left|(t_1-s)^\beta-(t_2-s)^\beta\right|}{(t_1-s)^\beta(t_2-s)^\beta}|h'(x)(s)|ds+\int\limits_{t_1}^{t_2}|(t_2-s)^{-\beta}||h'(x)(s)|ds\right\}\le$$

$$\le\frac{\Lambda_1}{\Gamma(1-\beta)}\left\{\int\limits_0^{t_1}\frac{\left|(t_1-s)^\beta-(t_2-s)^\beta\right|}{(t_1-s)^\beta(t_2-s)^\beta}ds+\int\limits_{t_1}^{t_2}\left|(t_2-s)^{-\beta}\right|ds\right\}\|\phi\|L_1\Omega(r).$$

Obviously, the right-hand side of each of the above two inequalities tends to zero independently of $x\in\mathcal{B}_\rho$ as $t_2-t_1\to 0$. Since F satisfies the above assumptions, therefore it follows by the Arzelá–Ascoli theorem that $F\colon C\big([0,1],\mathbb{R}\big)\to\mathcal{P}\big(C([0,1],\mathbb{R})\big)$ is completely continuous.

In next step, we show that Ω_F *is upper semicontinuous.* To this end it is sufficient to show that Ω_F *has a closed graph,* by Lemma 2.1. Let $x_n\to x_*, h_n\in\Omega_F(x_n)$ and $h_n\to h_*$. Then we need to show that $h_*\in\Omega_F(x_*)$. Associated with $h_n\in\Omega_F(x_n)$, there exists $v_n\in S_{F,x_n}$ such that, for each $t\in[0,1]$, we get

$$h_n(t)=\int\limits_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}v_n(s)ds+\frac{\Delta_2t-\Delta_3}{\Delta_1\Delta_2}\left[\sum_{i=1}^{m-2}\delta_i\int\limits_0^{\sigma_i}\frac{(\sigma_i-s)^{q-2}}{\Gamma(q-1)}v_n(s)ds+\right.$$

$$\left.+\sum_{j=1}^{p-2}\gamma_j\int\limits_{\xi_j}^{\eta_j}\left(\int\limits_0^s\frac{(s-u)^{q-2}}{\Gamma(q-1)}v_n(u)du\right)ds-d\int\limits_0^1\frac{(1-s)^{q-2}}{\Gamma(q-1)}v_n(s)ds\right]+$$

$$+\frac{1}{\Delta_2}\left[\sum_{i=1}^{m-2}\alpha_i\int\limits_0^{\sigma_i}\frac{(\sigma_i-s)^{q-1}}{\Gamma(q)}v_n(s)ds+\right.$$

$$\left.+\sum_{j=1}^{p-2}r_j\int\limits_{\xi_j}^{\eta_j}\left(\int\limits_0^s\frac{(s-u)^{q-1}}{\Gamma(q)}v_n(u)du\right)ds-b\int\limits_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}v_n(s)ds\right].$$

Thus it suffices to show that there exists $v_*\in S_{F,x_*}$ such that, for each $t\in[0,1]$,

$$h_*(t)=\int\limits_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}v_*(s)ds+\frac{\Delta_2t-\Delta_3}{\Delta_1\Delta_2}\left[\sum_{i=1}^{m-2}\delta_i\int\limits_0^{\sigma_i}\frac{(\sigma_i-s)^{q-2}}{\Gamma(q-1)}v_*(s)ds+\right.$$

$$+\sum_{j=1}^{p-2}\gamma_j\int\limits_{\xi_j}^{\eta_j}\left(\int\limits_0^s\frac{(s-u)^{q-2}}{\Gamma(q-1)}v_*(u)du\right)ds-d\int\limits_0^1\frac{(1-s)^{q-2}}{\Gamma(q-1)}v_*(s)ds\Bigg]+$$

$$+\frac{1}{\Delta_2}\left[\sum_{i=1}^{m-2}\alpha_i\int\limits_0^{\sigma_i}\frac{(\sigma_i-s)^{q-1}}{\Gamma(q)}v_*(s)ds+\right.$$

$$\left.+\sum_{j=1}^{p-2}r_j\int\limits_{\xi_j}^{\eta_j}\left(\int\limits_0^s\frac{(s-u)^{q-1}}{\Gamma(q)}v_*(u)du\right)ds-b\int\limits_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}v_*(s)ds\right].$$

