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# CERTAIN SUBCLASSES OF MEROMORPHICALLY $q$-STARLIKE FUNCTIONS ASSOCIATED WITH THE $q$-DERIVATIVE OPERATORS*

# ДЕЯКІ ПІДКЛАСИ МЕРОМОРФНИХ $q$-ЗІРКОВИХ ФУНКЦІЙ, ПОВ'ЯЗАНІ З $q$-ПОХІДНИМИ ОПЕРАТОРАМИ

The purpose of the present paper is to establish several general results concerning the partial sums of meromorphically starlike functions defined here by means of a certain class of $q$-derivative (or $q$-difference) operators. The familiar concept of neighborhood for meromorphic functions are also considered. Moreover, by using a Ruscheweyh-type $q$-derivative operator, we define and study another new class of functions emerging from the class of normalized meromorphic functions.

Метою цієї статті є отримання кількох загальних результатів, що пов'язані з частковими сумами мероморфних зіркових функцій, які визначаються за допомогою деякого класу $q$-похідних (або $q$-різницевих) операторів. Також розглянуто відоме поняття околу для мероморфних функцій. Крім того, за допомогою $q$-похідного оператора типу Рушевая визначається та вивчається новий клас функцій, який виводиться з класу нормалізованих мероморфних функцій.

**1. Introduction and definition.** Let the class of functions $f$ which are analytic in the open unit disk

$$\mathbb{U} = \left\{ z : z \in \mathbb{C} \text{ and } |z| < 1 \right\}$$

be denoted by $\mathcal{H}(\mathbb{U})$. Also, by $\mathcal{A}$ we denote the subclass of the analytic functions $f$ in $\mathcal{H}(\mathbb{U})$ satisfying the following normalization condition:

$$f(0) = f'(0) - 1 = 0.$$

Equivalently, the function $f \in \mathcal{A}$ has the Taylor–Maclaurin series expansion given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \forall z \in \mathbb{U}. \tag{1.1}$$

Let $\mathcal{S}$ be the subclass of analytic function class $\mathcal{A}$, consisting of all univalent functions in $\mathbb{U}$.

A function $f \in \mathcal{A}$ is said to be starlike in $\mathbb{U}$, if it satisfies the following inequality:

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \quad \forall z \in \mathbb{U},$$

where, for example, $\Re(z)$ denotes the real part of $z \in \mathbb{C}$.We denote by $\mathcal{S}^*$ all such starlike functions in the open unit disk $\mathbb{U}$.

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For any two functions $f$ and $g$, which are analytic in $\mathbb{U}$, we say that the function $f$ is subordinate to $g$, written as follows:

$$f \prec g \qquad \text{or} \qquad f(z) \prec g(z),$$

if there is a Schwarz function $w$, which is analytic in $\mathbb{U}$ with

$$w(0) = 0 \qquad \text{and} \qquad \big|w(z)\big| < 1,$$

such that

$$f(z) = g\big(w(z)\big).$$

Furthermore, for the function $g$, which is univalent in $\mathbb{U}$, it follows that

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \iff f(0) = g(0) \qquad \text{and} \qquad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Next, for a function $f \in \mathcal{A}$ given by (1.1) and another function $g \in \mathcal{A}$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \quad \forall z \in \mathbb{U},$$

the convolution (or the Hadamard product) of the functions $f$ and $g$ is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z) \quad \forall z \in \mathbb{U}.$$

Let $\mathcal{P}$ denote the class of analytic functions $p$ normalized by

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$$

such that

$$\Re\{p(z)\} > 0 \quad \forall z \in \mathbb{U}.$$

We now recall some essential definitions and concept details of the $q$-calculus, which are used in this paper. We suppose throughout this paper that $0 < q < 1$ and that

$$\mathbb{N} = \{1, 2, 3, \ldots\} = \mathbb{N} \setminus \{0\}, \quad \mathbb{N}_0 = \{0, 1, 2, \ldots\}.$$

**Definition 1.** *Let $q \in (0,1)$ and define the q-number $[\lambda]_q$ by*

$$[\lambda]_q = \begin{cases} \dfrac{1 - q^{\lambda}}{1 - q}, & \lambda \in \mathbb{C}, \\ \sum_{k=0}^{n-1} q^k = 1 + q + q^2 + \ldots + q^{n-1}, & \lambda = n \in \mathbb{N}. \end{cases}$$

**Definition 2.** *Let $q \in (0,1)$ and define the q-factorial $[n]_q!$ by*

$$[n]_q! = \begin{cases} 1, & n = 0, \\ \prod_{k=1}^{n-1} [k]_q, & n \in \mathbb{N}. \end{cases}$$

