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ON APPLICATION OF SLOWLY VARYING FUNCTIONS WITH REMAINDER IN THE THEORY OF MARKOV BRANCHING PROCESSES WITH MEAN ONE AND INFINITE VARIANCE

ПРО ЗАСТОСУВАННЯ ПОВІЛЬНО ЗМІННИХ ФУНКЦІЙ ІЗ ЗАЛИШКОМ У ТЕОРІЇ МАРКОВСЬКИХ РОЗГАЛУЖЕНИХ ПРОЦЕСІВ З ОДИНИЧНИМ МАТЕМАТИЧНИМ ОЧІКУВАННЯМ ТА НЕСКІНЧЕННОЮ ДИСПЕРСІЄЮ

We investigate an application of slowly varying functions (in sense of Karamata) in the theory of Markov branching processes. We treat the critical case so that the infinitesimal generating function of the process has the infinite second moment, but it regularly varies with the remainder. We improve the basic lemma of the theory of critical Markov branching processes and refine known limit results.

Досліджується застосування повільно змінних функцій (у сенсі Карамати) в теорії марковських розгалужених процесів. Критичний випадок трактується так, що інфінітезимальна генеруюча функція процесу має нескінченний другий момент, але регулярно змінюється з залишком. Покращено основну лему теорії критичних марковських розгалужених процесів та уточнено відомі граничні результати.

1. Introduction and main results. ***1.1. Preliminaries.*** We consider the Markov branching process (MBP) to be the homogeneous continuous-time Markov process $\{Z(t), t \geq 0\}$ with the state space $\mathcal{S}_0 = \{0\} \cup \mathcal{S}$, where $\mathcal{S} \subset \mathbb{N}$ and $\mathbb{N} = \{1, 2, \ldots\}$. The transition probabilities of the process

$$P_{ij}(t) := \mathbb{P}\{Z(t) = j \mid Z(0) = i\}$$

satisfy the following branching property:

$$P_{ij}(t) = P_{1j}^{i*}(t) \quad \text{for all} \quad i, j \in \mathcal{S}, \tag{1.1}$$

where the asterisk denotes convolution. Here transition probabilities $P_{1j}(t)$ are expressed by relation

$$P_{1j}(\varepsilon) = \delta_{1j} + a_j\varepsilon + o(\varepsilon) \quad \text{as} \quad \varepsilon \downarrow 0, \tag{1.2}$$

where δ_{ij} is Kronecker's delta function and $\{a_j\}$ are intensities of individuals transformation such that $a_j \geq 0$ for $j \in \mathcal{S}_0 \backslash \{1\}$ and

$$0 < a_0 < -a_1 = \sum_{j \in \mathcal{S}_0 \backslash \{1\}} a_j < \infty.$$

The MBP was defined first by Kolmogorov and Dmitriev [8] (for more detailed information see [2] (Ch. III) and [5] (Ch. V)).

Defining the generating function (GF) $F(t;s) = \sum_{j \in \mathcal{S}_0} P_{1j}(t)s^j$ it follows from (1.1) and (1.2) that the process $\{Z(t)\}$ is determined by the infinitesimal GF $f(s) = \sum_{j \in \mathcal{S}_0} a_j s^j$ for $s \in [0, 1)$. Moreover, it follows from (1.2) that GF $F(t;s)$ is unique solution of the backward Kolmogorov equation $\partial F/\partial t = f(F)$ with the boundary condition $F(0;s) = s$ (see [2, p. 106]). If $m := := \sum_{j \in \mathcal{S}} j a_j = f'(1-)$ is finite, then $F(t;1) = 1$ and due to Kolmogorov equation it can be calculated that $\mathbb{E}[Z(t)|Z(0) = i] = \sum_{j\in\mathcal{S}} jP_{ij}(t) = ie^{mt}$. Last formula shows that long-term properties of MBP are various depending on value of parameter m. Hence, the MBP is classified as *critical* if $m = 0$ and *sub-critical* or *supercritical* if $m < 0$ or $m > 0$, respectively. Monographs [1 – 3, 5] are general references for mentioned and other classical facts on theory of MBP.