Let us consider the linear operator $\Theta\colon L^1\big([0,1],\mathbb{R}\big)\to C\big([0,1],\mathbb{R}\big)$ given by

$$v\mapsto\Theta(v)(t)=\int\limits_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}v(s)ds+\frac{\Delta_2t-\Delta_3}{\Delta_1\Delta_2}\left[\sum_{i=1}^{m-2}\delta_i\int\limits_0^{\sigma_i}\frac{(\sigma_i-s)^{q-2}}{\Gamma(q-1)}v(s)ds+\right.$$

$$\left.+\sum_{j=1}^{p-2}\gamma_j\int\limits_{\xi_j}^{\eta_j}\left(\int\limits_0^s\frac{(s-u)^{q-2}}{\Gamma(q-1)}v(u)du\right)ds-d\int\limits_0^1\frac{(1-s)^{q-2}}{\Gamma(q-1)}v(s)ds\right]+$$

$$+\frac{1}{\Delta_2}\left[\sum_{i=1}^{m-2}\alpha_i\int\limits_0^{\sigma_i}\frac{(\sigma_i-s)^{q-1}}{\Gamma(q)}v(s)ds+\right.$$

$$\left.+\sum_{j=1}^{p-2}r_j\int\limits_{\xi_j}^{\eta_j}\left(\int\limits_0^s\frac{(s-u)^{q-1}}{\Gamma(q)}v(u)du\right)ds-b\int\limits_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}v(s)ds\right].$$

Since

$$\|h_n(t)-h_*(t)\|=$$

$$=\left\|\int\limits_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}(v_n(s)-v_*(s))ds+\frac{\Delta_2t-\Delta_3}{\Delta_1\Delta_2}\left[\sum_{i=1}^{m-2}\delta_i\int\limits_0^{\sigma_i}\frac{(\sigma_i-s)^{q-2}}{\Gamma(q-1)}\big(v_n(s)-v_*(s)\big)ds+\right.\right.$$

$$\left.+\sum_{j=1}^{p-2}\gamma_j\int\limits_{\xi_j}^{\eta_j}\left(\int\limits_0^s\frac{(s-u)^{q-2}}{\Gamma(q-1)}\big(v_n(u)-v_*(u)\big)du\right)ds-d\int\limits_0^1\frac{(1-s)^{q-2}}{\Gamma(q-1)}\big(v_n(s)-v_*(s)\big)ds\right]+$$

$$+\frac{1}{\Delta_2}\left[\sum_{i=1}^{m-2}\alpha_i\int\limits_0^{\sigma_i}\frac{(\sigma_i-s)^{q-1}}{\Gamma(q)}\big(v_n(s)-v_*(s)\big)ds+\right.$$

$$+\sum_{j=1}^{p-2} r_j \int_{\xi_j}^{\eta_j} \left(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} \big(v_n(u) - v_*(u)\big) du \right) ds - b \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} (v_n(s) - v_*(s)) ds \Bigg] \Bigg\| \to 0$$

as $n \to \infty$, therefore, it follows by Lemma 2.2 that $\Theta \circ S_F$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,x_n})$. As $x_n \to x_*$, we obtain

$$h_*(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} v_*(s) ds + \frac{\Delta_2 t - \Delta_3}{\Delta_1 \Delta_2} \left[\sum_{i=1}^{m-2} \delta_i \int_0^{\sigma_i} \frac{(\sigma_i - s)^{q-2}}{\Gamma(q-1)} v_*(s) ds + \right.$$

$$\left. + \sum_{j=1}^{p-2} \gamma_j \int_{\xi_j}^{\eta_j} \left(\int_0^s \frac{(s-u)^{q-2}}{\Gamma(q-1)} v_*(u) du \right) ds - d \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} v_*(s) ds \right] +$$

$$+ \frac{1}{\Delta_2} \left[\sum_{i=1}^{m-2} \alpha_i \int_0^{\sigma_i} \frac{(\sigma_i - s)^{q-1}}{\Gamma(q)} v_*(s) ds + \right.$$

$$\left. + \sum_{j=1}^{p-2} r_j \int_{\xi_j}^{\eta_j} \left(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} v_*(u) du \right) ds - b \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} v_*(s) ds \right]$$

for some $v_* \in S_{F,x_*}$.