**Definition 3.** *Let $q \in (0,1)$ and define $q$-Pochhammer symbol $[t]_{q,n}$, $t \in \mathbb{C}$, $n \in \mathbb{N}_0$, by*

$$[t]_{q,n} = \frac{(q^t;q)_n}{(1-q)^n} = \begin{cases} 1, & n = 0, \\ [t]_q[t+1]_q[t+2]_q \ldots [t+n-1]_q, & n \in \mathbb{N}. \end{cases}$$

Moreover, the $q$-gamma-function $\Gamma_q(z)$ may be defined here by the following recurrence relation:

$$\Gamma_q(z+1) = [z]_q\Gamma_q(z) \quad \text{and} \quad \Gamma_q(1) = 1.$$

**Definition 4** [20, 21]**.** *The $q$-derivative (or the $q$-difference) $(D_q f)$ of a function $f$ is defined, in a given subset of $\mathbb{C}$, by*

$$(D_q f)(z) = \begin{cases} \dfrac{f(z) - f(qz)}{(1-q)z}, & z \neq 0, \\ f'(0), & z = 0, \end{cases} \tag{1.2}$$

*provided that $f'(0)$ exists.*

We note from Definition 4 that

$$\lim_{q \to 1-} (D_q f)(z) = \lim_{q \to 1-} \frac{f(z) - f(qz)}{(1-q)z} = f'(z)$$

for a differentiable function $f$ in a given subset of $\mathbb{C}$. It is readily seen from (1.1) and (1.2) that

$$(D_q f)(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}.$$

A number of subclasses of the normalized analytic function class $\mathcal{A}$ in Geometric Function Theory have been studied already from different viewpoints (see, for example, [7, 8, 11, 12, 15]). The above-defined $q$-calculus provides an important tool in order to investigate several subclasses of the class $\mathcal{A}$. The $q$-derivative (or the $q$-difference) operator $D_q$ was first used in Geometrc Function Theory by Ismail et al. [19] in order to study the $q$-analogue of the class $\mathcal{S}^*$ of starlike functions in $\mathbb{U}$ (see Definition 5 below). However, initial usages of the $q$-calculus in the context of Geometric Function Theory were presented systematically, and the basic (or $q$-) hypergeometric functions were first used in Geometric Function Theory, in a book chapter by Srivastava (see, for details, [31, p. 347] and also [1, 2, 13, 14, 18, 24, 26, 33, 34, 36 – 38]).

**Definition 5** [19]**.** *A function $f \in \mathcal{A}$ is said to belong to the class $\mathcal{S}_q^*$ of $q$-starlike functions if*

$$f(0) = f'(0) - 1 = 0$$

*and*

$$\left| \frac{z(D_q f)(z)}{f(z)} - \frac{1}{1-q} \right| \leq \frac{1}{1-q}.$$

It is readily observed that, as $q \to 1-$, the closed disk given by

$$\left| w - \frac{1}{1-q} \right| \leq \frac{1}{1-q}$$

becomes the right-half plane and the class $\mathcal{S}_q^*$ of $q$-starlike functions reduces to the familiar class $\mathcal{S}^*$ of starlike functions in $\mathbb{U}$.

We next let $\mathcal{M}$ denote the class of functions $f$ of the form

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n, \tag{1.3}$$

which are analytic in the *punctured* open unit disk

$$\mathbb{U}^* = \big\{ z : z \in \mathbb{C} \text{ and } 0 < |z| < 1 \big\} = \mathbb{U} \setminus \{0\}.$$

A function $f \in \mathcal{M}$ is said to be in the class $\mathcal{MS}(\alpha)$ of meromorphically starlike functions of order $\alpha$ if it satisfies the following inequality:

$$-\Re\left( \frac{zf'(z)}{f(z)} \right) > \alpha \quad \forall z \in \mathbb{U}, \quad 0 \leq \alpha < 1.$$

Next, analogous to Definition 4, we extend the notion of the $q$-derivative (or the $q$-difference) operator $D_q$ to a function $f$ given by (1.3) from the above-defined class $\mathcal{M}$ and also introduce the class $\mathcal{MS}_q(\alpha)$. Indeed, for $f \in \mathcal{M}$ given by (1.3), the $q$-derivative (or the $q$-difference) $D_q f$ is given by

$$(D_q f)(z) = \frac{f(z) - f(qz)}{(1-q)z} = -\frac{1}{qz^2} + \sum_{n=0}^{\infty} [n]_q a_n z^{n-1} \quad \forall z \in \mathbb{U}^*.$$

**Definition 6.** *A function $f \in \mathcal{M}$ is said to be in the class $\mathcal{MS}_q(\alpha)$ $0 \leq \alpha < 1$ if it satisfies the following condition*:

$$\left| \frac{\left( -\dfrac{z(D_q f)(z)}{f(z)} \right) - \alpha}{1-\alpha} - \frac{1}{1-q} \right| \leq \frac{1}{1-q}. \tag{1.4}$$

Throughout this paper, we use the notation $\mathcal{MS}_q(\alpha)$ for the class of meromorphically $q$-starlike functions of order $\alpha$.