In the paper we consider the critical case. Let $R(t;s) = 1 - F(t;s)$ and

$$q(t) := R(t;0) = \mathbb{P}\{\mathcal{H} > t\},$$

where the variable $\mathcal{H} = \inf\{t\colon Z(t) = 0\}$ denotes an extinction time of MBP. Then $q(t)$ is the survival probability of the process. Sevastyanov [11] proved that if $f'''(1-) < \infty$, then the following asymptotic representation holds:

$$\frac{1}{R(t;s)} - \frac{1}{1-s} = \frac{f''(1-)}{2}t + \mathcal{O}(\ln t) \quad \text{as} \quad t \to \infty \tag{1.3}$$

for all $s \in [0,1)$ (see [11, p. 72]).

Later on Zolotarev [12] has found a principally new result on asymptotic representation of $q(t)$ without the assumption of $f''(1-) < \infty$. Namely, providing that $g(x) = f(1-x)$ is a regularly varying function at zero that is

$$\lim_{x\downarrow 0} \frac{xg'(x)}{g(x)} = \gamma$$

with index $1 < \gamma = 1 + \alpha \leq 2$, he has proved that

$$\frac{q(t)}{f(1-q(t))} \sim \alpha t \quad \text{as} \quad t \to \infty. \tag{1.4}$$

Further, we assume that the infinitesimal GF $f(s)$ has the following representation:

$$f(s) = (1-s)^{1+\nu}\mathcal{L}\left(\frac{1}{1-s}\right) \tag{1.5}$$

for all $s \in [0,1)$, where $0 < \nu < 1$ and $\mathcal{L}(x)$ is slowly varying (SV) function at infinity (in sense of Karamata, see [10]).

Pakes [9], in connection with the proof of limit theorems has established, that if the condition (1.5) holds, then

$$\frac{1}{R(t;s)} = U\left(t + V\left(\frac{1}{1-s}\right)\right), \tag{1.6}$$

where $V(x) = \mathcal{M}(1 - 1/x)$ and $\mathcal{M}(s)$ is GF of invariant measure of MBP that is $\mathcal{M}(s) = \sum_{j\in\mathcal{S}} \mu_j s^j$ and $\sum_{i\in\mathcal{S}} \mu_i P_{ij}(t) = \mu_j$, $j \in \mathcal{S}$. Function $U(y)$ is the inverse of $V(x)$. The formula (1.6) gives an alternative relation to (1.4):

$$q(t) = \frac{1}{U(t)}.$$

The following lemma is a version of more recent result that was proved in [6] (second part statement of Lemma 1), in which the character of asymptotical decreasing of the function $R(t;s)$ seems to be more explicit rather than in (1.6).

Lemma 1.1. *If the condition* (1.5) *holds, then*

$$R(t;s) = \frac{\mathcal{N}(t)}{(\nu t)^{1/\nu}} \left[1 - \frac{M(t;s)}{\nu t}\right], \tag{1.7}$$

where

$$\mathcal{N}^{\nu}(t)\mathcal{L}\left(\frac{(\nu t)^{1/\nu}}{\mathcal{N}(t)}\right) \longrightarrow 1 \quad as \quad t \to \infty. \tag{1.8}$$

Here $M(t;0) = 0$ for all $t > 0$ and $M(t;s) \to \mathcal{M}(s)$ as $t \to \infty$, where $\mathcal{M}(s)$ is GF of invariant measures of MBP and

$$\mathcal{M}(s) = \int\limits_{1}^{1/(1-s)} \frac{dx}{x^{1-\nu}\mathcal{L}(x)}.$$

1.2. Aim and basic assumptions. The representation (1.5) implies that the second moment $2b := f''(1-) = \infty$. If $b < \infty$, then it takes place with $\nu = 1$ and $\mathcal{L}(t) \to b$ as $t \to \infty$ and we can write asymptotic formula in type of (1.3). This circumstance suggests that we can look for some sufficient condition such that an asymptotic relation similar to (1.3) will be true provided that (1.5) holds. So the aim of the paper is to improve the Lemma 1.1 and thereafter to refine (1.4) and to improve some earlier well-known results by imposing an additional condition on the function $\mathcal{L}(s)$.

