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NEW CHARACTERIZATIONS FOR DIFFERENCES OF COMPOSITION OPERATORS BETWEEN WEIGHTED-TYPE SPACES IN THE UNIT BALL *

НОВІ ХАРАКТЕРИСТИКИ РІЗНИЦЬ ОПЕРАТОРІВ КОМПОЗИЦІЇ МІЖ ВАГОВИМИ ПРОСТОРАМИ В ОДИНИЧНІЙ КУЛІ

We present some asymptotically equivalent expressions to the essential norm of differences of composition operators acting on weighted-type spaces of holomorphic functions in the unit ball of $\mathbb{C}^N$. Especially, the descriptions in terms of $\langle z, \zeta\rangle^m$ are described, from which the sufficient and necessary conditions of compactness follows immediately. Also, we characterize the boundedness of these operators.

Запропоновано асимптотично еквівалентні вирази для суттєвої норми різниць операторів композиції, які діють у вагових просторах голоморфних функцій в одиничній кулі з $\mathbb{C}^N$. Зокрема, наведено опис у термінах $\langle z, \zeta\rangle^m$, з якого безпосередньо випливають необхідні та достатні умови компактності. Крім того, охарактеризовано обмеженість цих операторів.

1. Introduction. Let $\mathbb{C}^N$ denote the Euclidean space of complex dimension $N (N \geq 1)$. For $z = (z_1, \ldots, z_N)$ and $w = (w_1, \ldots, w_N)$ in $\mathbb{C}^N$, $\langle z, w\rangle = \sum_{j=1}^N z_j \overline{w_j}$ and $|z| = \sqrt{\langle z, z\rangle}$. $\mathbb{B}$ is the open unit ball of $\mathbb{C}^N$ with boundary $\partial\mathbb{B}$. $H(\mathbb{B})$ and $S(\mathbb{B})$ represent the class of holomorphic functions and analytic self-maps on $\mathbb{B}$, respectively. For $\varphi, \psi \in S(\mathbb{B})$, the difference of composition operator associated to φ and ψ is defined by $(C_\varphi - C_\psi)f = f \circ \varphi - f \circ \psi$ for all $f \in H(\mathbb{B})$.

For $0 < \alpha < \infty$, let H_α^∞ be the weighted-type space of holomorphic functions f on $\mathbb{B}$ satisfying

$$\|f\|_\alpha = \sup_{z\in\mathbb{B}} (1 - |z|^2)^\alpha |f(z)| < \infty.$$

With the norm $\|f\|_{H_\alpha^\infty} = |f(0)| + \|f\|_\alpha$, the weighted-type space becomes a Banach space.

For any point $a \in \mathbb{B} - \{0\}$, the involutive automorphism Φ_a is defined by

$$\Phi_a(z) = \frac{a - P_a(z) - s_a Q_a(z)}{1 - \langle z, a\rangle}, \qquad z \in \mathbb{B},$$

where $s_a = \sqrt{1 - |a|^2}$, and $P_a(z) = \dfrac{\langle z, a\rangle}{|a|^2} a$ is the orthogonal projection from $\mathbb{C}^N$ onto the one dimensional subspace $[a]$ generated by a, $Q_a(z) = z - P_a(z)$. When $a = 0, \Phi_a(z) = -z$. It is well-known that Φ_a interchanges the points 0 and a, that is, $\Phi_a(0) = a, \Phi_a(a) = 0$. For $z,\ w \in \mathbb{B}$, the pseudohyperbolic distance between z and w is defined by $\rho(z, w) = |\Phi_w(z)|$. For the simplicity, we write $\rho(z) = \rho(\varphi(z), \psi(z))$.

Let X and Y be Banach spaces and $T : X \to Y$ be a bounded linear operator. The essential norm of T is the distance form T to the sets of compact operators, that is, $\|T\|_{e,X\to Y} = \inf\{\|T - K\|_{X\to Y} : K$ is compact from X to $Y\}$. Notice that $\|T\|_e = 0$ if and only if the operator T is compact, so the

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estimate on $\|T\|_e$ will lead to a condition for the operator T to be compact. For the results in this topic, we refer the interested readers to the recent papers such as [1, 2, 9, 15, 17].

In 2009, Wulan et al. [18] (Theorem 2) obtained a new result about the compactness of composition operator on the classical Bloch space in the unit disk in terms of the sequence $\{z^n\}_{n=1}^{\infty}$. After that, Ruhan Zhao [19] (Corollary 4.4) showed that $\|C_\varphi\|_{e,\mathcal{B}^\alpha\to\mathcal{B}^\beta} \asymp \limsup_{n\to\infty} n^{\alpha-1}\|C_\varphi z^n\|_\beta$ for $0 < \alpha,\ \beta < \infty$. So, $C_\varphi\colon \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is compact if and only if $\limsup_{n\to\infty} n^{\alpha-1}\|C_\varphi z^n\|_\beta = 0$. Subsequently to this, strong interest has arisen to describe some properties of composition operator on Bloch-type spaces. For the results in the unit disk, one can refer to [4, 10, 13, 14, 18]. Then some mathematicians have contributed to development of this new characterizations in the unit ball and polydisk for some operators (see, e.g., [3, 5 – 8] and their references therein). In papers [11, 12, 16], on the unit disk, such new descriptions for differences of classical linear operators was obtained. But as far as we all known, there has no such characterizations for differences of any classical linear operators in the unit ball, so these problems are in desired need of response. In this paper, we pay our attention to start with the investigations for the differences of composition operators acting form α-weighted-type space to β-weighted-type space.

This paper is organized as follows. The boundedness of $C_\varphi - C_\psi\colon H_\alpha^\infty \to H_\beta^\infty$ is exhibited in Section 2 and then its essential norm is estimated in Section 3. In summary, this paper has systematic exposition of equivalent conditions for the differences of composition operators from H_α^∞ to H_β^∞.