***Remark* 1.** It is easily seen that

$$\lim_{q \to 1-} \mathcal{MS}_q(\alpha) =: \mathcal{MS}(\alpha) \quad \text{and} \quad \lim_{q \to 1-} \mathcal{MS}_q(0) =: \mathcal{MS},$$

where $\mathcal{MS}$ is the function class which was introduced and studied by Clunie (see [10]).

Since the work in the meromorphically univalent case has parallel to that of the analytically univalent case, one is tempted to search for results analogous to those of Silverman [30] for meromorphically univalent functions in $\mathbb{U}^*$. Thus, in this paper, we are motivated essentially by the works [4, 9, 16, 29, 30] (see also [23, 25, 27, 35]). We propose to investigate the ratio of a function of the form (1.3) to its sequence of partial sums given by

$$f_k(z) = \frac{1}{z} + \sum_{n=0}^{k} a_n z^n, \quad k \in \mathbb{N}, \tag{1.5}$$

when the coefficients are sufficiently small. We will determine sharp lower bounds for

$$\Re\left(\frac{f(z)}{f_k(z)}\right), \quad \Re\left(\frac{f_k(z)}{f(z)}\right), \quad \Re\left(\frac{(D_q f)(z)}{(D_q f_k)(z)}\right) \quad \text{and} \quad \Re\left(\frac{(D_q f_k)(z)}{(D_q f)(z)}\right).$$

Furthermore, in this paper, we introduce the $(\xi, q)$-neighborhood of a function $f \in \mathcal{M}$ of the form (1.3) by means of the following definition.

**Definition 7.** *For $\xi \geq 0$, $0 \leq \alpha < 1$ and $f \in \mathcal{M}$ given by* (1.3), *we define the $(\xi, q)$-neighborhood of the function $f$ by*

$$N_{(\xi,q)}(f) = \left\{ g : g \in \mathcal{M},\ g(z) = \frac{1}{z} + \sum_{n=0}^{\infty} b_n z^n \ \textit{and} \ \sum_{n=0}^{\infty} \mathcal{L}(n, q, \alpha)|a_k - b_k| \leq \xi \right\}, \tag{1.6}$$

*where*

$$\mathcal{L}(n, q, \alpha) = \frac{\big(2[n]_q + (1+q)\alpha\big)q}{q - 1 + (1+q)(1 - \alpha q)}, \quad n \in \mathbb{N}_0. \tag{1.7}$$

**2. Main results and their demonstration.** First of all, we give a sufficient condition for a function $f \in \mathcal{M}$ of the form (1.3) to be in the class $\mathcal{MS}_q(\alpha)$.

**Theorem 1.** *Let*

$$\frac{1}{q} - \alpha > 0.$$

*Suppose also that the function $f \in \mathcal{M}$ is given by* (1.3). *If*

$$\sum_{n=0}^{\infty} \big([n]_q + \alpha\big)|a_n| \leq \frac{1}{q} - \alpha, \tag{2.1}$$

*then $f \in \mathcal{MS}_q(\alpha)$.*

***Proof.*** Let $f \in \mathcal{M}$. Then, from (1.4) we have

$$\left| \frac{z(D_q f)(z)}{f(z)} + \frac{1 - \alpha q}{1 - q} \right| =$$

$$= \left| -\frac{\dfrac{1}{qz} + \sum_{n=0}^{\infty} [n]_q a_n z^n + \dfrac{1 - \alpha q}{1 - q}\left(\dfrac{1}{z} + \sum_{n=0}^{\infty} a_n z^n\right)}{\dfrac{1}{z} + \sum_{n=0}^{\infty} a_n z^n} \right| \leq$$

$$\leq \frac{\dfrac{1-\alpha q}{1-q} - \dfrac{1}{q} + \sum_{n=0}^{\infty} \left([n]_q + \dfrac{1-\alpha q}{1-q}\right)|a_n|}{1 + \sum_{n=0}^{\infty} |a_n|}. \tag{2.2}$$

This last expression in (2.2) is bounded above by $\dfrac{1-\alpha}{1-q}$ if the condition (2.1) is satisfied.