Finally, we show there exists an open set $U \subseteq C([0,1],\mathbb{R})$ with $x \notin \Omega_F(x)$ for any $\theta \in (0,1)$ and all $x \in \partial U$. Let $\theta \in (0,1)$ and $x \in \theta\Omega_F(x)$. Then there exists $v \in L^1([0,1],\mathbb{R})$ with $v \in S_{F,x}$ such that, for $t \in [0,1]$, we get

$$|x(t)| \le \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} |v(s)| ds + \max_{t \in [0,1]} \frac{|\Delta_2 t - \Delta_3|}{|\Delta_1 \Delta_2|} \left[\sum_{i=1}^{m-2} |\delta_i| \int_0^{\sigma_i} \frac{(\sigma_i - s)^{q-2}}{\Gamma(q-1)} |v(s)| ds + \right.$$

$$\left. + \sum_{j=1}^{p-2} |\gamma_j| \int_{\xi_j}^{\eta_j} \left(\int_0^s \frac{(s-u)^{q-2}}{\Gamma(q-1)} |v(u)| du \right) ds + |d| \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} |v(s)| ds \right] +$$

$$+ \frac{1}{\Delta_2} \left[\sum_{i=1}^{m-2} |\alpha_i| \int_0^{\sigma_i} \frac{(\sigma_i - s)^{q-1}}{\Gamma(q)} |v(s)| ds + \right.$$

$$\left. + \sum_{j=1}^{p-2} |r_j| \int_{\xi_j}^{\eta_j} \left(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} |v(u)| du \right) ds + |b| \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} |v(s)| ds \right] \le$$

$$\leq \left(\frac{1}{\Gamma(q+1)} + \max_{t\in[0,1]} \frac{|\Delta_2 t - \Delta_3|}{|\Delta_1 \Delta_2|} \left[\sum_{i=1}^{m-2} |\delta_i| \frac{\sigma_i^{q-1}}{\Gamma(q)} + \sum_{j=1}^{p-2} \frac{|\gamma_j|}{\Gamma(q+1)} |\eta_j^q - \xi_j^q| + \frac{|d|}{\Gamma(q)} \right] + \right.$$

$$\left. + \frac{1}{|\Delta_2|} \left[\sum_{i=1}^{m-2} |\alpha_i| \frac{\sigma_i^q}{\Gamma(q+1)} + \sum_{j=1}^{p-2} \frac{|r_j|}{\Gamma(q+2)} |\eta_j^{q+1} - \xi_j^{q+1}| + \frac{|b|}{\Gamma(q+1)} \right] \right) \|\phi\| L_1 \Omega\big(\|x\|_X\big) \leq$$

$$\leq \Lambda \|\phi\| L_1 \Omega\big(\|x\|_X\big),$$

which on taking the norm for $t \in [0,1]$ yields

$$\|x\| \leq \Lambda \|\phi\| L_1 \Omega\big(\|x\|_X\big).$$

In a similar manner, one can obtain that $|x'(t)| \leq \Lambda_1 \|\phi\| L_1 \Omega(\|x\|_X)$. In consequence, we have

$$\|{}^c D^\beta x(t)| \leq \int_0^t \frac{(t-s)^{-\beta}}{\Gamma(1-\beta)} \|x'(s)| ds \leq \frac{\Lambda_1}{\Gamma(2-\beta)} \|\phi\| L_1 \Omega(\|x\|_X).$$

Hence,

$$\|x\|_X = \|x\| + \|{}^c D^\beta x\| \leq \left(\Lambda + \frac{\Lambda_1}{\Gamma(2-\beta)} \right) \|\phi\| L_1 \Omega(\|x\|_X), \tag{3.6}$$

which implies that

$$\frac{\|x\|_X}{\left(\Lambda + \dfrac{\Lambda_1}{\Gamma(2-\beta)} \right) \|\phi\| L_1 \Omega(\|x\|_X)} \leq 1.$$

In view of (H_3), there exists M such that $\|x\| \neq M$. Let us set

$$U = \big\{ x \in C\big([0,1], \mathbb{R}\big) \colon \|x\| < M \big\}.$$

Note that the operator $\Omega_F \colon \overline{U} \to \mathcal{P}\big(C([0,1],\mathbb{R})\big)$ is upper semicontinuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x \in \theta \Omega_F(x)$ for some $\theta \in (0,1)$. Consequently, by Lemma 2.3, we deduce that Ω_F has a fixed point $x \in \overline{U}$ which is a solution of the problem (1.1).

Theorem 3.1 is proved.

Now we prove the existence of solutions for the problem (1.1) with a nonconvex valued right-hand side of the inclusion (Lipschitz case) by applying a fixed point theorem for multivalued maps due to Covitz and Nadler [32]. Let us briefly recall some preliminary concepts needed to establish the desired result.