Throughout this paper, we will use the symbol C to denote a finite positive number, and it may differ from one occurrence to the other. For two positive quantities A and B, the notations $A \preceq B$, $A \succeq B$ and $A \asymp B$ mean that $A \le CB$, $A \ge CB$ and $A/C \le B \le CA$ for some positive numbers C, respectively. Besides, $\mathbb{N}$ denotes the set of all positive integers.

2. Boundedness of $C_\varphi - C_\psi\colon H_\alpha^\infty \to H_\beta^\infty$. In this section, we give the characterization for the boundedness of the operator $C_\varphi - C_\psi\colon H_\alpha^\infty \to H_\beta^\infty$. For any $a \in \mathbb{B}$, we define the following families test functions:

$$f_a(z) = \frac{(1-|a|^2)^\alpha}{(1-\langle z, a\rangle)^{2\alpha}}$$

and

$$g_{\varphi(a)}(z) = \frac{(1-|\varphi(a)|^2)^\alpha}{(1-\langle z, \varphi(a)\rangle)^{2\alpha}} \frac{\langle \Phi_{\varphi(a)}(z), \Phi_{\varphi(a)}(\psi(a))\rangle}{|\Phi_{\varphi(a)}(\psi(a))|},$$

$$g_{\psi(a)}(z) = \frac{(1-|\psi(a)|^2)^\alpha}{(1-\langle z, \psi(a)\rangle)^{2\alpha}} \frac{\langle \Phi_{\psi(a)}(z), \Phi_{\psi(a)}(\varphi(a))\rangle}{|\Phi_{\psi(a)}(\varphi(a))|}.$$

It is easy to prove that $\|g_{\varphi(a)}\|_{H_\alpha^\infty} \asymp \|g_{\psi(a)}\|_{H_\alpha^\infty} \preceq \|f_a\|_{H_\alpha^\infty} = 1$. For the sake of convenience, we use the notation as below

$$\mathcal{T}_\alpha^\beta \varphi(z) = \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha}.$$

The main result in this section is the following theorem.

Theorem 2.1. *Let* $0 < \alpha, \beta < \infty, \varphi, \psi \in S(\mathbb{B})$. *Then the following statements are equivalent*:

(i) $C_\varphi - C_\psi \colon H^\infty_\alpha \to H^\infty_\beta$ *is bounded*,

(ii$_1$) $\sup\limits_{z\in\mathbb{B}} \mathcal{T}^\beta_\alpha \varphi(z)\rho(z) + \sup\limits_{z\in\mathbb{B}} \left|\mathcal{T}^\beta_\alpha \varphi(z) - \mathcal{T}^\beta_\alpha \psi(z)\right| < \infty$,

(ii$_2$) $\sup\limits_{z\in\mathbb{B}} \mathcal{T}^\beta_\alpha \psi(z)\rho(z) + \sup\limits_{z\in\mathbb{B}} \left|\mathcal{T}^\beta_\alpha \varphi(z) - \mathcal{T}^\beta_\alpha \psi(z)\right| < \infty$,

(iii) $\sup\limits_{a\in\mathbb{B}} \|(C_\varphi - C_\psi)f_a\|_{H^\infty_\beta} + \sup\limits_{a\in\mathbb{B}} \max\left\{\|(C_\varphi - C_\psi)g_{\varphi(a)}\|_{H^\infty_\beta},\ (C_\varphi - C_\psi)g_{\psi(a)}\|_{H^\infty_\beta}\right\} < \infty$,

(iv) $\sup\limits_{m\in\mathbb{N}} \sup\limits_{\zeta\in\partial\mathbb{B}} m^\alpha \|(C_\varphi - C_\psi)\langle\cdot,\zeta\rangle^m\|_{H^\infty_\beta} < \infty$.

In order to prove this result, we need some lemmas. For the first one, it was originally proved in [19, 20].

Lemma 2.1. *Let* $0 < \alpha < \infty$, $m \in \mathbb{N}$ *and* $0 \le x \le 1$. *Set* $r_m = \left(\dfrac{m-1}{m-1+2\alpha}\right)^{1/2}$ *for* $m \ge 2$ *and* $r_m = 0$ *for* $m = 1$. *Then* $H_{m,\alpha}(x) = x^{m-1}(1-x^2)^\alpha$ *has the following properties*:

(i) $$\max_{0\le x\le 1} H_{m,\alpha}(x) = H_{m,\alpha}(r_m) = \begin{cases} 1, & m = 1, \\ \left(\dfrac{m-1}{m-1+2\alpha}\right)^{(m-1)/2} \left(\dfrac{2\alpha}{m-1+2\alpha}\right)^\alpha, & m \ge 2, \end{cases}$$

and $\lim_{m\to\infty} m^\alpha \max_{0\le x\le 1} H_{m,\alpha}(x) = \left(\dfrac{2\alpha}{e}\right)^\alpha$,

(ii) *for* $m \ge 1, H_{m,\alpha}$ *is increasing on* $[0, r_m]$ *and decreasing on* $[r_m, 1]$,

(iii) *for* $m \ge 1, H_{m,\alpha}$ *is decreasing on* $[r_m, r_{m+1}]$,

and $$\min_{x\in[r_m, r_{m+1}]} H_{m,\alpha}(x) = H_{m,\alpha}(r_{m+1}) = \left(\frac{m}{m+2\alpha}\right)^{(m-1)/2} \left(\frac{2\alpha}{m+2\alpha}\right)^\alpha.$$

Consequently,

$$\lim_{m\to\infty} m^\alpha \min_{x\in[r_m, r_{m+1}]} H_{m,\alpha}(x) = \left(\frac{2\alpha}{e}\right)^\alpha.$$

Lemma 2.2. *Let* $0 < \alpha < \infty, m \in \mathbb{N}$. *Then, for each* $\zeta \in \partial\mathbb{B}$, *we have*

$$\lim_{m\to\infty} m^\alpha \|\langle\cdot,\zeta\rangle^m\|_{H^\infty_\alpha} = \left(\frac{2\alpha}{e}\right)^\alpha. \tag{2.1}$$