Theorem 1 is proved.

Our next result is based upon Definition 7.

**Theorem 2.** *Let $\xi \geq 0$ and $f \in \mathcal{M}$ given by* (1.3) *satisfy the following condition*:

$$\frac{f(z) + \varepsilon z^{-1}}{1+\varepsilon} \in \mathcal{MS}_q(\alpha) \tag{2.3}$$

*for any complex number $\varepsilon$ such that $|\varepsilon| < \xi$. Then*

$$N_{(\xi,q)}(f) \subset \mathcal{MS}_q(\alpha). \tag{2.4}$$

***Proof.*** By noting that the condition (1.4) can be written as follows:

$$\left| \frac{\dfrac{z(D_q)f(z)}{f(z)} + 1}{\dfrac{z(D_q f)(z)}{f(z)} + (1+q)\alpha - 1} \right| < 1, \tag{2.5}$$

it is easy to see from the condition (2.5) that $g(z) \in \mathcal{MS}_q(\alpha)$ if and only if

$$\frac{z(D_q g)(z) + g(z)}{z(D_q g)(z) + \big((1+q)\alpha - 1\big)g(z)} \neq \sigma \quad \forall z \in \mathbb{U}, \quad \sigma \in \mathbb{C}, \quad |\sigma| = 1,$$

which is equivalent to

$$\frac{(g * h)(z)}{z^{-1}} \neq 0 \quad \forall z \in \mathbb{U}. \tag{2.6}$$

The function $h(z)$, which is involved in (2.6), is given by

$$h(z) = \frac{1}{z} + \sum_{n=0}^{\infty} \Upsilon_n z^n$$

and

$$\Upsilon_n := \frac{\big([n]_q + 1 - \big([n]_q + (1+q)\alpha - 1\big)\sigma\big)q}{q - 1 + (1+q)(1-\alpha q)\sigma}. \tag{2.7}$$

It follows from (2.7) that

$$|\Upsilon_n| = \left| \frac{[n]_q + 1 - \big(\big([n]_q + (1+q)\alpha - 1\big)\sigma\big)\, q}{q - 1 + (1+q)(1-\alpha q)\sigma} \right| \leq$$

$$\leq \frac{\big([n]_q + 1 + \big([n]_q + (1+q)\alpha - 1\big)|\sigma|\big)\, q}{q - 1 + (1+q)(1-\alpha q)|\sigma|} =$$

$$= \frac{(2[n]_q + (1+q)\alpha)q}{q - 1 + (1+q)(1-\alpha q)} =: \mathcal{L}(n,q,\alpha), \qquad |\sigma| = 1, \quad n \in \mathbb{N}_0.$$

Now, if $f \in \mathcal{M}$ given by (1.3) satisfies the condition (2.3), we deduce from (2.6) that

$$\frac{(g * h)(z)}{z^{-1}} \neq \varepsilon \qquad |\varepsilon| < \xi, \quad \xi > 0, \tag{2.8}$$

or, equivalently, the condition in (2.8) can be written as follows:

$$\left|\frac{(g * h)(z)}{z^{-1}}\right| \geq \xi, \qquad z \in \mathbb{U}, \quad \xi > 0. \tag{2.9}$$

Next, if we suppose that

$$\mathfrak{q}(z) = \frac{1}{z} + \sum_{n=0}^{\infty} d_n z^n \in N_{(\xi,q)}(f),$$

it follows from (1.6) that

$$\left|\frac{\big((\mathfrak{q} - f) * h\big)(z)}{z^{-1}}\right| = \left|\sum_{n=0}^{\infty}(d_n - a_n)\Upsilon_n z^{n+1}\right| \leq$$

$$\leq |z| \sum_{n=0}^{\infty} \frac{2[n]_q + (1+q)\alpha}{q - 1 + (1+q)\,(1-\alpha q)} |d_n - a_n| < \xi. \tag{2.10}$$

Upon combining (2.9) and (2.10), we easily see that

$$\left|\frac{(\mathfrak{q} * h)(z)}{z^{-1}}\right| = \left|\frac{\big([f + (\mathfrak{q} - f] * h)\big)(z)}{z^{-1}}\right| \geq$$

$$\geq \left|\frac{(f * h)(z)}{z^{-1}}\right| - \left|\frac{\big((\mathfrak{q} - f) * h\big)(z)}{z^{-1}}\right| > 0. \tag{2.11}$$

The inequality in (2.11) now implies that

$$\left|\frac{(\mathfrak{q} * h)(z)}{z^{-1}}\right| \neq 0.$$

Consequently, we have

$$\mathfrak{q}(z) \in \mathcal{MS}_q(\alpha),$$

which completes the proof of Theorem 2.