Let

$$\Lambda(y) := y^{\nu}\mathcal{L}\left(\frac{1}{y}\right)$$

for $y \in (0,1]$ and rewrite (1.5) as

$$[f_\nu]: \, f(1-y) = y\Lambda(y).$$

Note that the function $y\Lambda(y)$ is positive, tends to zero and has a monotone derivative so that $y\Lambda'(y)/\Lambda(y) \to \nu$ as $y \downarrow 0$ (see [3, p. 401]). Thence it is natural to write

$$[\Lambda_\delta]: \, \frac{y\Lambda'(y)}{\Lambda(y)} = \nu + \delta(y),$$

where $\delta(y)$ is continuous and $\delta(y) \to 0$ as $y \downarrow 0$.

Throughout the paper $[f_\nu]$ and $[\Lambda_\delta]$ are our *basic assumptions*.

Since $\mathcal{L}(\lambda x)/\mathcal{L}(x) \to 1$ as $x \to \infty$ for each $\lambda > 0$, we can write

$$\frac{\mathcal{L}(\lambda x)}{\mathcal{L}(x)} = 1 + \varrho(x), \tag{1.9}$$

where $\varrho(x) \to 0$ as $x \to \infty$. If there is some positive function $g(x)$ so that $g(x) \to 0$ and $\varrho(x) = \mathcal{O}(g(x))$ as $x \to \infty$, then $\mathcal{L}(x)$ is said to be *SV-function with remainder* at infinity (see [3, p. 185], condition SR1). As we can see below, if the function $\delta(y)$ is known, it will be possible to estimate a decreasing rate of the remainder $\varrho(x)$.

Using that $\Lambda(1) = \mathcal{L}(1) = a_0$ integration $[\Lambda_\delta]$ yields

$$\Lambda(y) = a_0 y^{\nu} \exp \int_1^y \frac{\delta(u)}{u}\, du.$$

Therefore, we have

$$\mathcal{L}\left(\frac{1}{y}\right) = a_0 \exp \int_1^y \frac{\delta(u)}{u}\, du.$$

Changing variable as $u = 1/t$ in the integrand gives

$$\mathcal{L}(x) = a_0 \exp \int_1^x \frac{\varepsilon(t)}{t}\, dt, \tag{1.10}$$

where $\varepsilon(t) = -\delta(1/t)$ and $\varepsilon(t) \to 0$ as $t \to \infty$. It follows from (1.9) and (1.10) that

$$\frac{\mathcal{L}(\lambda x)}{\mathcal{L}(x)} = \exp \int_x^{\lambda x} \frac{\varepsilon(t)}{t}\, dt = 1 + \varrho(x) \quad \text{as} \quad x \to \infty$$

for each $\lambda > 0$, where $\varrho(x) \to 0$ as $x \to \infty$. Thus,

$$\int_x^{\lambda x} \frac{\varepsilon(t)}{t}\, dt = \ln[1 + \varrho(x)] = \varrho(x) + \mathcal{O}\left(\varrho^2(x)\right) \quad \text{as} \quad x \to \infty.$$

Applying the mean value theorem to the left-hand side of the last equality, we can assert that

$$\varepsilon(x) = \mathcal{O}(\varrho(x)) \quad \text{as} \quad x \to \infty. \tag{1.11}$$

Thus, the assumption $[\Lambda_\delta]$ provides that $\mathcal{L}(s)$ to be an SV-function at infinity with the remainder in the form of $\varrho(x) = \mathcal{O}(\delta(1/x))$ as $x \to \infty$.

1.3. Results. Our results appear due to an improvement of the Lemma 1.1 under the basic assumptions. Let

$$\mho(t; s) := \int_0^t \delta(R(u; s))du, \quad \mho(t) := \mho(t; 0).$$

Needless to say $R(t; s) \to 0$ as $t \to \infty$, due to (1.7). Therefore, since $\delta(y) \to 0$ as $y \downarrow 0$, we make sure of

$$\frac{\mho(t; s)}{t} = \frac{1}{t} \int_0^t \delta(R(u; s))du = o(1) \quad \text{as} \quad t \to \infty.$$

Thus, $\mho(t; s) = o(t)$ as $t \to \infty$. Herewith a more important interest represents the special case when

$$\delta(y) = \Lambda(y). \tag{1.12}$$

Remark. The case (1.12) implies that $\mathcal{L}(x)$ be an SV-function at infinity with the remainder in the form of

$$\varrho(x) = \mathcal{O}\left(\frac{\mathcal{L}(x)}{x^{\nu}}\right) \quad \text{as} \quad x \to \infty.$$

So, under the condition (1.12) our results appear for all SV-functions at infinity with remainder $\varrho(x)$ in the form above.