Proof. For any $\zeta \in \partial\mathbb{B}$,

$$\|\langle\cdot,\zeta\rangle^m\|_{H^\infty_\alpha} = \sup_{z\in\mathbb{B}} (1-|z|^2)^\alpha |\langle z,\zeta\rangle^m| \le \sup_{z\in\mathbb{B}} (1-|z|^2)^\alpha |z|^m = \sup_{0\le r\le 1} (1-r^2)^\alpha r^m,$$

and, on the other hand,

$$\sup_{z\in\mathbb{B}} (1-|z|^2)^\alpha |\langle z,\zeta\rangle|^m \ge \sup_{0\le r\le 1} (1-|r\zeta|^2)^\alpha |\langle r\zeta,\zeta\rangle|^m = \sup_{0\le r\le 1} (1-r^2)^\alpha r^m.$$

Thus,

$$m^\alpha \|\langle\cdot,\zeta\rangle^m\|_{H^\infty_\alpha} = m^\alpha \sup_{0\le r\le 1} (1-r^2)^\alpha r^m =$$

$$= \left(\frac{m}{m+1}\right)^\alpha (m+1)^\alpha \sup_{0\le r\le 1} (1-r^2)^\alpha r^m.$$

It follows from Lemma 2.1 (i) that (2.1) holds.

Lemma 2.2 is proved.

We will also make use of the following lemma. For the proof, see the original source [6].

Lemma 2.3. *Let* $f \in H_\alpha^\infty$. *Then*

$$|(1-|z|^2)^\alpha f(z) - (1-|w|^2)^\alpha f(w)| \le C\|f\|_{H_\alpha^\infty}\rho(z,w)$$

for all $z, w \in \mathbb{B}$.

Lemma 2.4. *Let* $0 < \alpha,\ \beta < \infty,\ \varphi,\ \psi \in S(\mathbb{B})$. *Then the following inequalities hold*:

(i) $\displaystyle\sup_{z\in\mathbb{B}} \mathcal{T}_\alpha^\beta\varphi(z)\rho(z) \le \sup_{a\in\mathbb{B}} \|(C_\varphi - C_\psi)f_a\|_{H_\beta^\infty} + \sup_{a\in\mathbb{B}} \|(C_\varphi - C_\psi)g_{\varphi(a)}\|_{H_\beta^\infty},$

(ii) $\displaystyle\sup_{z\in\mathbb{B}} \mathcal{T}_\alpha^\beta\psi(z)\rho(z) \le \sup_{a\in\mathbb{B}} \|(C_\varphi - C_\psi)f_a\|_{H_\beta^\infty} + \sup_{a\in\mathbb{B}} \|(C_\varphi - C_\psi)g_{\psi(a)}\|_{H_\beta^\infty},$

(iii) $\displaystyle\sup_{z\in\mathbb{B}} \left|\mathcal{T}_\alpha^\beta\varphi(z) - \mathcal{T}_\alpha^\beta\psi(z)\right| \le \sup_{a\in\mathbb{B}} \|(C_\varphi - C_\psi)f_a\|_{H_\beta^\infty} + \sup_{a\in\mathbb{B}} \|(C_\varphi - C_\psi)g_{\psi(a)}\|_{H_\beta^\infty}.$

Proof. For any $a \in \mathbb{B}$, we have

$$\|(C_\varphi - C_\psi)f_{\varphi(a)}\|_{H_\beta^\infty} = \sup_{z\in\mathbb{B}}(1-|z|^2)^\beta |f_{\varphi(a)}(\varphi(z)) - f_{\varphi(a)}(\psi(z))| \ge$$

$$\ge (1-|a|^2)^\beta |f_{\varphi(a)}(\varphi(a)) - f_{\varphi(a)}(\psi(a))| \ge$$

$$\ge \mathcal{T}_\alpha^\beta\varphi(a) - \frac{(1-|\varphi(a)|^2)^\alpha(1-|\psi(a)|^2)^\alpha}{|1-\langle\psi(a),\varphi(a)\rangle|^{2\alpha}}\mathcal{T}_\alpha^\beta\psi(a)$$

and

$$\|(C_\varphi - C_\psi)g_{\varphi(a)}\|_{H_\beta^\infty} \ge (1-|a|^2)^\beta |g_{\varphi(a)}(\varphi(a)) - g_{\varphi(a)}(\psi(a))| =$$

$$= (1-|a|^2)^\beta \frac{(1-|\varphi(a)|^2)^\alpha}{|1-\langle\psi(a),\varphi(a)\rangle|^{2\alpha}}\rho(a) =$$

$$= \frac{(1-|\varphi(a)|^2)^\alpha(1-|\psi(a)|^2)^\alpha}{|1-\langle\psi(a),\varphi(a)\rangle|^{2\alpha}}\mathcal{T}_\alpha^\beta\psi(a)\rho(a).$$

Thus,

$$\mathcal{T}_\alpha^\beta\varphi(a)\rho(a) \le \|(C_\varphi - C_\psi)f_{\varphi(a)}\|_{H_\beta^\infty}\rho(a) + \|(C_\varphi - C_\psi)g_{\varphi(a)}\|_{H_\beta^\infty} \le$$

$$\le \|(C_\varphi - C_\psi)f_{\varphi(a)}\|_{H_\beta^\infty} + \|(C_\varphi - C_\psi)g_{\varphi(a)}\|_{H_\beta^\infty}, \tag{2.2}$$

where the last inequality follows from $\rho(a) \le 1$. Analogously, we deduce that

$$\mathcal{T}_\alpha^\beta\psi(a)\rho(a) \le \|(C_\varphi - C_\psi)f_{\psi(a)}\|_{H_\beta^\infty} + \|(C_\varphi - C_\psi)g_{\psi(a)}\|_{H_\beta^\infty}. \tag{2.3}$$