We now derive the partial sums for the function class $\mathcal{MS}_q(\alpha)$.

**Theorem 3.** *Let $f \in \mathcal{M}$ given by* (1.3) *and define the partial sum $f_k(z)$ of the function $f$ by* (1.5), *where an empty sum is interpreted* (*as usual*) *to be nil. If*

$$\sum_{n=0}^{\infty} \mathcal{L}(n,q,\alpha)|a_n| \leq 1, \tag{2.12}$$

*then*

$$f(z) \in \mathcal{MS}_q(\alpha), \tag{2.13}$$

$$\Re\left(\frac{f(z)}{f_k(z)}\right) \geq 1 - \frac{1}{\mathcal{L}(k+1,q,\alpha)} \quad \forall z \in \mathbb{U}, \quad k \in \mathbb{N}, \tag{2.14}$$

*and*

$$\Re\left(\frac{f_k(z)}{f(z)}\right) \geq \frac{\mathcal{L}(k+1,q,\alpha)}{1+\mathcal{L}(k+1,q,\alpha)} \quad \forall z \in \mathbb{U}, \quad k \in \mathbb{N}, \tag{2.15}$$

*where* $\mathcal{L}(n,q,\alpha),\ n \in \mathbb{N}_0,$ *is defined by* (1.7). *The bound in* (2.14) *and* (2.15) *are sharp.*

***Proof.*** First of all, we set

$$f_1(z) = \frac{1}{z}$$

and we know that

$$\frac{f_1(z) + \varepsilon z^{-1}}{1+\varepsilon} = \frac{1}{z} \in \mathcal{MS}_q(\alpha).$$

Also, from (2.12), we can easily see that

$$\sum_{n=0}^{\infty} \mathcal{L}(n,q,\alpha)|a_n - 0| \leq 1, \tag{2.16}$$

where $\mathcal{L}(n,q,\alpha),\ n \in \mathbb{N}_0,$ is given by (1.7). Inequality in (2.16) now implies that $f \in N_{(1,q)}(z^{-1})$. From Theorem 2, we conclude that

$$f(z) \in N_{(1,q)}(z^{-1}) \subset \mathcal{MS}_q(\alpha).$$

We deduce that the assertion (2.13) holds true.

Next, it is easy to verify that

$$\mathcal{L}(k+1,q,\alpha) > \mathcal{L}(k,q,\alpha) > 1.$$

Thus, we find

$$\sum_{n=0}^{k} |a_n| + \mathcal{L}(k+1,q,\alpha) \sum_{n=0}^{\infty} |a_n| \leq \sum_{n=k+1}^{\infty} \mathcal{L}\left(n+1,q,\alpha\right)|a_n| \leq 1. \tag{2.17}$$

If we set

$$h_1(z) = \mathcal{L}(k+1,q,\alpha)\left\{\frac{f(z)}{f_k(z)} - \left(1 - \frac{1}{\mathcal{L}(k+1,q,\alpha)}\right)\right\} =$$

$$= 1 + \frac{\mathcal{L}(k+1,q,\alpha)\sum_{n=k+1}^{\infty} a_n z^{n+1}}{1+\sum_{n=0}^{k} a_n z^{n+1}}. \tag{2.18}$$

It follows from (2.17) and (2.18) that

$$\left|\frac{h_1(z)-1}{h_1(z)+1}\right| \leq \frac{\mathcal{L}(k+1,q,\alpha)\sum_{n=k+1}^{\infty}|a_n|}{2-2\sum_{n=0}^{k}|a_n|-\mathcal{L}(k+1,q,\alpha)\sum_{n=k+1}^{\infty}|a_n|} \leq 1 \quad \forall z\in\mathbb{U}. \tag{2.19}$$

Clearly, the inequality in (2.19) now shows that

$$\Re\big(h_1(z)\big)\geq 0 \quad \forall z\in\mathbb{U}. \tag{2.20}$$

Thus, by combining (2.17) and (2.20), we deduce that the assertion (2.14) holds true.

Next, by taking

$$f(z)=\frac{1}{z}-\frac{z^{n+1}}{\mathcal{L}(k+1,q,\alpha)}, \tag{2.21}$$

we easily observe that

$$\frac{f(z)}{f_k(z)}=1-\frac{z^{n+2}}{\mathcal{L}(k+1,q,\alpha)}\to 1-\frac{1}{\mathcal{L}(k+1,q,\alpha)}, \quad z\to 1-,$$

which shows that the bound in (2.14) is best possible for each $k\in\mathbb{N}$.

