DOI: 10.37863/umzh.v73i7.495

UDC 515.12

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RICCI SOLITON BIHARMONIC HYPERSURFACES IN THE EUCLIDEAN SPACE

БІГАРМОНІЧНІ ГІПЕРПОВЕРХНІ СОЛІТОНІВ РІЧЧІ В ЕВКЛІДОВОМУ ПРОСТОРІ

We investigate biharmonic Ricci soliton hypersurfaces (M^n, g, ξ, λ) whose potential field ξ satisfies certain conditions. We obtain a result based on the average scalar curvature of the compact Ricci soliton hypersurface M^n where ξ is a general vector field. Then we prove that there are no proper biharmonic Ricci soliton hypersurfaces in the Euclidean space E^{n+1} provided that the potential field ξ is either a principal vector in grad $H^{\perp}$ or $\xi = \dfrac{\operatorname{grad} H}{|\operatorname{grad} H|}$.

Вивчаються бігармонічні гіперповерхні солітонів Річчі (M^n, g, ξ, λ), поле потенціалу ξ яких задовольняє певні умови. Отриманий результат базується на середній скалярній кривині гіперповерхні M^n компактного солітону Річчі, де ξ розглядається як узагальнене векторне поле. Після цього доведено, що не існує нетривіальних бігармонічних гіперповерхонь солітонів Річчі в евклідовому просторі E^{n+1}, якщо поле потенціалу ξ є або головним вектором у grad $H^{\perp}$, або $\xi = \dfrac{\operatorname{grad} H}{|\operatorname{grad} H|}$.

1. Introduction. The conception of biharmonic maps was introduced by Eells and Lemair [7] in 1983. It is denoted by $C^{\infty}(M,N)$ the space of smooth maps $\varphi\colon (M,g) \to (N,h)$ between two Riemannian manifolds. A biharmonic map $\varphi \in C^{\infty}(M,N)$ is a critical points of the bienergy functional

$$E_2\colon C^{\infty}(M,N) \to R, \qquad E_2(\varphi) = \frac{1}{2}\int\limits_M |\tau(\varphi)|^2 d\nu_g,$$

where $\tau(\varphi) = \operatorname{trac} \nabla d\varphi$ is tension field of φ. Actually, the Euler–Lagrange equation correlate to the bienergy is given by the vanishing of the bitension field

$$\tau_2(\varphi) = -\Delta\tau(\varphi) - \operatorname{trac} R^N(d\varphi(.), \tau(\varphi))d\varphi(.) = 0,$$

where R^N is curvature tensor of N. Infact, bihamonic immersions are special class of biharmonic maps. An isometric immersion $\varphi\colon (M^n,g) \longrightarrow (N^m,h)$ is called biharmonic if and only if the mean curvature vector field $\overrightarrow{H}$ satisfies:

$$0 = \triangle\overrightarrow{H} + \operatorname{trac} R^N(d\varphi(.), \overrightarrow{H})d\varphi(.).$$

Additionally, in Euclidean space biharmonic submanifold and biharmonic immersion are coinsided with each other. Also, it should be noticed that biharmonic submanifold was introduced by B. Y. Chen in the middle of 1980s. At first, it was proved the biharmonic surfaces in three dimensions Euclidean space E^3 are minimal in 1985 [2]. Later on, others geometer deal with the result and extended it. Infact, the result was developed by I. Dimitric [6] and T. Hasanis and T. Vlachos [9]. More precisely, Dimitric got progress on the result where the biharmonic hypersurfaces of E^m have at most two distinct principle curvatures. Also, T. Hasanis and T. Vlachos have extend the result when they have proved biharmonic hypersurfaces in E^4 are minimal. Consequently, according to the outcome, a challenging conjecture was made by Chen [3]:

original biharmonic conjecture: ”the only biharmonic submanifolds of Euclidean space are minimal ones”.

Later on, it was proven that biharmonic hypersurfaces in hyperbolic n-space $H^n(-1)$ with at most two distinct principle curvatures are minimal [1]. Hence, according to the result, they made the following generalization of Chen’s conjecture in [1]. Generalized Chen’s conjecture: ”any biharmonic submanifold of a Reimannian manifold with nonpositive sectional curvature is minimal. Moreover, Maeta [11] made another generalized Chen’s conjecture: ”the only k-harmonic submanifolds of a Euclidean space are the minimal ones”.