Theorem 1.1. *Under the basic assumptions*

$$q(t) = \frac{\mathcal{N}(t)}{(\nu t)^{1/\nu}}\left(1 - \frac{\mho(t)}{\nu^2 t} + o\left(\frac{\mho(t)}{t}\right)\right) \quad \textit{as} \quad t \to \infty, \tag{1.13}$$

here and everywhere $\mathcal{N}(t)$ *is the SV-function satisfying* (1.8). *In addition, if* (1.12) *holds, then*

$$q(t) = \frac{\mathcal{N}(t)}{(\nu t)^{1/\nu}}\left(1 - \frac{\ln\left[a_0 \nu t + 1\right]}{\nu^3 t} + o\left(\frac{\ln t}{t}\right)\right) \quad \textit{as} \quad t \to \infty. \tag{1.14}$$

Theorem 1.2. *Under the basic assumptions*

$$(\nu t)^{1+1/\nu} P_{11}(t) = \frac{\mathcal{N}(t)}{a_0}\left(1 - \frac{1+\nu}{\nu^2}\frac{\mho(t)}{t} + o\left(\frac{\mho(t)}{t}\right)\right) \tag{1.15}$$

as $t \to \infty$. *In addition, if* (1.12) *holds, then*

$$(\nu t)^{1+1/\nu} P_{11}(t) = \frac{\mathcal{N}(t)}{a_0}\left(1 - \frac{1+\nu}{\nu^3}\frac{\ln\left[a_0 \nu t + 1\right]}{t} + o\left(\frac{\ln t}{t}\right)\right) \tag{1.16}$$

as $t \to \infty$.

Let $\mathbb{P}_i\{*\} := \mathbb{P}\{* \mid Z(0) = i\}$ and consider a conditional distribution

$$\mathbb{P}_i^{\mathcal{H}(t+u)}\{*\} := \mathbb{P}_i\left\{* \middle| t + u < \mathcal{H} < \infty\right\}.$$

It was shown in [7] that the probability measure

$$\mathcal{Q}_{ij}(t) := \lim_{u \to \infty} \mathbb{P}_i^{\mathcal{H}(t+u)}\{Z(t) = j\} = \frac{j}{i} P_{ij}(t) \tag{1.17}$$

defines the continuous-time Markov chain $\{W(t), t \geq 0\}$ with states space $\mathcal{E} \subset \mathbb{N}$, called the *Markov Q-process* (MQP). According to the definition

$$\mathcal{Q}_{ij}(t) = \mathbb{P}_i\{Z(t) = j \mid \mathcal{H} = \infty\},$$

so MQP can be interpreted as MBP with non degenerating trajectory in remote future.

In a term of GF the equality (1.17) can be written as following:

$$G_i(t; s) := \sum_{j \in \mathcal{E}} \mathcal{Q}_{ij}(t) s^j = \left[F(t; s)\right]^{i-1} G(t; s), \tag{1.18}$$

where GF $G(t; s) := G_1(t; s) = \mathbb{E}\left[s^{W(t)} \mid W(0) = 1\right]$ and

$$G(t; s) = -s\,\frac{\partial R(t; s)}{\partial s} \quad \text{for all} \quad t \geq 0.$$

Combining the backward and the forward Kolmogorov equation we write it in the next form

$$G(t;s) = s\,\frac{f(F(t;s))}{f(s)} \quad \text{for all} \quad t \geq 0. \tag{1.19}$$

Since $F(t;s) \to 1$ as $t \to \infty$ uniformly for all $s \in [0,1)$ according to (1.18) it is suffice to consider the case $i = 1$.