Taking the supremum about $a \in \mathbb{B}$ in (2.2) and (2.3), we obtain

(i) $\displaystyle\sup_{a\in\mathbb{B}} \mathcal{T}_\alpha^\beta\varphi(a)\rho(a) \le \sup_{a\in\mathbb{B}}\left(\|(C_\varphi - C_\psi)f_{\varphi(a)}\|_{H_\beta^\infty} + \|(C_\varphi - C_\psi)g_{\varphi(a)}\|_{H_\beta^\infty}\right) \le$

$\displaystyle\le \sup_{a\in\mathbb{B}} \|(C_\varphi - C_\psi)f_a\|_{H_\beta^\infty} + \sup_{a\in\mathbb{B}} \|(C_\varphi - C_\psi)g_{\varphi(a)}\|_{H_\beta^\infty}$

and

(ii) $\sup\limits_{a\in\mathbb{B}} \mathcal{T}_\alpha^\beta \psi(a)\rho(a) \le \sup\limits_{a\in\mathbb{B}} \|(C_\varphi - C_\psi) f_a\|_{H_\beta^\infty} + \sup\limits_{a\in\mathbb{B}} \|(C_\varphi - C_\psi) g_{\psi(a)}\|_{H_\beta^\infty}$.

On the other hand, by Lemma 2.3 we note that

$$\|(C_\varphi - C_\psi) f_{\varphi(a)}\|_{H_\beta^\infty} \ge$$

$$\ge (1-|a|^2)^\beta |f_{\varphi(a)}(\varphi(a)) - f_{\varphi(a)}(\psi(a))| =$$

$$= (1-|a|^2)^\beta \left| \frac{1}{(1-|\varphi(a)|^2)^\alpha} - \frac{(1-|\varphi(a)|^2)^\alpha}{(1-\langle \psi(a), \varphi(a)\rangle)^{2\alpha}} \right| \ge$$

$$\ge \left| \mathcal{T}_\alpha^\beta \varphi(a) - \mathcal{T}_\alpha^\beta \psi(a) \right| - \left| \mathcal{T}_\alpha^\beta \psi(a) - \frac{(1-|a|^2)^\beta (1-|\varphi(a)|^2)^\alpha}{(1-\langle \psi(a), \varphi(a)\rangle)^{2\alpha}} \right| =$$

$$= \left| \mathcal{T}_\alpha^\beta \varphi(a) - \mathcal{T}_\alpha^\beta \psi(a) \right| -$$

$$- \mathcal{T}_\alpha^\beta \psi(a) \left| (1-|\varphi(a)|^2)^\alpha f_{\varphi(a)}(\varphi(a)) - (1-|\psi(a)|^2)^\alpha f_{\varphi(a)}(\psi(a)) \right| \succeq$$

$$\succeq \left| \mathcal{T}_\alpha^\beta \varphi(a) - \mathcal{T}_\alpha^\beta \psi(a) \right| - \mathcal{T}_\alpha^\beta \psi(a) \rho(a). \tag{2.4}$$

So together with (ii), we arrive at

(i) $\sup\limits_{a\in\mathbb{B}} \left| \mathcal{T}_\alpha^\beta \varphi(a) - \mathcal{T}_\alpha^\beta \psi(a) \right| \preceq \sup\limits_{a\in\mathbb{B}} \Big(\|(C_\varphi - C_\psi) f_{\varphi(a)}\|_{H_\beta^\infty} + \mathcal{T}_\alpha^\beta \psi(a)\rho(a) \Big) \preceq$

$\preceq \sup\limits_{a\in\mathbb{B}} \|(C_\varphi - C_\psi) f_a\|_{H_\beta^\infty} + \sup\limits_{a\in\mathbb{B}} \|(C_\varphi - C_\psi) g_{\psi(a)}\|_{H_\beta^\infty}$.

Lemma 2.4 is proved.

Lemma 2.5. *Let* $0 < \alpha,\ \beta < \infty,\ \varphi,\ \psi \in S(\mathbb{B})$. *Then the following inequalities hold*:

(i) $\sup\limits_{a\in\mathbb{B}} \|(C_\varphi - C_\psi) f_a\|_{H_\beta^\infty} \preceq \sup\limits_{m\in\mathbb{N}} \sup\limits_{\zeta\in\partial\mathbb{B}} m^\alpha \|(C_\varphi - C_\psi)\langle\cdot,\zeta\rangle^m\|_{H_\beta^\infty}$,

(ii) $\sup\limits_{a\in\mathbb{B}} \max\{\|(C_\varphi - C_\psi) g_{\varphi(a)}\|_{H_\beta^\infty}, (C_\varphi - C_\psi) g_{\psi(a)}\|_{H_\beta^\infty}\} \preceq$

$\preceq \sup\limits_{m\in\mathbb{N}} \sup\limits_{\zeta\in\partial\mathbb{B}} m^\alpha \|(C_\varphi - C_\psi)\langle\cdot,\zeta\rangle^m\|_{H_\beta^\infty}$.