Just as above, we set that

$$h_2(z)=(1+\mathcal{L}(k+1,q,\alpha))\left\{\frac{f_k(z)}{f(z)}-\frac{\mathcal{L}(k+1,q,\alpha)}{1+\mathcal{L}(k+1,q,\alpha)}\right\}=$$

$$=1-\frac{(1+\mathcal{L}(k+1,q,\alpha))\sum_{n=k+1}^{\infty}a_nz^{n+1}}{1+\sum_{n=0}^{\infty}a_nz^{n+1}}. \tag{2.22}$$

By the virtue of (2.17) and (2.22), we conclude that

$$\left|\frac{h_2(z)-1}{h_2(z)+1}\right| \leq \frac{\big(1+\mathcal{L}(k+1,q,\alpha)\big)\sum_{n=k+1}^{\infty}|a_n|}{2-2\sum_{n=0}^{k}|a_n|+\big(1-\mathcal{L}(k+1,q,\alpha)\big)\sum_{n=k+1}^{\infty}|a_n|} \leq 1 \quad \forall z\in\mathbb{U},$$

which shows that

$$\Re\big(h_2(z)\big)\geq 0 \quad \forall z\in\mathbb{U}. \tag{2.23}$$

Finally, upon combining (2.22) and (2.23), we readily get the assertion (2.15) of Theorem 3. The bound in (2.15) is sharp with the extremal function $f(z)$ given by (2.21).

Theorem 3 is proved.

In its special case when $q\to 1-$, Theorem 3 yields the following known result proved by Cho and Owa ([9], see also Remark 1).

***Corollary* 1** [9]**.** *If the function $f$ of the form* (1.3) *satisfies the following condition:*

$$\sum_{n=0}^{\infty}(n+\alpha)\,|a_n|\leq 1-\alpha,$$

*then*

$$f\in\mathcal{MS}(\alpha),$$

$$\Re\left(\frac{f(z)}{f_k(z)}\right) \geq \frac{k+2\alpha}{k+1+\alpha} \quad \forall z \in \mathbb{U}, \quad k \in \mathbb{N},$$

*and*

$$\Re\left(\frac{f_k(z)}{f(z)}\right) \geq \frac{k+1+\alpha}{k+2} \quad \forall z \in \mathbb{U}, \quad k \in \mathbb{N}.$$

The proof of Theorem 4 below is similar to that of Theorem 3, so we have chosen to omit the analogous details.

**Theorem 4.** *Let* $f \in \mathcal{M}$ *given by* (1.3) *and define the partial sum* $f_k(z)$ *of* $f$ *by* (1.5). *If the condition* (2.12) *holds true*, *then*

$$\Re\left(\frac{(D_q f)(z)}{(D_q f_k)(z)}\right) \geq 1 - \frac{[k+1]_q}{\mathcal{L}(k+1,q,\alpha)} \tag{2.24}$$

*and*

$$\Re\left(\frac{(D_q f_k)(z)}{(D_q f)(z)}\right) \geq \frac{\mathcal{L}(k+1,q,\alpha)}{[k+1]_q + \mathcal{L}(k+1,q,\alpha)}, \tag{2.25}$$

*where* $\mathcal{L}(n,q,\alpha)$, $n \in \mathbb{N}_0$, *is given in* (1.7) *and the bounds in* (2.24) *and* (2.25) *are sharp with the extremal function given by* (2.21).

As an application of Theorem 4 (with $\alpha = 0$), we immediately deduce Corollary 2 below.

***Corollary* 2.** *If the function* $f \in \mathcal{M}$ *given by* (1.3) *satisfies the condition* (2.12) *with* $\alpha = 0$, *then*

$$\Re\left(\frac{(D_q f)(z)}{(D_q f_k)(z)}\right) \geq 1 - \frac{[k+1]_q}{\mathcal{L}(k+1,q,0)}$$

*and*

$$\Re\left(\frac{(D_q f_k)(z)}{(D_q f)(z)}\right) \geq \frac{\mathcal{L}(k+1,q,0)}{[k+1]_q + \mathcal{L}(k+1,q,0)},$$

*where* $\mathcal{L}(n,q,\alpha)$, $n \in \mathbb{N}_0$, *is given by* (1.7).

In the limit case when $q \to 1-$, Theorem 4 yields the following known result.