Let $H_d \colon \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R} \cup \{\infty\}$ be a mapping defined by

$$H_d(A,B) = \max\Big\{ \sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b) \Big\},$$

where $d(A,b) = \inf_{a\in A} d(a;b)$ and $d(a,B) = \inf_{b\in B} d(a;b)$. Then $(\mathcal{P}_{b,cl}(X), H_d)$ is a metric space and $(\mathcal{P}_{cl}(X), H_d)$ is a generalized metric space (see [21]).

Definition 3.2. *A multivalued operator* $N: X \to \mathcal{P}_{cl}(X)$ *is called* γ*-Lipschitz if and only if there exists* $\gamma > 0$ *such that* $H_d(N(x), N(y)) \le \gamma d(x, y)$ *for each* $x, y \in X$, *and a contraction if and only if it is* γ*-Lipschitz with* $\gamma < 1$.

Lemma 3.1 [32]. *Let* (X, d) *be a complete metric space. If* $N: X \to \mathcal{P}_{cl}(X)$ *is a contraction, then* $\operatorname{Fix} N \neq \varnothing$.

Theorem 3.2. *Suppose that the following assumptions hold*:

(A_1) $F: [0,1] \times \mathbb{R}^3 \to \mathcal{P}_{cp}(\mathbb{R})$ *is such that* $F(\cdot, x(t), {}^cD^\beta x(t), I^\gamma x(t)): [0,1] \to \mathcal{P}_{cp}(\mathbb{R})$ *is measurable for each* $x \in \mathbb{R}$;

(A_2) $H_d(F(t,x,y,z), F(t,\bar{x},\bar{y},\bar{z})) \le \varrho(t)\big[|x-\bar{x}| + |y-\bar{y}| + |z-\bar{z}|\big]$ *for almost all* $t \in [0,1]$ *and* $x, y, z, \bar{x}, \bar{y}, \bar{z} \in \mathbb{R}$ *with* $\varrho \in C\big([0,1], \mathbb{R}^+\big)$ *and* $d(0, F(t,0,0,0)) \le \varrho(t)$ *for almost all* $t \in [0,1]$. *Then the boundary-value problem* (1.1) *has at least one solution on* $[0,1]$ *if*

$$\|\varrho\| L_1 \left(\Lambda + \frac{\Lambda_1}{\Gamma(2-\beta)}\right) < 1,$$

where Λ, Λ_1 *and* L_1 *are defined by* (3.2)–(3.4).

Proof. Notice that the set $S_{F,x}$ is nonempty for each $x \in C([0,1], \mathbb{R})$ by the assumption (A_1), so F has a measurable selection (see Theorem III.6 [33]). Now we show that the operator $\Omega_F: C\big([0,1],\mathbb{R}\big) \to \mathcal{P}\big(C([0,1],\mathbb{R})\big)$ (defined in the beginning of the proof of Theorem 3.1) satisfies the assumptions of Lemma 3.1. To show that $\Omega_F(x) \in \mathcal{P}_{cl}\big((C[0,1],\mathbb{R})\big)$ for each $x \in C\big([0,1],\mathbb{R}\big)$, let $\{u_n\}_{n\ge 0} \in \Omega_F(x)$ be such that $u_n \to u$ $(n \to \infty)$ in $C([0,1],\mathbb{R})$. Then $u \in C\big([0,1],\mathbb{R}\big)$ and there exists $v_n \in S_{F,x}$ such that, for each $t \in [0,1]$,

$$u_n(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} v_n(s)ds + \frac{\Delta_2 t - \Delta_3}{\Delta_1 \Delta_2} \left[\sum_{i=1}^{m-2} \delta_i \int_0^{\sigma_i} \frac{(\sigma_i - s)^{q-2}}{\Gamma(q-1)} v_n(s)ds + \right.$$

$$\left. + \sum_{j=1}^{p-2} \gamma_j \int_{\xi_j}^{\eta_j} \left(\int_0^s \frac{(s-u)^{q-2}}{\Gamma(q-1)} v_n(u)du\right) ds - d \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} v_n(s)ds \right] +$$

$$+ \frac{1}{\Delta_2} \left[\sum_{i=1}^{m-2} \alpha_i \int_0^{\sigma_i} \frac{(\sigma_i - s)^{q-1}}{\Gamma(q)} v_n(s)ds + \right.$$

$$\left. + \sum_{j=1}^{p-2} r_j \int_{\xi_j}^{\eta_j} \left(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} v_n(u)du\right) ds - b \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} v_n(s)ds \right].$$