Recently, authors in [10] have shown that the Hopf biharmonic hypersurfaces in complex Euclidean space C^{n+1} are minimal. Also, they proved that pseudo Hopf biharmonic hypersurface in unit sphere S^{2n+1} is either a hypersphere $S^{2n}\left(\frac{1}{\sqrt{2}}\right)$ or a Clifford hypersurface $S^{n_1}\left(\frac{1}{\sqrt{2}}\right)\times$ $\times\, S^{n_2}\left(\frac{1}{\sqrt{2}}\right)$, where $n_1+n_2=2n$. Indeed, it was a small progress on the biharmonic conjecture for hypersurfaces too. In view of the above aspect, studying biharmonic hypersurface with geometric condition is reasonable. The geometry of Ricci soliton manifolds have been intensively studied by many geometers, for instance, see the paper was written by Chen – Yen and S. Deshmukh [4]. Indeed, there was a classification of Ricci solitons on Euclidean hypersurfaces. Furthermore, S. Deshmukh deal with the geometry of Ricci soliton that to find condition under which it is an Einstein manifold [5].

In this paper, we study about proper biharmonic Ricci soliton hypersurface (M^n,g,ξ,λ) in Euclidean space E^{n+1}, somehow whose potential field ξ has an important role, to obtain the following result. At first, we got a result about compact Ricci soliton hypersurfaces in Euclidean space with respect to average scaler curvature S, which is defined $Av(S)=1/\mathrm{vol}(M)\int_M S d\nu$, where Ricci soliton vector field ξ is general. Then, it was shown that a nonexisting proper biharmonic Ricci soliton hypersurface (M^n,g,ξ,λ), in Euclidean space E^{n+1} where either ξ is in $\mathrm{grad} H^{\perp}$ or $\xi=\dfrac{\mathrm{grad}\, H}{|\mathrm{grad}\, H|}$.

2. Preliminaries. In this section, we recall some fundamental definition for the theorem of Ricci soliton biharmonic hypersurfaces which are immersed in an Euclidean space E^{n+1}.

Let $x\colon M^n\longrightarrow E^{n+1}$ be an isometric immersion of n-dimensional hypersurface (M,g) into the Euclidean space E^{n+1}. Let ∇ and $\overline{\nabla}$ stand for Levi – Civita connections on M^n and E^{n+1}, respectively. Let X and Y are tangent vector fields on M also N is considered a locally unit normal vector field to M in E^{n+1}. Then Gauss and Weingarten formulas are

$$\overline{\nabla}_X Y=\nabla_X Y+h(X,Y),$$

$$\overline{\nabla}_X N=-AX,$$

where A is Weingarten operator and h is the second fundamental form of M. The mean curvature vector field $\overrightarrow{H}$ of M^n is defined

$$\overrightarrow{H}=\frac{1}{n}(\mathrm{trace}\, A)N.$$

Assume that $\overrightarrow{H} = HN$ and H implies the mean curvature. One of considerable equation in differential geometry is $\triangle x = -n\overrightarrow{H}$, where $\triangle$ Laplace – Beltrame operator is defined $\triangle = -\mathrm{trace}\,\nabla^2$. An isometric immersion $x\colon M^n \longrightarrow E^{n+1}$ is called biharmonic if and only if $\triangle\overrightarrow{H} = 0$. With respect to $\overrightarrow{H} = HN$ we have

$$0 = \triangle\overrightarrow{H} = 2A(\mathrm{grad}\, H) + nH\mathrm{grad}\, H + (\triangle H + H\mathrm{trace}\, A^2).$$

So, by identifying the tangent and normal part of above equation, we arrived at necessary and sufficient condition for M^n to be biharmonic hypersurface in Euclidean space E^{n+1} in following:

$$\begin{gathered}\triangle H + H\mathrm{trace}\, A^2 = 0,\\ 2A(\mathrm{grad}\, H) + nH\mathrm{grad}\, H = 0.\end{gathered} \tag{1}$$

***Remark* 1.** Obviously, any minimal immersion, i.e., $H = 0$, is biharmonic. The nonharmonic biharmonic immersions are called proper-biharmonic.