Proof. For $\alpha > 0$, recall that

$$\frac{1}{(1-\langle z,a\rangle)^{2\alpha}} = \sum_{k=0}^\infty \frac{\Gamma(k+2\alpha)}{k!\Gamma(2\alpha)} \langle z,a\rangle^k,$$

then we express f_a into Maclaurin expansion as follows:

$$f_a(z) = (1-|a|^2)^\alpha \sum_{k=0}^\infty \frac{\Gamma(k+2\alpha)}{k!\Gamma(2\alpha)} \langle z,a\rangle^k.$$

If $a = 0,\ f_a(z) \equiv 1$, (i) holds obvious. If $a \neq 0$, then

$$\|(C_\varphi - C_\psi) f_a\|_{H_\beta^\infty} \le$$

$$\leq (1-|a|^2)^\alpha \sum_{k=0}^{\infty} \frac{\Gamma(k+2\alpha)}{k!\Gamma(2\alpha)} \|(C_\varphi - C_\psi)\langle\cdot, a\rangle^k\|_{H^\infty_\beta} \leq \tag{2.5}$$

$$\leq (1-|a|^2)^\alpha \sum_{k=0}^{\infty} \frac{\Gamma(k+2\alpha)}{k!\Gamma(2\alpha)} |a|^k k^{-\alpha} k^\alpha \|(C_\varphi - C_\psi)\langle\cdot, \frac{a}{|a|}\rangle^k\|_{H^\infty_\beta} \leq$$

$$\leq (1-|a|^2)^\alpha \sum_{k=0}^{\infty} \frac{\Gamma(k+2\alpha)}{k!\Gamma(2\alpha)} |a|^k k^{-\alpha} \sup_{m\in\mathbb{N}} \sup_{\zeta\in\partial\mathbb{B}} m^\alpha \|(C_\varphi - C_\psi)\langle\cdot, \zeta\rangle^m\|_{H^\infty_\beta}. \tag{2.6}$$

By Stirling's formula, $\dfrac{\Gamma(k+\alpha)}{k!\Gamma(\alpha)} \asymp k^{\alpha-1}$ as $k\to\infty$. It follows that

$$\frac{\Gamma(k+2\alpha)}{k!\Gamma(2\alpha)} k^{-\alpha} \asymp k^{\alpha-1} \quad \text{as} \quad k\to\infty.$$

Hence,

$$\sum_{k=0}^{\infty} \frac{\Gamma(k+2\alpha)}{k!\Gamma(2\alpha)} |a|^k k^{-\alpha} \asymp \sum_{k=0}^{\infty} k^{\alpha-1}|a|^k \asymp \sum_{k=0}^{\infty} \frac{\Gamma(k+\alpha)}{k!\Gamma(\alpha)} |a|^k \asymp \frac{1}{(1-|a|)^\alpha}, \tag{2.7}$$

which combine with (2.6), we conclude (i).

Next, we prove the inequality (ii). When $\varphi(a)=0$, $g_{\varphi(a)}(z) = \dfrac{\langle z, \psi(a)\rangle}{|\psi(a)|}$, then

$$\|(C_\varphi - C_\psi) g_{\varphi(a)}\|_{H^\infty_\beta} = \left\|(C_\varphi - C_\psi)\left\langle \cdot, \frac{\psi(a)}{|\psi(a)|}\right\rangle\right\|_{H^\infty_\beta} \leq$$

$$\leq \sup_{\zeta\in\partial\mathbb{B}} \|(C_\varphi - C_\psi)\langle\cdot,\zeta\rangle\|_{H^\infty_\beta} \leq \sup_{m\in\mathbb{N}} \sup_{\zeta\in\partial\mathbb{B}} m^\alpha \|(C_\varphi - C_\psi)\langle\cdot,\zeta\rangle^m\|_{H^\infty_\beta}.$$

For $\varphi(a)\neq 0$,

$$g_{\varphi(a)}(z) = \frac{(1-|\varphi(a)|^2)^\alpha}{(1-\langle z,\varphi(a)\rangle)^{2\alpha}} \frac{\langle \Phi_{\varphi(a)}(z) - \Phi_{\varphi(a)}(\psi(a)) + \Phi_{\varphi(a)}(\psi(a)), \Phi_{\varphi(a)}(\psi(a))\rangle}{|\Phi_{\varphi(a)}(\psi(a))|} =$$

$$= f_{\varphi(a)}(z)\rho(a) + f_{\varphi(a)}(z) \frac{\langle \Phi_{\varphi(a)}(z) - \Phi_{\varphi(a)}(\psi(a)), \Phi_{\varphi(a)}(\psi(a))\rangle}{|\Phi_{\varphi(a)}(\psi(a))|},$$

thus, for any $a\in\mathbb{B}$, we have

$$\|(C_\varphi - C_\psi) g_{\varphi(a)}\|_{H^\infty_\beta} \leq \|(C_\varphi - C_\psi) f_{\varphi(a)}\|_{H^\infty_\beta} + 2\|(C_\varphi - C_\psi) f_{\varphi(a)}\|_{H^\infty_\beta} \preceq$$

$$\preceq \sup_{m\in\mathbb{N}} \sup_{\zeta\in\partial\mathbb{B}} m^\alpha \|(C_\varphi - C_\psi)\langle\cdot,\zeta\rangle^m\|_{H^\infty_\beta}. \tag{2.8}$$

Here we used the fact that $\left|\dfrac{\langle \Phi_{\varphi(a)}(z) - \Phi_{\varphi(a)}(\psi(a)), \Phi_{\varphi(a)}(\psi(a))\rangle}{|\Phi_{\varphi(a)}(\psi(a))|}\right| \leq 2.$

Similarly, the inequality

$$\|(C_\varphi - C_\psi) g_{\psi(a)}\|_{H^\infty_\beta} \preceq \sup_{m\in\mathbb{N}} \sup_{\zeta\in\partial\mathbb{B}} m^\alpha \|(C_\varphi - C_\psi)\langle\cdot,\zeta\rangle^m\|_{H^\infty_\beta} \tag{2.9}$$

can easily be obtained by the methods used in the proof of (2.8). Taking the supremum about $a \in \mathbb{B}$ in (2.8) and (2.9), (ii) comes ture.

Lemma 2.5 is proved.

***Proof of Theorem* 2.1.** The implications (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii$_1$) or (ii$_2$) follow from Lemmas 2.4 and 2.5. We next prove (i)$\Rightarrow$ (iv) and (ii)$\Rightarrow$ (i).