As F has compact values, we pass onto a subsequence (if necessary) to obtain that v_n converges to v in $L^1([0,1],\mathbb{R})$. Thus, $v \in S_{F,x}$ and, for each $t \in [0,1]$, we have

$$u_n(t) \to u(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} v(s)ds + \frac{\Delta_2 t - \Delta_3}{\Delta_1 \Delta_2} \left[\sum_{i=1}^{m-2} \delta_i \int_0^{\sigma_i} \frac{(\sigma_i - s)^{q-2}}{\Gamma(q-1)} v(s)ds + \right.$$

$$+\sum_{j=1}^{p-2}\gamma_j\int\limits_{\xi_j}^{\eta_j}\left(\int\limits_0^s\frac{(s-u)^{q-2}}{\Gamma(q-1)}v(u)du\right)ds-d\int\limits_0^1\frac{(1-s)^{q-2}}{\Gamma(q-1)}v(s)ds\Bigg]+$$

$$+\frac{1}{\Delta_2}\Bigg[\sum_{i=1}^{m-2}\alpha_i\int\limits_0^{\sigma_i}\frac{(\sigma_i-s)^{q-1}}{\Gamma(q)}v(s)ds+$$

$$+\sum_{j=1}^{p-2}r_j\int\limits_{\xi_j}^{\eta_j}\left(\int\limits_0^s\frac{(s-u)^{q-1}}{\Gamma(q)}v(u)du\right)ds-b\int\limits_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}v(s)ds\Bigg].$$

Hence, $u\in\Omega_F(x)$.

Next we show that there exists $\delta<1$ such that

$$H_d(\Omega_F(x),\Omega_F(\bar{x}))\leq\delta\|x-\bar{x}\|_X \quad \text{for each} \quad x,\bar{x}\in C^2\big([0,1],\mathbb{R}\big).$$

Let $x,\bar{x}\in C^2\big([0,1],\mathbb{R}\big)$ and $h_1\in\Omega_F(x)$. Then there exists $v_1(t)\in F(t,x(t),{}^cD^\beta x(t),I^\gamma x(t))$ such that, for each $t\in[0,1]$,

$$h_1(t)=\int\limits_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}v_1(s)ds+\frac{\Delta_2 t-\Delta_3}{\Delta_1\Delta_2}\Bigg[\sum_{i=1}^{m-2}\delta_i\int\limits_0^{\sigma_i}\frac{(\sigma_i-s)^{q-2}}{\Gamma(q-1)}v_1(s)ds+$$

$$+\sum_{j=1}^{p-2}\gamma_j\int\limits_{\xi_j}^{\eta_j}\left(\int\limits_0^s\frac{(s-u)^{q-2}}{\Gamma(q-1)}v_1(u)du\right)ds-d\int\limits_0^1\frac{(1-s)^{q-2}}{\Gamma(q-1)}v_1(s)ds\Bigg]+$$

$$+\frac{1}{\Delta_2}\Bigg[\sum_{i=1}^{m-2}\alpha_i\int\limits_0^{\sigma_i}\frac{(\sigma_i-s)^{q-1}}{\Gamma(q)}v_1(s)ds+$$

$$+\sum_{j=1}^{p-2}r_j\int\limits_{\xi_j}^{\eta_j}\left(\int\limits_0^s\frac{(s-u)^{q-1}}{\Gamma(q)}v_1(u)du\right)ds-b\int\limits_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}v_1(s)ds\Bigg].$$

By (A_2), we have

$$H_d\big(F(t,x,y,z),F(t,\bar{x},\bar{y},\bar{z})\big)\leq\varrho(t)\big[|x-\bar{x}|+|y-\bar{y}|+|z-\bar{z}|\big].$$

So, there exists $w\in F(t,\bar{x},\bar{y},\bar{z})$ such that

$$|v_1(t)-w|\leq\varrho(t)\big[|x(t)-\bar{x}(t)|+|y(t)-\bar{y}(t)|+|z(t)-\bar{z}(t)|\big], \quad t\in[0,1].$$

Define $\mathcal{V}\colon[0,1]\to\mathcal{P}(\mathbb{R})$ by

$$\mathcal{V}(t)=\Big\{w\in\mathbb{R}\colon|v_1(t)-w|\leq\varrho(t)\big[|x(t)-\bar{x}(t)|+|y(t)-\bar{y}(t)|+|z(t)-\bar{z}(t)|\big]\Big\}.$$

Since the multivalued operator $\mathcal{V}(t)\cap F(t,\bar{x},\bar{y},\bar{z})$ is measurable (Proposition III.4 [33]), there exists a function $v_2(t)$ which is a measurable selection for $\mathcal{V}(t)\cap F(t,\bar{x},\bar{y},\bar{z})$. So, $v_2(t)\in F(t,\bar{x},\bar{y},\bar{z})$ and, for each $t\in[0,1]$, we get $|v_1(t)-v_2(t)|\le \varrho(t)\big[|x(t)-\bar{x}(t)|+|y(t)-\bar{y}(t)|+|z(t)-\bar{z}(t)|\big]$.