The significant type of smooth vector field on a Riemannian manifold (M, g) is the vector field that defines a Ricci soliton. A smooth vector field ξ on a Riemannian manifold (M, g) is called to define a Ricci soliton if it satisfies

$$\frac{1}{2}(\pounds_\xi g)(X, Y) + \mathrm{Ric}\,(X, Y) = \lambda g(X, Y), \qquad X, Y \in \chi(M), \tag{2}$$

where $\pounds_\xi$ denotes the Lie derivative in the direction of the vector field ξ, Ric is Ricci tensor of (M, g) and λ is a real number. A Ricci soliton manifold is denoted by (M, g, ξ, λ) and say the vector field ξ the potential field of the Ricci soliton. Also, the Ricci soliton is named shrinking, steady or expanding with respect to $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$, respectively. The Ricci soliton is called trivial when ξ is Killing or zero, so in each case the metric is Einsteinian. If the potential vector field ξ be the gradient of some smooth function f on M, the (M, g, ξ, λ) is called gradient Ricci soliton such that is denoted by (M, g, f, λ) and say the smooth function f the potential function. The gradient Ricci soliton (M, g, f, λ) is named trivial provided that the potential function f be a constant. Automatically the trivial gradient Ricci solitons are trivial Ricci solitons due to $\xi = \nabla f$.

In order to show the significant role of Ricci soliton vector field in our principal theorem, we ended this section with following proposition, where ξ is general.

Proposition 1. *Let (M^n, g, ξ, λ) be a compact Ricci soliton hypersurface in Euclidean space E^{n+1}. Then the Ricci soliton hypersurface is expanding, steady or shrinking provided that the average scaler curvature $Av(S)$ of M^n be positive, zero or negative, respectively.*

Proof. Suppose that $\{e_i\}$ be an appropriate orthogonal frame field on M^n that the Weingarten operator takes form $Ae_i = \mu_i e_i$ for $1 \leq i \leq n$. Let H be mean curvature vector field of M^n in E^{n+1} too. Now, by applying the equation (2) we have

$$\begin{gathered}g(e_i, \nabla_{e_i}\xi) = \lambda g(e_i, e_i) - \mathrm{Ricci}\,(e_i, e_i) = \\ = \lambda + nH\mu_i - {\mu_i}^2,\end{gathered}$$

it yields that

$$\sum_i^n g(e_i, \nabla_{e_i}\xi) = n\lambda + n^2H^2 - \sum_i^n {\mu_i}^2,$$

$$\operatorname{div}\xi = n\lambda + n^2H^2 - |h|^2 = n\lambda + S,$$

where $|h|^2$ and $S = n^2H^2 - |h|^2$ are square of second fundamental form length and scaler curvature of M^n, respectively. For a Riemannian manifold (M, g) we have that average scaler curvature S of M as $Av(S) = 1/\operatorname{vol}(M)\int_M s d\nu$. Then, from the last equation and take to account that M^n is compact, we get

$$0 = \int\limits_{M^n} \operatorname{div}\xi d\nu = n\lambda \int\limits_{M^n} d\nu + \int\limits_{M^n} S d\nu = n\lambda \operatorname{vol}(M^n) + Av(S)\operatorname{vol}(M^n),$$

and $\lambda = -\dfrac{Av(S)}{n}$. As it was claimed average scaler curvature, determined the type of compact Ricci soliton hypersurfaces in Euclidean space as an expanding, steady or shrinking one.

Proposition 1 is proved.

In this short note, it will be shown how potential field ξ is used in order to get a little progress on Chen's conjecture.

3. Ricci soliton biharmonic hypersurface in Euclidean space E^{n+1}. In this section, we are going to show that a proper biharmonic Ricci soliton connected hypersurfaces (M^n, g, ξ, λ) in E^{n+1} can not be existing, where the potential vector field ξ is either in grad$H^{\perp}$ or specially $\xi = \dfrac{\operatorname{grad} H}{|\operatorname{grad} H|}$.

Now we suppose that the mean curvature is non constant. Taking it to account that, if we have grad$H \equiv 0$, so $H =$ constant. Then, according to the first condition of biharmonicity in equation (1) and due to M^n is a proper biharmonic hypersurface, that is, $H \neq 0$. Hence, it implies that M^n is a totally geodesic hypersurface in Euclidean space. Then it is a part of hyperplane in E^{n+1}. Consequently, we obtained that M^n is a steady Ricci soliton biharmonic hypersurface in this case. Nevertheless, there exists a point $p \in M$, where grad$H \neq 0$ at p. So, there is a open subset U of M^n such that grad$H \neq 0$ on U. Actually, the second view of biharmonic condition recall that grad$H$ is an eigenvector corresponding to eigenvalue $\dfrac{-n}{2}H$. The Weingarten operator A takes following form in the appropriate local frame field $\{e_1, \ldots, e_n\}$:

$$A = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}, \tag{3}$$

where λ_i is eigenvalue of shape operator A corresponding to eigenvector e_i. Without loss of generality, we suppose that e_1 in the direction of $\operatorname{grad} H$. Assume that grad$H$ is given by

$$\operatorname{grad} H = \sum_{i=1}^{n} e_i(H)e_i.$$

It is followed that

$$e_1(H) \neq 0, \qquad e_i(H) = 0, \qquad i = 2, \ldots, n.$$

Also, it is written

$$\nabla_{e_i} e_j = \sum_{k=1}^{n} \omega_{ij}^k e_k, \tag{4}$$

where ω_{ij}^k is called Cartan coefficient. Then computing the compatibility conditions

$$\nabla_{e_k} \langle e_i, e_j \rangle = 0,$$

that denotes

$$\omega_{ki}^i = 0, \qquad i = j, \tag{5}$$

$$\omega_{ki}^j + \omega_{kj}^i = 0, \qquad i \neq j, \qquad i, j, k = 1, \ldots, n. \tag{6}$$

Moreover, from the Codazzi equation we have

$$(\nabla_{e_k} A) e_i = (\nabla_{e_i} A) e_k,$$

$$\nabla_{e_k} A e_i - A \nabla_{e_k} e_i = \nabla_{e_i} A e_k - A \nabla_{e_i} e_k.$$

Now take the above equation, (3), (4) and we get

$$e_k(\lambda_i) e_i + (\lambda_i - \lambda_j) \omega_{ki}^j e_j = e_i(\lambda_k) e_k + (\lambda_k - \lambda_j) \omega_{ik}^j e_j.$$

We multiply both side of above equation to e_j, then we arrived at following equation:

$$e_i(\lambda_j) = (\lambda_i - \lambda_j) \omega_{ji}^j, \tag{7}$$

$$(\lambda_i - \lambda_j) \omega_{ki}^j = (\lambda_k - \lambda_j) \omega_{ik}^j, \tag{8}$$

for distinct $i,\ j,\ k = 1, \ldots, n$. From $\lambda_1 = -\dfrac{n}{2} H$ and (4), we obtain

$$e_1(\lambda_1) \neq 0, \qquad e_i(\lambda_1) = 0, \quad i = 2, \ldots, n, \tag{9}$$

and

$$[e_i, e_j] \lambda_1 = 0, \qquad 2 \leq i, j \leq n, \quad i \neq j,$$

which implies

$$\omega_{ij}^1 = \omega_{ji}^1 \tag{10}$$

for distinct $i,\ j = 2, \ldots, n$. It is claimed that $\lambda_j \neq \lambda_1$ for $j = 2, \ldots, n$ [8]. Since, if $\lambda_j = \lambda_1$ for $j \neq 1$, utilize (7) and put $i = 1$. Then

$$0 = (\lambda_1 - \lambda_j) \omega_{j1}^j = e_1(\lambda_j) = e_1(\lambda_1),$$

which contradicts to (9). For $j = 1$ and $k,\ i \neq 1$, from (8), we get

$$(\lambda_i - \lambda_1) \omega_{ki}^1 = (\lambda_k - \lambda_1) \omega_{ik}^1,$$

which together with (10) yield

$$\omega_{ij}^1 = 0, \qquad i \neq j, \quad i, j = 2, \ldots, n. \tag{11}$$

Combining (11) with equation (6), we obtain $\omega_{i1}^j = 0,\ i \neq j\ ,\ i, j = 2, \ldots, n.$

Taking all the information in to account and summarizing them, we have the following lemma.

Lemma 1. *Suppose that M be a biharmonic hypersurface in Euclidean space E^{n+1} with non-constant mean curvature, whose Weingarten operator is given by* (3) *with respect to an orthogonal frame $\{e_1, \ldots, e_n\}$. Then*

$$\nabla_{e_1} e_i = 0, \qquad 1 \leq i \leq n,$$

$$\nabla_{e_i} e_1 = -\omega_{ii}^1 e_1, \qquad i = 2, \ldots, n,$$

$$\nabla_{e_i} e_i = \sum_{k=1, i \neq k}^{n} \omega_{ii}^k e_k,$$

$$\nabla_{e_i} e_j = \sum_{k=2}^{n} \omega_{ij}^k e_k,$$

where ∇ denote Livi – Civita connection on M and ω_{ij}^k satisfies the equation (4).