(i) $\Rightarrow$ (iv). Suppose that $C_\varphi - C_\psi \colon H^\infty_\alpha \to H^\infty_\beta$ is bounded. For any $m \in \mathbb{N}$ and $\zeta \in \partial\mathbb{B}$, consider the function $h_{m,\zeta}(z) = \dfrac{\langle z,\zeta\rangle^m}{\|\langle\cdot,\zeta\rangle^m\|_{H^\infty_\alpha}}$, then it is easy to see that $h_{m,\zeta} \in H^\infty_\alpha$ with $\|h_{m,\zeta}\|_{H^\infty_\alpha} = 1$. Note that from Lemma 2.2, there is a constant $C > 0$ independent of m and ζ such that $\|\langle\cdot,\zeta\rangle^m\|_{H^\infty_\alpha} \le Cm^{-\alpha}$. Combining with the boundedness of $C_\varphi - C_\psi \colon H^\infty_\alpha \to H^\infty_\beta$, it follows that

$$\infty > \|C_\varphi - C_\psi\|_{H^\infty_\alpha \to H^\infty_\beta} \ge \|(C_\varphi - C_\psi)h_{m,\zeta}\|_{H^\infty_\beta} = \frac{\|(C_\varphi - C_\psi)(\langle\cdot,\zeta\rangle^m)\|_{H^\infty_\beta}}{\|\langle\cdot,\zeta\rangle^m\|_{H^\infty_\alpha}} \succeq$$

$$\succeq m^\alpha \|(C_\varphi - C_\psi)(\langle\cdot,\zeta\rangle^m)\|_{H^\infty_\beta},$$

for any $m \in \mathbb{N}$ and $\zeta \in \partial\mathbb{B}$. Which shows the statement (i) $\Rightarrow$ (iv).

(ii$_1$) $\Rightarrow$ (i). For any $f \in H^\infty_\alpha$, we employ Lemma 2.3 to show that

$$\|(C_\varphi - C_\psi)f\|_{H^\infty_\beta} = \sup_{z\in\mathbb{B}} (1-|z|^2)^\beta |f(\varphi(z)) - f(\psi(z))| \le$$

$$\le \sup_{z\in\mathbb{B}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha} \left|(1-|\varphi(z)|^2)^\alpha f(\varphi(z)) - (1-|\psi(z)|^2)^\alpha f(\psi(z))\right| +$$

$$+ \sup_{z\in\mathbb{B}} \left| \frac{(1-|z|^2)^\beta (1-|\psi(z)|^2)^\alpha f(\psi(z))}{(1-|\varphi(z)|^2)^\alpha} - (1-|z|^2)^\beta f(\psi(z)) \right| \preceq$$

$$\preceq \sup_{z\in\mathbb{B}} \mathcal{T}^\beta_\alpha \varphi(z)\rho(z) + \sup_{z\in\mathbb{B}} (1-|\psi(z)|^2)^\alpha |f(\psi(z))| \left| \mathcal{T}^\beta_\alpha \varphi(z) - \mathcal{T}^\beta_\alpha \psi(z) \right| \preceq$$

$$\preceq \sup_{z\in\mathbb{B}} \mathcal{T}^\beta_\alpha \varphi(z)\rho(z) + \sup_{z\in\mathbb{B}} \left| \mathcal{T}^\beta_\alpha \varphi(z) - \mathcal{T}^\beta_\alpha \psi(z) \right| < \infty. \tag{2.10}$$

Thus, $C_\varphi - C_\psi \colon H^\infty_\alpha \to H^\infty_\beta$ is bounded. Therefore, (i), (ii$_1$), (iii), (iv) are equivalent. The equivalence of statements (i), (ii$_2$), (iii), (iv) can be proved in a similar manner.

Theorem 2.1 is proved.

3. Essential norm of $C_\varphi - C_\psi \colon H^\infty_\alpha \to H^\infty_\beta$. In this section, we turn our attention to the estimations for essential norm of $C_\varphi - C_\psi \colon H^\infty_\alpha \to H^\infty_\beta$. The proof of the main assertion relies on the following two lemmas.

Lemma 3.1. *Let* $0 < \alpha, \beta < \infty, \varphi, \psi \in S(\mathbb{B})$. *Then the following inequalities hold*:

(i) $\displaystyle\limsup_{|\varphi(z)|\to 1} \mathcal{T}^\beta_\alpha \varphi(z)\rho(z) \preceq \limsup_{|a|\to 1} \|(C_\varphi - C_\psi)f_a\|_{H^\infty_\beta} + \limsup_{|\varphi(a)|\to 1} \|(C_\varphi - C_\psi)g_{\varphi(a)}\|_{H^\infty_\beta}$,

(ii) $\displaystyle\limsup_{|\psi(z)|\to 1} \mathcal{T}^\beta_\alpha \psi(z)\rho(z) \preceq \limsup_{|a|\to 1} \|(C_\varphi - C_\psi)f_a\|_{H^\infty_\beta} + \limsup_{|\psi(a)|\to 1} \|(C_\varphi - C_\psi)g_{\psi(a)}\|_{H^\infty_\beta}$,

(iii) $$\limsup_{\min\{|\varphi(z)|,|\psi(z)|\}\to 1}\left|\mathcal{T}_\alpha^\beta\varphi(z)-\mathcal{T}_\alpha^\beta\psi(z)\right|\preceq$$

$$\preceq\limsup_{|a|\to 1}\|(C_\varphi-C_\psi)f_a\|_{H_\beta^\infty}+\limsup_{|\psi(a)|\to 1}\|(C_\varphi-C_\psi)g_{\psi(a)}\|_{H_\beta^\infty}.$$

Proof. From the inequalities (2.2) – (2.4) the assertion follows easily.