Theorem 1.3. *Under the basic assumptions*

$$(\nu t)^{1+1/\nu} G(t;s) = \pi(s)\mathcal{N}(t)(1 + \rho(t;s)), \tag{1.20}$$

where the function $\pi(s)$ has an expansion in powers of s with nonnegative coefficients so that $\pi(s) = \sum_{j\in\mathcal{E}} \pi_j s^j$ and $\{\pi_j, j \in \mathcal{E}\}$ is an invariant measure for MQP. Moreover, it has a form of

$$\pi(s) = \frac{s}{(1-s)^{1+\nu}}\,\mathcal{L}_\pi\left(\frac{1}{1-s}\right), \tag{1.21}$$

where $\mathcal{L}_\pi(*) = \mathcal{L}^{-1}(*)$. Furthermore, $\rho(t;s) = o(1)$ as $t \to \infty$. In addition, if (1.12) *holds, then*

$$\rho(t;s) = -\frac{1+\nu}{\nu^3}\,\frac{\ln\left[\Lambda(1-s)\nu t + 1\right]}{t} + o\left(\frac{\ln t}{t}\right) \quad as \quad t \to \infty. \tag{1.22}$$

Note that in accordance with Tauberian theorem for the power series (see [4, p. 513], Ch. XIII, § 5, Theorem 5) the relation (1.21) implies

$$\sum_{j=1}^{n} \pi_j \sim \frac{1}{\Gamma(2+\nu)} n^{1+\nu} \mathcal{L}_\pi(n) \quad \text{as} \quad n \to \infty,$$

where $\Gamma(*)$ is Euler's gamma function and $(\mathcal{L}_\pi \mathcal{L})(*) = 1$.

Let $D(t;x) := \mathbb{P}\{q(t)W(t) \leq x\}$. In [6] (Theorem 21) it was proved, that if $[f_\nu]$ holds, then

$$\lim_{t\to\infty} D(t;x) = D(x),$$

where

$$\Psi(\theta) := \int_0^\infty e^{-\theta x} dD(x) = \frac{1}{(1+\theta^\nu)^{1+1/\nu}}.$$

Theorem 1.4. *Let*

$$\Delta(t;\theta) := \left|\int_0^\infty e^{-\theta x} dD(t;x) - \Psi(\theta)\right|.$$

If the basic assumptions and (1.12) *hold, then*

$$\sup_{\theta\in(0,\infty)} \Delta(t;\theta) = \frac{1+\nu}{\nu^3}\,\frac{\ln t}{t}(1 + o(1)) \quad as \quad t \to \infty. \tag{1.23}$$

Theorem 1.4 yields that from Berry – Esseen type inequality (see [4, p. 616], Ch. XVI, § 3, Lemma 2) follows the following corollary.

***Corollary* 1.1.** *Under the conditions of Theorem* 1.4

$$\sup_{x\in(0,\infty)} |D(t;x) - D(x)| = \mathcal{O}\left(\frac{\ln t}{t}\right) \quad as \quad t \to \infty.$$

2. Auxiliaries. The following lemma improves the statement of the Lemma 1.1.

Lemma 2.1. *Under the basic assumptions*

$$\frac{1}{\Lambda(R(t;s))} - \frac{1}{\Lambda(1-s)} = \nu t + \int_0^t \delta(R(u;s))du. \tag{2.1}$$

If, in addition, (1.12) *holds, then*

$$\frac{1}{\Lambda(R(t;s))} - \frac{1}{\Lambda(1-s)} = \nu t + \frac{1}{\nu}\ln\nu(t;s) + o(\ln\nu(t;s)) \tag{2.2}$$

as $t \to \infty$, *where* $\nu(t;s) = \Lambda(1-s)\nu t + 1$.

Proof. From $[\Lambda_\delta]$ we write

$$\frac{R\Lambda'(R)}{\Lambda(R)} = \nu + \delta(R), \tag{2.3}$$

since $R = R(t;s) \to 0$ as $t \to \infty$. By the backward Kolmogorov equation $\partial F/\partial t = f(F)$ and considering representation $[f_\nu]$ the relation (2.3) becomes

$$\frac{d\Lambda(R)}{dt} = -\frac{\Lambda(R)}{R} f(1-R)(\nu + \delta(R)) = -\Lambda^2(R)(\nu + \delta(R)).$$

Therefore,

$$d\left[\frac{1}{\Lambda(R)} - \nu t\right] = \delta(R)dt. \tag{2.4}$$

Integrating (2.4) from 0 to t, we obtain (2.1).