***Corollary* 3** [9]**.** *If the function* $f$ *of the form* (1.3) *satisfies the following condition*:

$$\sum_{n=0}^{\infty}(n+\alpha)|a_n| \leq 1-\alpha,$$

*then*

$$\Re\left(\frac{f'(z)}{f_k'(z)}\right) \geq 1 - \frac{(k+1)(1-\alpha)}{k+1+\alpha} \quad \forall z \in \mathbb{U}, \quad k \in \mathbb{N},$$

*and*

$$\Re\left(\frac{f_k'(z)}{f'(z)}\right) \geq \frac{k+1+\alpha}{2(k+1)-k\alpha} \quad \forall z \in \mathbb{U}, \quad k \in \mathbb{N}.$$

**3. Ruscheweyh-type $q$-derivative operator for meromorphic functions.** In this section, by using a Ruscheweyh-type $q$-derivative operator, we define and study a new class of functions emerging from the class $\mathcal{M}$ of normalized meromorphic functions. We also investigate the results analogous to those that have been proved in the preceding section.

Analogues of the Ruscheweyh derivative for analytic functions (see, for details, [28]), Al-Amiri [3] studied what he called the $m$-order Ruscheweyh-type derivative. Subsequently, Ganigi and Uralegaddi [17] introduced the meromorphic analogue of this derivative. More recently, Kanas and Răducanu [22] introduced the Ruscheweyh derivative operator for analytic functions by using the $q$-derivative operator. We propose to define a $q$-extension of the meromorphic analogue of the Ruscheweyh derivative by using the $q$-derivative operator.

**Definition 8.** *For $f \in \mathcal{M}$, the meromorphic analogue of the Ruscheweyh-type $q$-derivative operator is defined by*

$$\mathcal{MR}_q^\delta f(z) = f(z) * \phi(q, \delta+1; z) = \frac{1}{z} + \sum_{n=1}^{\infty} \psi_n(\delta) a_n z^n, \qquad z \in \mathbb{U}^*, \quad \delta > -1, \tag{3.1}$$

*where*

$$\phi(q, \delta+1; z) = \frac{1}{z} + \sum_{n=1}^{\infty} \psi_n(\delta) z^n$$

*and*

$$\psi_n(\delta) = \frac{[\delta+n+1]_q!}{[n+1]_q![\delta]_q!}, \quad n \in \mathbb{N}. \tag{3.2}$$

It is easily seen from (3.1) that

$$\mathcal{MR}_q^0 f(z) = f(z), \qquad \mathcal{MR}_q^1 f(z) - [2]_q \mathcal{MR}_q^0 f(qz) = z D_q f(z)$$

and

$$\mathcal{MR}_q^m f(z) = \frac{z^{-1} D_q\big(z^{m+1} f(z)\big)}{[m]_q!}, \quad m \in \mathbb{N}.$$

We also note that

$$\lim_{q \to 1-} \phi(q, \delta+1; z) = \frac{1}{z(1-z)^{\delta+1}}$$

and

$$\lim_{q \to 1-} \mathcal{MR}_q^\delta f(z) = f(z) * \frac{1}{z(1-z)^{\delta+1}},$$

which is the familiar Ruscheweyh derivative operator for meromorphic functions introduced and studied in [5, 6].

**Definition 9.** *A function $f \in \mathcal{M}$ is said to be in the class $\mathcal{MS}_q^\delta(\alpha)$, $0 \le \alpha < 1$, if it satisfies the following condition*:

$$\left| \frac{\left( -\dfrac{z D_q\left(\mathcal{MR}_q^\delta f(z)\right)}{\mathcal{MR}_q^\delta f(z)} \right) - \alpha}{1-\alpha} - \frac{1}{1-q} \right| \le \frac{1}{1-q}.$$

***Remark* 2.** First of all, we see that

$$\mathcal{MS}_q^0(\alpha) = \mathcal{MS}_q(\alpha),$$

where $\mathcal{MS}_q(\alpha)$ is the function class in Definition 6. Secondly, we have

$$\lim_{q \to 1-} \mathcal{MS}_q^0(0) = \mathcal{MS},$$

where $\mathcal{MS}$ is the function class, which was introduced and studied by Clunie (see [10]).

The following results can be proved by using the arguments similar to those that were already use in Section 2, so we choose to omit the details of our proof of Theorems 5 – 8 below.

**Theorem 5.** *Let*

$$\frac{1}{q} - \alpha > 0.$$

*Suppose also that the function* $f \in \mathcal{M}$ *is given by* (1.3). *If*

$$\sum_{n=0}^{\infty} \big([n]_q + \alpha\big)\psi_n|a_n| \leq \frac{1}{q} - \alpha,$$

*then* $f \in \mathcal{MS}_q^{\delta}(\alpha)$.