Lemma 3.2. *Let* $0<\alpha,\ \beta<\infty,\ \varphi,\ \psi\in S(\mathbb{B}),\ C_\varphi-C_\psi\colon H_\alpha^\infty\to H_\beta^\infty$ *is bounded. Then the following inequalities hold*:

(i) $\limsup_{|a|\to 1}\|(C_\varphi-C_\psi)f_a\|_{H_\beta^\infty}\preceq\limsup_{m\to\infty}\sup_{\zeta\in\partial\mathbb{B}}m^\alpha\|(C_\varphi-C_\psi)\langle\cdot,\zeta\rangle^m\|_{H_\beta^\infty}$,

(ii) $\max\{\limsup_{|\varphi(a)|\to 1}\|(C_\varphi-C_\psi)g_{\varphi(a)}\|_{H_\beta^\infty},\limsup_{|\psi(a)|\to 1}\|(C_\varphi-C_\psi)g_{\psi(a)}\|_{H_\beta^\infty}\}\preceq$

$\preceq\limsup_{m\to\infty}\sup_{\zeta\in\partial\mathbb{B}}m^\alpha\|(C_\varphi-C_\psi)\langle\cdot,\zeta\rangle^m\|_{H_\beta^\infty}$.

Proof. For any $a\in\mathbb{B}$ and each positive integer N, employing (2.5) we obtain

$$\|(C_\varphi-C_\psi)f_a\|_{H_\beta^\infty}\le(1-|a|^2)^\alpha\sum_{k=0}^\infty\frac{\Gamma(k+2\alpha)}{k!\Gamma(2\alpha)}|a|^k\left\|(C_\varphi-C_\psi)\left\langle\cdot,\frac{a}{|a|}\right\rangle^k\right\|_{H_\beta^\infty}\le$$

$$\le(1-|a|^2)^\alpha\sum_{k=0}^N\frac{\Gamma(k+2\alpha)}{k!\Gamma(2\alpha)}|a|^k\sup_{\zeta\in\partial\mathbb{B}}\|(C_\varphi-C_\psi)\langle\cdot,\zeta\rangle^k\|_{H_\beta^\infty}+$$

$$+(1-|a|^2)^\alpha\sum_{k=N+1}^\infty\frac{\Gamma(k+2\alpha)}{k!\Gamma(2\alpha)}|a|^kk^{-\alpha}\sup_{m\ge N+1}\sup_{\zeta\in\partial\mathbb{B}}m^\alpha\|(C_\varphi-C_\psi)\langle\cdot,\zeta\rangle^m\|_{H_\beta^\infty}\triangleq$$

$$\triangleq J_1+J_2.$$

For $k\in\{0,1,\ldots,N\}$, since $\langle z,\zeta\rangle^k\in H_\alpha^\infty$, for all $\zeta\in\partial\mathbb{B}$ and $C_\varphi-C_\psi\colon H_\alpha^\infty\to H_\beta^\infty$ is bounded, then

$$\sup_{\zeta\in\partial\mathbb{B}}\|(C_\varphi-C_\psi)\langle\cdot,\zeta\rangle^k\|_{H_\beta^\infty}<\infty.$$

Hence,

$$\limsup_{|a|\to 1}J_1=0.$$

On the other hand, noting (2.7) we have

$$J_2\preceq\sup_{m\ge N+1}\sup_{\zeta\in\partial\mathbb{B}}m^\alpha\|(C_\varphi-C_\psi)\langle\cdot,\zeta\rangle^m\|_{H_\beta^\infty},$$

which leads to

$$\limsup_{|a|\to 1}J_2\preceq\sup_{m\ge N+1}\sup_{\zeta\in\partial\mathbb{B}}m^\alpha\|(C_\varphi-C_\psi)\langle\cdot,\zeta\rangle^m\|_{H_\beta^\infty}.$$

Thus, (i) holds. Next based on the result in (2.8), it follows that

$$\limsup_{|\varphi(a)|\to 1} \|(C_\varphi - C_\psi) g_{\varphi(a)}\|_{H^\infty_\beta} \preceq \limsup_{|\varphi(a)|\to 1} \|(C_\varphi - C_\psi) f_{\varphi(a)}\|_{H^\infty_\beta} \preceq$$

$$\preceq \limsup_{|a|\to 1} \|(C_\varphi - C_\psi) f_a\|_{H^\infty_\beta} \preceq$$

$$\preceq \limsup_{m\to\infty} \sup_{\zeta\in\partial\mathbb{B}} m^\alpha \|(C_\varphi - C_\psi)\langle\cdot,\zeta\rangle^m\|_{H^\infty_\beta}.$$

Similarly, we can prove that

$$\limsup_{|\varphi(a)|\to 1} \|(C_\varphi - C_\psi) g_{\psi(a)}\|_{H^\infty_\beta} \preceq \limsup_{m\to\infty} \sup_{\zeta\in\partial\mathbb{B}} m^\alpha \|(C_\varphi - C_\psi)\langle\cdot,\zeta\rangle^m\|_{H^\infty_\beta}.$$

Thus, we conclude (ii).

Lemma 3.2 is proved.

The following characterization about the essential norm of $C_\varphi - C_\psi \colon H^\infty_\alpha \to H^\infty_\beta$ appears to be useful for our purposes. For a proof, see Theorem 2 in [17].

Lemma 3.3. *Let* $0 < \alpha,\ \beta < \infty,\ \varphi,\ \psi \in S(\mathbb{B})$ *such that* $\max\{\|\varphi_1\|_\infty, \|\varphi_2\|_\infty\} = 1$. *If* $C_\varphi,\ C_\psi \colon H^\infty_\alpha \to H^\infty_\beta$ *are bounded operators, then the essential norm* $\|C_\varphi - C_\psi\|_{e,H^\infty_\alpha\to H^\infty_\beta}$ *is equivalent to the maximum of the following expressions*:

(i) $\limsup_{|\varphi(z)|\to 1} \mathcal{T}^\beta_\alpha \varphi(z)\rho(z)$,

(ii) $\limsup_{|\psi(z)|\to 1} \mathcal{T}^\beta_\alpha \psi(z)\rho(z)$,

(iii) $\limsup_{\min\{|\varphi(z)|,|\psi(z)|\}\to 1} \left|\mathcal{T}^\beta_\alpha \varphi(z) - \mathcal{T}^\beta_\alpha \psi(z)\right|$.