For each $t\in[0,1]$, let us define

$$h_2(t)=\int\limits_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}v_2(s)ds+\frac{\Delta_2 t-\Delta_3}{\Delta_1\Delta_2}\left[\sum_{i=1}^{m-2}\delta_i\int\limits_0^{\sigma_i}\frac{(\sigma_i-s)^{q-2}}{\Gamma(q-1)}v_2(s)ds+\right.$$

$$\left.+\sum_{j=1}^{p-2}\gamma_j\int\limits_{\xi_j}^{\eta_j}\left(\int\limits_0^s\frac{(s-u)^{q-2}}{\Gamma(q-1)}v_2(u)du\right)ds-d\int\limits_0^1\frac{(1-s)^{q-2}}{\Gamma(q-1)}v_2(s)ds\right]+$$

$$+\frac{1}{\Delta_2}\left[\sum_{i=1}^{m-2}\alpha_i\int\limits_0^{\sigma_i}\frac{(\sigma_i-s)^{q-1}}{\Gamma(q)}v_2(s)ds+\right.$$

$$\left.+\sum_{j=1}^{p-2}r_j\int\limits_{\xi_j}^{\eta_j}\left(\int\limits_0^s\frac{(s-u)^{q-1}}{\Gamma(q)}v_2(u)du\right)ds-b\int\limits_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}v_2(s)ds\right].$$

Thus,

$$\big|h_1(t)-h_2(t)\big|=\int\limits_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}|v_1(s)-v_2(s)|ds+$$

$$+\max_{t\in[0,1]}\frac{|\Delta_2 t-\Delta_3|}{|\Delta_1\Delta_2|}\left[\sum_{i=1}^{m-2}|\delta_i|\int\limits_0^{\sigma_i}\frac{(\sigma_i-s)^{q-2}}{\Gamma(q-1)}|v_1(s)-v_2(s)|ds+\right.$$

$$+\sum_{j=1}^{p-2}|\gamma_j|\int\limits_{\xi_j}^{\eta_j}\left(\int\limits_0^s\frac{(s-u)^{q-2}}{\Gamma(q-1)}|v_1(u)-v_2(u)|du\right)ds+$$

$$\left.+|d|\int\limits_0^1\frac{(1-s)^{q-2}}{\Gamma(q-1)}|v_1(s)-v_2(s)|ds\right]+\frac{1}{|\Delta_2|}\left[\sum_{i=1}^{m-2}|\alpha_i|\int\limits_0^{\sigma_i}\frac{(\sigma_i-s)^{q-1}}{\Gamma(q)}|v_1(s)-v_2(s)|ds+\right.$$

$$\left.+\sum_{j=1}^{p-2}|r_j|\int\limits_{\xi_j}^{\eta_j}\left(\int\limits_0^s\frac{(s-u)^{q-1}}{\Gamma(q)}|v_1(u)-v_2(u)|du\right)ds+|b|\int\limits_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}|v_1(s)-v_2(s)|(s)ds\right]\le$$

$$\le\|\varrho\|\left(\frac{1}{\Gamma(q+1)}+\max_{t\in[0,1]}\frac{|\Delta_2 t-\Delta_3|}{|\Delta_1\Delta_2|}\left[\sum_{i=1}^{m-2}|\delta_i|\frac{\sigma_i^{q-1}}{\Gamma(q)}+\sum_{j=1}^{p-2}\frac{|\gamma_j|}{\Gamma(q+1)}|\eta_j^q-\xi_j^q|+\frac{|d|}{\Gamma(q)}\right]+\right.$$

$$+\frac{1}{|\Delta_2|}\left[\sum_{i=1}^{m-2}|\alpha_i|\frac{\sigma_i^q}{\Gamma(q+1)}+\sum_{j=1}^{p-2}\frac{|r_j|}{\Gamma(q+2)}|\eta_j^{q+1}-\xi_j^{q+1}|++\frac{|b|}{\Gamma(q+1)}\right]\Bigg)\Big[\|x-\bar{x}\|_X+$$

$$+\frac{1}{\Gamma(\gamma+1)}\|x-\bar{x}\|_X\Big]\leq\|\varrho\|\Lambda L_1\|x-\bar{x}\|_X.$$