To prove (2.2) we should calculate integral in (2.1) putting $\delta(y) = \Lambda(y)$. Write

$$\frac{1}{\Lambda(R(t;s))} - \frac{1}{\Lambda(1-s)} = \nu t + \int_0^t \Lambda(R(u;s))du. \tag{2.5}$$

Since $\Lambda(y) = y^\nu \mathcal{L}(1/y)$ and $R(t;s) \to 0$ as $t \to \infty$ for $s \in [0,1)$, the integral in the right-hand side of (2.5) is $o(t)$. Hence

$$\Lambda(R(t;s)) = \frac{\Lambda(1-s)}{\nu(t;s)} + o\left(\frac{\Lambda(1-s)}{\nu(t;s)}\right) \quad \text{as} \quad t \to \infty,$$

where $\nu(t;s) = \Lambda(1-s)\nu t + 1$. Therefore,

$$\mho(t;s) = \int_0^t \Lambda(R(u;s))du = \frac{1}{\nu}\ln\nu(t;s) + o(\ln\nu(t;s)) \quad \text{as} \quad t \to \infty. \tag{2.6}$$

This together with (2.5) implies (2.2).

Lemma 2.1 is proved.

In the proof of our results we also will essentially use the following lemma.

Lemma 2.2. *Let*

$$\phi(y) := y - yK(y),$$

where $K(y) \to 0$ *as* $y \downarrow 0$. *If, in addition, to the basic assumptions* (1.12) *holds, then*

$$\mathcal{L}\left(\frac{1}{\phi(y)}\right) = \mathcal{L}\left(\frac{1}{y}\right)(1 + \mathcal{O}(\Lambda(y))) \quad \textit{as} \quad y \downarrow 0. \tag{2.7}$$

Proof. Since the function $\mathcal{L}(x) = x^{\nu}\Lambda(1/x)$ is differentiable, by virtue of the mean value theorem we have

$$\mathcal{L}\left(\frac{x}{1-K}\right) - \mathcal{L}(x) = \mathcal{L}'\left(\frac{1-\gamma K}{1-K}x\right)\frac{K}{1-K}\,x, \tag{2.8}$$

where $K := K(1/x)$ and $0 < \gamma < 1$. Since $\varrho(x) = \mathcal{O}(\mathcal{L}(x)/x^{\nu})$, from (1.10) and (1.11) it follows that

$$\mathcal{L}'(u) = \mathcal{L}(u)\frac{\varepsilon(u)}{u} = \mathcal{O}\left(\frac{\mathcal{L}^2(u)}{u^{1+\nu}}\right) \quad \text{as} \quad u \to \infty. \tag{2.9}$$

Denote $u = (1-\gamma K)\,x/(1-K)$. Since $K(1/x) \to 0$, then $u \sim x$ and $\mathcal{L}(u) \sim \mathcal{L}(x)$ as $x \to \infty$. Therefore after using (2.9) in the equality (2.8) and some elementary transformations the assertion (2.7) readily follows.

Lemma 2.2 is proved.

3. Proofs of results. ***Proof of Theorem* 1.1.** Putting $s = 0$ in (2.1), we have

$$\frac{1}{\Lambda(q(t))} = \nu t + \frac{1}{a_0} + \mho(t) \tag{3.1}$$

and by elementary arguments we get to assertion (1.13). Similarly putting $s = 0$ in (2.2), we obtain (1.14).

Theorem 1.1 is proved.

***Proof of Theorem* 1.2.** Considering together the backward and the forward Kolmogorov equations and seeing $[f_{\nu}]$, we write

$$\frac{\partial F(t;s)}{\partial s} = \frac{f(1-R(t;s))}{f(s)} = \frac{R(t;s)\Lambda(R(t;s))}{f(s)}.$$

Thence at $s = 0$ we deduce

$$P_{11}(t) = \frac{q(t)\Lambda(q(t))}{a_0}.$$

Hence using (1.13) and (1.14) the relations (1.15) and (1.16) easily follow.

Theorem 1.2 is proved.