Generally, a biharmonic hypersurface in Euclidean space E^{n+1} satisfies the equation (1), where $\operatorname{grad} H$ is an eigenvector of Wiengarten operator.

We are going to prove the following theorem according to the significant point that M is a proper biharmonic hypersurface in E^{n+1}.

Theorem 1. *In Euclidean space E^{n+1} does not exist a proper biharmonic Ricci soliton hypersurface (M^n, g, ξ, λ) provided that the potential field ξ either be a principal vector in* $\operatorname{grad} H^{\perp}$, *or specially* $\xi = \dfrac{\operatorname{grad} H}{|\operatorname{grad} H|}$.

Proof. Let at a point $p \in M^n$ we have $\operatorname{grad} H \neq 0$. So, there exists an open subset $U \subset M^n$ which $\operatorname{grad} H \neq 0$ there. Suppose that $\{e_1, e_2, \ldots, e_n\}$ be an appropriate orthonormal local frame field at p such that Wiengarten operator A takes form (3). According to equation (1) it can be let that $e_1 = \dfrac{\operatorname{grad} H}{|\operatorname{grad} H|}$. Now, by applying the equation (2), we have

$$(\pounds_\xi g)(e_i, e_i) = \lambda g(e_i, e_i) - \operatorname{Ric}(e_i, e_i) = g(\nabla_{e_i} \xi, e_i).$$

From above equation we obtain

$$g(\nabla_{\operatorname{grad} H} \xi, \operatorname{grad} H) = \lambda g(\operatorname{grad} H, \operatorname{grad} H) - \operatorname{Ric}(\operatorname{grad} H, \operatorname{grad} H).$$

According to assumption $\operatorname{grad} H = |\operatorname{grad} H| e_1$, it yields

$$g(\nabla_{e_1} \xi, e_1) |\operatorname{grad} H|^2 = (\lambda g(e_1, e_1) - \operatorname{Ric}(e_1, e_1)) |\operatorname{grad} H|^2,$$

also $\operatorname{Ric}(e_1, e_1) = -\dfrac{3n^2 H^2}{4}$. Hence, we arrived at

$$g(e_1, \nabla_{e_1} \xi) = \left(\lambda + \frac{3n^2 H^2}{4}\right). \tag{12}$$

Now we have two following cases.

Case 1. The potential field $\xi = \dfrac{\operatorname{grad} H}{|\operatorname{grad} H|}$.

Obviously, $g(e_1, \nabla_{e_1}\xi) = 0$, where $\xi = e_1$ by applying Lemma 1 that $\nabla_{e_1}\xi = 0$. Then, according to the right-hand side of equation (12), we obtained a contradiction due to grad$H \neq 0$ and λ is constant. Therefore, the theorem was proven as it was claimed in special case.

Case 2. The potential field ξ is in $\mathrm{grad}\, H^{\perp}$.

Suppose that potential field ξ is in grad$H^{\perp}$. With respect to the assumption grad$H$ is an principal vector corresponding to eigenvalue $-\dfrac{n}{2}H$. Then, using equation (2), where we suppose $\xi = \sum_{t=2}^{n} \alpha_t e_t$. On the one hand, rewrite the left-hand side of equation (2) and apply Lemma 1, we arrived at

$$g\left(e_1, \nabla_{e_1} \sum_{t=2}^{n} \alpha_t e_t\right) = g\left(e_1, \sum_{t=2}^{n}(e_1(\alpha_t)e_t + (\alpha_t)\nabla_{e_1}e_t)\right) =$$

$$= g\left(e_1, \sum_{t=2}^{n} e_1(\alpha_t)e_t\right) + g\left(e_1, \sum_{t=2}^{n} \alpha_t \nabla_{e_1} e_t\right) = 0.$$

On the other hand, it is observed that the left-hand side of above equation is equal to $\left(\lambda + \dfrac{3n^2H^2}{4}\right) = 0$ according to equation (12), which yields it is impossible, where the mean curvature H is not constant. Consequently, we obtained that proper biharmonic Ricci soliton hypersurfaces (M^n, g, ξ, λ) in Euclidean space are not existing whenever the potential field ξ is in grad$H^{\perp}$.

Theorem 1 is proved.

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Received 22.11.18