Hence, $\|h_1-h_2\|\leq\|\varrho\|\Lambda L_1\|x-\bar{x}\|_X$. In a similar manner, we obtain

$$|h_1'(t)-h_2'(t)|\leq\|\varrho\|\left\{\frac{1}{\Gamma(q)}+\frac{1}{|\Delta_1|}\left[\sum_{i=1}^{m-2}|\delta_i|\frac{\sigma_i^{q-1}}{\Gamma(q)}+\sum_{j=1}^{p-2}\frac{|\gamma_j|}{\Gamma(q+1)}|\eta_j^q-\xi_j^q|+\frac{|d|}{\Gamma(q)}\right]\right\}\times$$

$$\times\left(1+\frac{1}{\Gamma(\gamma+1)}\right)\|x-\bar{x}\|_X=\|\varrho\|\Lambda_1L_1\|x-\bar{x}\|_X$$

and

$$\left|{}^cD^\beta h_1(t)-{}^cD^\beta h_2(t)\right|\leq\int_0^t\frac{(t-s)^{-\beta}}{\Gamma(1-\beta)}|h_1'(t)-h_2'(t)|ds\leq\frac{1}{\Gamma(2-\beta)}\|\varrho\|\Lambda_1L_1\|x-\bar{x}\|_X.$$

Thus,

$$\|h_1-h_2\|_X\leq\|\varrho\|L_1\left(\Lambda+\frac{\Lambda_1}{\Gamma(2-\beta)}\right)\|x-\bar{x}\|_X.$$

Analogously, interchanging the roles of x and $\overline{x}$, we obtain

$$H_d(\Omega_F(x),\Omega_F(\bar{x}))\leq\|\varrho\|L_1\left(\Lambda+\frac{\Lambda_1}{\Gamma(2-\beta)}\right)\|x-\bar{x}\|_X.$$

Since Ω_F is a contraction, it follows by Lemma 3.1 that Ω_F has a fixed point x which is a solution of (1.1).

Theorem 3.2 is proved.

Examples. Consider the nonlocal integro-multipoint boundary multivalued (inclusion) problem

$${}^cD^{8/5}x(t)\in F(t,x(t),{}^cD^{3/4}x(t),I^{1/2}x(t)),\quad t\in[0,1],$$

$$ax(0)+bx(1)=\sum_{i=1}^{3}\alpha_ix(\sigma_i)+\sum_{j=1}^{4}r_j\int_{\xi_j}^{\eta_i}x(s)ds, \tag{3.7}$$

$$cx'(0)+dx'(1)=\sum_{i=1}^{3}\delta_ix'(\sigma_i)+\sum_{j=1}^{4}\gamma_j\int_{\xi_j}^{\eta_j}x'(s)ds.$$

Here, $q=8/5,\ \beta=3/4,\ \gamma=1/2,\ m=5,\ p=6,\ \sigma_1=1/15,\ \sigma_2=2/15,\ \sigma_3=3/15,\ \xi_1=1/4,$ $\eta_1=5/16,\ \xi_2=6/16,\ \eta_2=7/16,\ \xi_3=8/16,\ \eta_3=9/16,\ \xi_4=10/16,\ \eta_4=11/16,\ \alpha_1=2,$ $\alpha_2=1,\ \alpha_3=1,\delta_1=2,\ \delta_2=3,\delta_3=-1,\ r_1=1,\ r_2=1,\ r_3=3,\ r_4=-2,\ \gamma_1=-3,$

$\gamma_2 = -1,\ \gamma_3 = 1,\ \gamma_4 = 2,\ a = 1,\ b = 1,\ c = -2,\ d = 4$. Using the given data in (3.1)–(3.4), we find that $\Delta_1 = -2.0625,\ \Delta_2 = -1.8125,\ \Delta_3 = 0.5939,\ \mu_1 = 4.0625,\ \mu_2 = 3.8125,\ \mu_3 = 0.4061, \Lambda = 6.6286, \Lambda_1 = 4.3363,\ L_1 = 2.1284$.