***Proof of Theorem* 1.3.** It follows from (1.19) and $[f_{\nu}]$ that

$$G(t;s) = \frac{R^{1+\nu}(t;s)}{f(s)}\,\mathcal{L}\left(\frac{1}{R(t;s)}\right). \tag{3.2}$$

On the other hand, (2.1) entails

$$R(t;s) = \frac{\mathcal{N}(t;s)}{(\nu t)^{1/\nu}}\left(1 - \frac{\mho(t;s)}{\nu^2 t}(1 + o(1))\right) \quad \text{as} \quad t \to \infty, \tag{3.3}$$

where $\mathcal{N}(t;s) := \mathcal{L}^{-1/\nu}\left(1/R(t;s)\right)$ and $\mho(t;s) = \int_0^t \delta(R(u;s))du = o(t)$ as $t \to \infty$. From (3.3) we conclude that

$$R(t;s) = q(t)\frac{\mathcal{N}(t;s)}{\mathcal{N}(t)}\left(1 - \frac{\mho(t;s)}{\nu^2 t}(1+o(1))\right) \quad \text{as} \quad t \to \infty.$$

Since $R(t;s)/q(t) \to 1$ uniformly for $s \in [0,1)$, then $\mathcal{N}(t;s)/\mathcal{N}(t) \to 1$ for all $s \in [0,1)$. But in accordance with (1.9) and (1.12)

$$\frac{\mathcal{L}\left(R^{-1}(t;s)\right)}{\mathcal{L}\left(q^{-1}(t)\right)} = 1 + \mathcal{O}\left(\frac{1}{t}\right)$$

and, therefore,

$$\frac{\mathcal{N}(t;s)}{\mathcal{N}(t)} = 1 + \mathcal{O}\left(\frac{1}{t}\right) \quad \text{as} \quad t \to \infty. \tag{3.4}$$

Combining $[f_\nu]$ and (3.2)–(3.4), we obtain

$$G(t;s) = \frac{\pi(s)\mathcal{N}(t)}{(\nu t)^{1+1/\nu}}\left(1 - \frac{1+\nu}{\nu^2}\frac{\mho(t;s)}{t}(1+o(1))\right) \quad \text{as} \quad t \to \infty. \tag{3.5}$$

The representation (1.20) with evanescent (1.22) follows from (2.6) and (3.5).

To show that $\pi(s)$ is GF of invariant measure, from (1.19) we obtain the following functional equation:

$$G(t+\tau;s) = \frac{G(t;s)}{F(t;s)}\, G(\tau; F(t;s)) \quad \text{for all} \quad \tau > 0,$$

since $F(t+\tau;s) = F(\tau;F(t;s))$ (see [11]). Then taking limit as $\tau \to \infty$ it follows from this equation that

$$\pi(s) = \frac{G(t;s)}{F(t;s)}\, \pi(F(t;s)).$$

This is equivalent to the equation

$$\pi_j = \sum_{i \in \mathcal{E}} \pi_i \mathcal{Q}_{ij}(t).$$

Thus, $\{\pi_j, j \in \mathcal{E}\}$ is an invariant measure for MQP.

Theorem 1.3 is proved.

Proof of Theorem **1.4.** Consider the Laplace transform

$$\Psi(t;\theta) := \mathbb{E}e^{-\theta q(t) W(t)} = G(t;\theta(t)),$$

where $\theta(t) = \exp\{-\theta q(t)\}$. From $[f_\nu]$ and (1.19) we write

$$\Psi(t;\theta) = \theta(t)\left(\frac{R(t;\theta(t))}{1-\theta(t)}\right)^{1+\nu} \frac{\mathcal{L}\left(1/R(t;\theta(t))\right)}{\mathcal{L}\left(1/(1-\theta(t))\right)}. \tag{3.6}$$

It follows from (2.2) that

$$\frac{1}{\Lambda(R(t;\theta(t)))} - \frac{1}{\Lambda(1-\theta(t))} = \nu t + \frac{1}{\nu}\ln\left[\Lambda(1-\theta(t))\nu t + 1\right] + o(\ln t) \tag{3.7}$$

as $t \to \infty$. Since $1 - e^{-x} \sim x - x^2/2$ as $x \to 0$, then according to our designation

$$\Lambda(1-\theta(t)) = \theta^{\nu} q^{\nu}(t) \mathcal{L}\left(\frac{1}{1-\theta(t)}\right)\left(1 - \frac{1}{2}\theta q(t)(1+o(1))\right)^{\nu}$$

as $t \to \infty$. By Lemma 2.2 with $K(y) = y/2$

$$\mathcal{L}\left(\frac{1}{1-\theta(t)}\right) = \mathcal{L}\left(\frac{1}{q(t)}\right)\left(1 + \mathcal{O}\left(\frac{1}{t}\right)\right) \quad \text{as} \quad t \to \infty. \tag{3.8}$$