Theorem 3.1. *Let* $0 < \alpha, \beta < \infty, \varphi, \psi \in S(\mathbb{B})$. *If the operators* $C_\varphi, C_\psi \colon H^\infty_\alpha \to H^\infty_\beta$ *are bounded, then the following equivalences hold*:

$$\|C_\varphi - C_\psi\|_{e,H^\infty_\alpha\to H^\infty_\beta} \approx$$

$$\approx \limsup_{|\varphi(z)|\to 1} \mathcal{T}^\beta_\alpha \varphi(z)\rho(z) + \limsup_{|\psi(z)|\to 1} \mathcal{T}^\beta_\alpha \psi(z)\rho(z) + \limsup_{\min\{|\varphi(z)|,|\psi(z)|\}\to 1} \left|\mathcal{T}^\beta_\alpha \varphi(z) - \mathcal{T}^\beta_\alpha \psi(z)\right| \approx$$

$$\approx \limsup_{|a|\to 1} \|(C_\varphi - C_\psi) f_a\|_{H^\infty_\beta} +$$

$$+ \max\{\limsup_{|\varphi(a)|\to 1} \|(C_\varphi - C_\psi) g_{\varphi(a)}\|_{H^\infty_\beta}, \limsup_{|\psi(a)|\to 1} \|(C_\varphi - C_\psi) g_{\psi(a)}\|_{H^\infty_\beta}\} \approx$$

$$\approx \limsup_{m\to\infty} \sup_{\zeta\in\partial\mathbb{B}} m^\alpha \|(C_\varphi - C_\psi)\langle\cdot,\zeta\rangle^m\|_{H^\infty_\beta}.$$

Proof. The boundedness of $C_\varphi - C_\psi \colon H^\infty_\alpha \to H^\infty_\beta$ comes easily from the boundedness of the operators C_φ and C_ψ from H^∞_α to H^∞_β. Thus, using the results in Lemmas 3.1 – 3.3, it suffices to prove that

$$\|C_\varphi - C_\psi\|_{e,H^\infty_\alpha\to H^\infty_\beta} \succeq \limsup_{m\to\infty} \sup_{\zeta\in\partial\mathbb{B}} m^\alpha \|(C_\varphi - C_\psi)\langle\cdot,\zeta\rangle^m\|_{H^\infty_\beta}.$$

Choose $f_{m,\zeta}(z) = \dfrac{\langle z,\zeta\rangle^m}{\|\langle\cdot,\zeta\rangle^m\|_{H_\alpha^\infty}}$, then $\|f_{m,\zeta}\|_{H_\alpha^\infty} = 1$ and $f_{m,\zeta} \to 0, m \to \infty$ weakly in H_α^∞. Thus, for any compact operator $K : H_\alpha^\infty \to H_\beta^\infty$, we have $\lim_{m\to\infty} \|f_{m,\zeta}\|_{H_\beta^\infty} = 0$. Hence,

$$\|C_\varphi - C_\psi - K\| \geq \limsup_{m\to\infty} \sup_{\zeta\in\partial\mathbb{B}} \|(C_\varphi - C_\psi - K)f_{m,\zeta}\|_{H_\beta^\infty} \geq$$

$$\geq \limsup_{m\to\infty} \sup_{\zeta\in\partial\mathbb{B}} \|(C_\varphi - C_\psi)f_{m,\zeta}\|_{H_\beta^\infty}.$$

Then, from Lemma 2.2, we obtain

$$\|C_\varphi - C_\psi\|_{e,H_\alpha^\infty\to H_\beta^\infty} \geq \limsup_{m\to\infty} \sup_{\zeta\in\partial\mathbb{B}} \|(C_\varphi - C_\psi)f_{m,\zeta}\|_{H_\beta^\infty} \succeq$$

$$\succeq \limsup_{m\to\infty} \sup_{\zeta\in\partial\mathbb{B}} m^\alpha \|(C_\varphi - C_\psi)\langle\cdot,\zeta\rangle^m\|_{H_\beta^\infty}.$$

Theorem 3.1 is proved.

In view of Theorem 3.1, it gives equivalent conditions about the compactness of $C_\varphi - C_\psi : H_\alpha^\infty \to H_\beta^\infty$.

***Corollary* 3.1.** *Let* $0 < \alpha, \beta < \infty, \varphi, \psi \in S(\mathbb{B})$. *If the operators* $C_\varphi, C_\psi : H_\alpha^\infty \to H_\beta^\infty$ *are bounded*, *then the following conditions are equivalent*:

(i) $C_\varphi - C_\psi : H_\alpha^\infty \to H_\beta^\infty$ *is compact*,

(ii) $\limsup_{|\varphi(z)|\to 1} \mathcal{T}_\alpha^\beta \varphi(z)\rho(z) + \limsup_{|\psi(z)|\to 1} \mathcal{T}_\alpha^\beta \psi(z)\rho(z) + \limsup_{\min\{|\phi(z)|,|\psi(z)|\}\to 1} \left|\mathcal{T}_\alpha^\beta \varphi(z) - \mathcal{T}_\alpha^\beta \psi(z)\right| = 0,$

(iii) $\limsup_{|a|\to 1} \|(C_\varphi - C_\psi)f_a\|_{H_\beta^\infty} +$

$$+ \max\{\limsup_{|\varphi(a)|\to 1} \|(C_\varphi - C_\psi)g_{\varphi(a)}\|_{H_\beta^\infty}, \limsup_{|\psi(a)|\to 1} \|(C_\varphi - C_\psi)g_{\psi(a)}\|_{H_\beta^\infty}\} = 0,$$

(iv) $\limsup_{m\to\infty} \sup_{\zeta\in\partial\mathbb{B}} m^\alpha \|(C_\varphi - C_\psi)\langle\cdot,\zeta\rangle^m\|_{H_\beta^\infty} = 0.$

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