***Remark* 3.** Upon letting $\delta = 0$ in Theorem 5, we are led to Theorem 1 of the preceding section.

**Theorem 6.** *For* $\xi \geq 0$, *and let the function* $f \in \mathcal{M}$ *given by* (1.3) *satisfy the following condition*:

$$\frac{f(z) + \varepsilon z^{-1}}{1 + \varepsilon} \in \mathcal{MS}_q^{\delta}(\alpha)$$

*for any complex number* $\varepsilon$ *such that* $|\varepsilon| < \xi$. *Then*

$$N_{(\xi,q)}(f) \subset \mathcal{MS}_q^{\delta}(\alpha).$$

**Theorem 7.** *Let the function* $f \in \mathcal{M}$ *be given by* (1.3) *and define the partial sum* $f_k(z)$ *of the function* $f$ *by* (1.5), *where an empty sum is interpreted* (*as usual*) *to be nil*. *If*

$$\sum_{n=0}^{\infty} \kappa_n(\alpha)|a_n| \leq 1, \tag{3.3}$$

*where*

$$\kappa_n(\alpha) = \frac{\big(2[n]_q + (1+q)\alpha\big)q\psi_n(\delta)}{q - 1 + (1+q)(1-\alpha q)} = \mathcal{L}(n,q,\alpha)q\psi_n(\delta), \quad n \in \mathbb{N}, \tag{3.4}$$

*in terms of* $\mathcal{L}(n,q,\alpha)$ *and* $\psi_n(\delta)$ *given by* (1.7) *and* (3.2), *respectively*, *then*

$$f(z) \in \mathcal{MS}_q^{\delta}(\alpha),$$

$$\Re\left(\frac{f(z)}{f_k(z)}\right) \geq 1 - \frac{1}{\kappa_{k+1}(\alpha)}, \qquad z \in \mathbb{U}, \quad k \in \mathbb{N}, \tag{3.5}$$

*and*

$$\Re\left(\frac{f_k(z)}{f(z)}\right) \geq \frac{\kappa_{k+1}(\alpha)}{1 + \kappa_{k+1}(\alpha)}, \qquad z \in \mathbb{U}, \quad k \in \mathbb{N}. \tag{3.6}$$

*The bounds in* (3.5) *and* (3.6) *are sharp.*

**Theorem 8.** *Let the function* $f \in \mathcal{M}$ *be given by* (1.3) *and define the partial sum* $f_k(z)$ *of the function* $f$ *by* (1.5). *If the condition* (3.3) *holds true*, *then*

$$\Re\left(\frac{D_q f(z)}{D_q f_k(z)}\right) \geq 1 - \frac{[k+1]_q}{\kappa_{k+1}(\alpha)} \tag{3.7}$$

*and*

$$\Re\left(\frac{(D_q f_k)(z)}{(D_q f)(z)}\right) \geq \frac{\kappa_{k+1}(\alpha)}{[k+1]_q + \kappa_{k+1}(\alpha)}, \tag{3.8}$$

*where* $\kappa_n(\alpha)$, $n \in \mathbb{N}_0$, *is given in* (3.4). *The bounds in* (3.7) *and* (3.8) *are sharp with the extremal function given by* (2.21).

In its special case when $\alpha = 0$, Theorem 8 yields the following corollary.

***Corollary* 4.** *Let the function* $f \in \mathcal{M}$, *given by* (1.3), *satisfy the condition* (3.3) *with* $\alpha = 0$. *Suppose also that the partial sum* $f_k(z)$ *of the function* $f$ *is defined by* (1.5). *Then*

$$\Re\left(\frac{(D_q f)(z)}{(D_q f_k)(z)}\right) \geq 1 - \frac{1}{q\psi_{k+1}(\delta)}$$

*and*

$$\Re\left(\frac{(D_q f_k)(z)}{(D_q f)(z)}\right) \geq \frac{q\psi_{k+1}(\delta)}{1 + q\psi_{k+1}(\delta)},$$

*where* $\psi_n(\delta)$ *is given by* (3.2).

**4. Conclusion.** For the triviality and inconsequential nature of the so-called $(p, q)$-variations of our $q$-results, with an obviously redundant (or superfluous) parameter $p$, the reader is referred to a recent survey-cum-expository review article by Srivastava [32, p. 340]. In this paper, we have established several general results involving the partial sums of meromorphically $q$-starlike functions defined here by means of a certain class of $q$-derivative (or $q$-difference) operators. We have also investigated the familiar concept of neighborhood for meromorphic functions. Moreover, by using a Ruscheweyh-type $q$-derivative operator, we have introduced and studied another new class of functions emerging from the class of normalized meromorphic functions.

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