Then

$$\Lambda(1-\theta(t)) = \theta^{\nu}\Lambda(q(t))\left(1 + \mathcal{O}\left(\frac{1}{t}\right)\right) \quad \text{as} \quad t \to \infty,$$

since $q(t) = \mathcal{O}\big(\mathcal{N}(t)/t^{1/\nu}\big)$ and $\nu < 1$. Thence considering (3.1)

$$\Lambda(1-\theta(t)) = \frac{\theta^{\nu}}{\nu t}\left(1 - \frac{1}{\nu^2}\frac{\ln t}{t}(1+o(1))\right) \quad \text{as} \quad t \to \infty. \tag{3.9}$$

By using (3.9), we can write (3.7) in the following form:

$$\frac{1}{\Lambda(R(t;\theta(t)))} = \nu t \frac{1+\theta^{\nu}}{\theta^{\nu}}\left(1 - \frac{1}{1+\theta^{\nu}}\frac{\ln t}{\nu^2 t}(1+o(1))\right)$$

and, therefore,

$$R(t;\theta(t)) = \frac{\mathcal{N}_{\theta}(t)}{(\nu t)^{1/\nu}}\frac{\theta}{(1+\theta^{\nu})^{1/\nu}}\left(1 - \frac{1}{1+\theta^{\nu}}\frac{\ln t}{\nu^3 t}(1+o(1))\right) \tag{3.10}$$

as $t \to \infty$, where $\mathcal{N}_{\theta}(t) := \mathcal{L}^{-1/\nu}\left(1/R(t;\theta(t))\right)$.

Since $R(t;s)/q(t) \to 1$ for all $s \in [0,1)$, then by force of (3.10) it is necessary that

$$\frac{R\left(t;\theta(t)\right)}{q(t)} \longrightarrow c(\theta) \quad \text{as} \quad t \to \infty,$$

where $|c(\theta)| < \infty$ at any fixed $\theta \in (0,\infty)$. Therefore, according to (1.9)

$$\frac{\mathcal{L}\left(R^{-1}(t;\theta(t))\right)}{\mathcal{L}\left(q^{-1}(t)\right)} = 1 + \mathcal{O}(\Lambda(q(t))) \quad \text{as} \quad t \to \infty \tag{3.11}$$

or the same

$$\frac{\mathcal{N}_{\theta}(t)}{\mathcal{N}(t)} = 1 + \mathcal{O}\left(\frac{1}{t}\right) \quad \text{as} \quad t \to \infty.$$

Thus, (3.10) becomes

$$R(t;\theta(t)) = \frac{\mathcal{N}(t)}{(\nu t)^{1/\nu}}\frac{\theta}{(1+\theta^{\nu})^{1/\nu}}\left(1 - \frac{1}{1+\theta^{\nu}}\frac{\ln t}{\nu^3 t}(1+o(1))\right) \tag{3.12}$$

as $t \to \infty$.

Further, by using (3.8) and (3.11), we can rewrite (3.6) as

$$\Psi(t;\theta) = \left(\frac{R\left(t;\theta(t)\right)}{1-\theta(t)}\right)^{1+\nu}\left(1 + \mathcal{O}\left(\frac{1}{t}\right)\right) \quad \text{as} \quad t \to \infty,$$

and by using (3.12), after some transformation we obtain

$$\Psi(t;\theta) = \Psi(\theta)\left(1 + \frac{\theta^{\nu}}{1+\theta^{\nu}}\frac{1+\nu}{\nu^3}\frac{\ln t}{t}(1+o(1))\right) \quad \text{as} \quad t \to \infty. \tag{3.13}$$

The assertion (1.23) follows from (3.13).

Theorem 1.4 is proved.

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