In order to illustrate Theorem 3.1, we take

$$F(t, x(t), {}^cD^{3/4}x(t), I^{1/2}x(t)) =$$

$$= \left[\frac{1}{\sqrt{t^2+144}}\left(\frac{1}{3}\sin(x(t)) + \frac{1}{2}\frac{|{}^cD^{3/4}x(t)|}{\left(1+|{}^cD^{3/4}x(t)|\right)} + \frac{1}{\pi}\tan^{-1}\left(I^{1/2}x(t)\right) + \frac{1}{2}\right),\right.$$

$$\left.\frac{1}{t+15}\left(\frac{1}{16}e^{-x^4(t)} + \frac{1}{5}\sin({}^cD^{1/2}x(t)) + \frac{1}{10}\frac{|I^{1/4}x(t)|}{\left(1+|I^{1/4}x(t)|\right)} + \frac{1}{2}\right)\right]. \tag{3.8}$$

It is easy to find that $\phi(t) = \dfrac{1}{\sqrt{t^2+144}}$ with $\|\phi\| = 1/12, \Omega(M) = 11/6$. By the condition (H_3), we find that $M > 3.7111$. Thus all the assumptions of Theorem 3.1 hold and consequently the problem (3.7) with F given by (3.8) has a solution on $[0, 1]$.

Next we illustrate Theorem 3.2 by taking the map

$$F(t, x(t), {}^cD^{3/4}x(t), I^{1/2}x(t)) =$$

$$= \left[\frac{1}{t+30}x(t) + \frac{1}{t^2+45}\cos(D^{3/4}x(t)) + \frac{1}{t^2+49}I^{1/2}x(t),\right.$$

$$\left.\frac{1}{t^2+64}\tan^{-1}(x(t)) + \frac{1}{t^2+36}{}^cD^{3/4}x(t) + \frac{1}{t^2+49}\tan^{-1}(I^{1/2}x(t)) + \frac{1}{32}\right]. \tag{3.9}$$

By the condition (A_2), we get $\varrho(t) = 1/(t+30)$ with $\|\varrho\| = 1/30$. Then $\|\varrho\| L_1[\Lambda + \Lambda_1/\Gamma(5/4)] \approx$ $\approx 0.8097 < 1$. Clearly the hypothesis of Theorem 3.2 is satisfied. Therefore, there exists at least one solution for the problem (3.7) with F given by (3.9) on $[0, 1]$.

4. Conclusions. We have addressed a more general problem of fractional order differential inclusions involving a multivalued map depending on the unknown function together with its lower-order fractional derivative and Riemann–Liouville integral, supplemented with non-separated boundary conditions containing finite many nonlocal points and strips on the given interval $[0, 1]$. The existence results obtained for the problem at hand are not only new but also yield several new results as special cases by fixing the parameters involved in the problem. Some of the these results are listed below.

We obtain the results for the inclusion problem with periodic/antiperiodic type boundary conditions of the form $x(0) = -(b/a)x(1),\ x'(0) = -(d/c)x'(1)$ by taking $r_j = \gamma_j = \alpha_j = \delta_j = 0$, $j = 1, \ldots, p$ in (1.1). Further, the results for antiperiodic boundary conditions follow with $(b/a) =$ $= 1 = (d/c)$.

Our results correspond to non-separated nonlocal multipoint and multistrip conditions, respectively, by taking $r_j = 0 = \gamma_j,\ j = 1, \ldots, p$, and $\alpha_j = 0 = \delta_j,\ j = 1, \ldots, p$, in (1.1).

For $a = c = 0, b = d = 1$, we get the results for the inclusion problem with terminal nonlocal multipoint and multistrip conditions:

$$x(1) = \sum_{i=1}^{m-2} \alpha_i x(\sigma_i) + \sum_{j=1}^{p-2} r_j \int_{\xi_j}^{\eta_i} x(s)ds, \qquad x'(1) = \sum_{i=1}^{m-2} \delta_i x'(\sigma_i) + \sum_{j=1}^{p-2} \gamma_j \int_{\xi_j}^{\eta_j} x'(s)ds.$$

Existence theorems for the inclusion problem with purely nonlocal multipoint and multistrip conditions follow by choosing $a = c = b = d = 0$ in the results of this paper.

In the scenario of generality of fractional order differential inclusions and boundary conditions, the present work is quite versatile in nature and significantly contributes to the existing literature on fractional order multivalued boundary-value problems. Moreover, a variety of new results follow from the ones obtained in this paper by specializing the parameters involved in the problem, which enhances the utility/scope of the work.

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Received 05.10.18,
after revision — 08.01.19