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GENERALIZATIONS OF STARLIKE HARMONIC FUNCTIONS DEFINED BY SĂLĂGEAN AND RUSCHEWEYH DERIVATIVES

УЗАГАЛЬНЕННЯ ЗІРКОПОДІБНИХ ГАРМОНІЧНИХ ФУНКЦІЙ, ЩО ВИЗНАЧЕНІ ПОХІДНИМИ САЛАГЕНА ТА РУШЕВЕЯ

We investigate some generalizations of the classes of harmonic functions defined by the Sălăgean and Ruscheweyh derivatives. By using the extreme-points theory, we obtain the coefficient-estimates distortion theorems and mean integral inequalities for these classes of functions.

Досліджено деякі узагальнення класів гармонічних функцій, що визначені похідними Салагена та Рушевея. З використанням теорії екстремальних точок отримано теореми про спотворення оцінок коефіцієнтів та нерівності для інтегральних середніх для цих класів функцій.

1. Preliminaries. Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} \colon |z| < 1\}$.

A continuous function $f = u + iv$ is a complex-valued harmonic function in a complex domain $\mathcal{G}$ if both u and v are real and harmonic in $\mathcal{G}$. In any simply-connected domain $D \subset \mathcal{G}$, we can write $f = h + \overline{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and orientation preserving in D is that $|h'(z)| > |g'(z)|$ in D (see [2]).

Let $\mathcal{H}$ denote the family of continuous complex-valued functions that are harmonic in U. Denote by $S_{\mathcal{H}}$ the family of functions $f \in \mathcal{H}$ of the form

$$f = h + \overline{g}, \quad h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = \sum_{k=2}^{\infty} b_k z^k, \tag{2}$$

which are univalent and orientation preserving in the open unit disc U. Thus, $f(z)$ is then given by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k + \overline{\sum_{k=2}^{\infty} b_k z^k}. \tag{3}$$

A function f of the form (3) is said to be in $\mathcal{S}^*_{\mathcal{H}}(\alpha)$ if and only if (see [2, 4, 5])

$$\frac{\partial}{\partial\theta}\Big(\arg f\Big(re^{i\theta}\Big)\Big) > \alpha, \quad 0 \le \theta < 2\pi, \quad |z| = r < 1, \quad 0 \le \alpha < 1. \tag{4}$$

Similarly, a function f of the form (3) is said to be in $\mathcal{S}^c_{\mathcal{H}}(\alpha)$ if and only if

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$$\frac{\partial}{\partial\theta}\left(\arg\frac{\partial}{\partial\theta}\left(f\left(re^{i\theta}\right)\right)\right) > \alpha, \quad 0 \le \theta < 2\pi, \quad |z| = r < 1. \tag{5}$$

We note that (see [7]) a harmonic function $f \in \mathcal{S}^*_{\mathcal{H}}(\alpha)$ if and only if

$$\Re\,\frac{J_{\mathcal{H}}f(z)}{f(z)} > \alpha, \quad |z| = r < 1, \quad \text{where} \quad J_{\mathcal{H}}f(z) = zh'(z) - \overline{zg'(z)}.$$

Definition 1 [1]. *For $f \in \mathcal{A}$, $\lambda \ge 0$ and $n \in \mathbb{N}$, the operator $\mathcal{D}^n_\lambda$ is defined by $\mathcal{D}^n_\lambda : \mathcal{A} \to \mathcal{A}$,*

$$\mathcal{D}^0_\lambda f(z) = f(z),$$

$$\mathcal{D}^{n+1}_\lambda f(z) = (1-\lambda)\mathcal{D}^n_\lambda f(z) + \lambda z(\mathcal{D}^n_\lambda f(z))' = \mathcal{D}_\lambda(\mathcal{D}^n_\lambda f(z)), \quad z \in U.$$

***Remark* 1.** If $f \in \mathcal{A}$, then

$$\mathcal{D}^n_\lambda f(z) = z + \sum_{k=2}^{\infty}[1+(k-1)\lambda]^n a_k z^k, \quad z \in U.$$

***Remark* 2.** For $\lambda = 1$ in the above definition we obtain the Sălăgean differential operator [13].

Definition 2 [12]. *For $f \in \mathcal{A}$, $n \in \mathbb{N}$, the operator $\mathcal{R}^n$ is defined by $\mathcal{R}^n : \mathcal{A} \to \mathcal{A}$,*

$$\mathcal{R}^0 f(z) = f(z),$$

$$(n+1)\mathcal{R}^{n+1} f(z) = z(\mathcal{R}^n f(z))' + n\mathcal{R}^n f(z), \quad z \in U.$$

***Remark* 3.** If $f \in \mathcal{A}$, then

$$\mathcal{R}^n f(z) = z + \sum_{k=2}^{\infty}\frac{(n+k-1)!}{n!(k-1)!}a_k z^k, \quad z \in U,$$

which is the Ruscheweyh differential operator [12].

Definition 3. *Let $\gamma, \lambda \ge 0$, $n \in \mathbb{N}$. Denote by $\mathcal{L}^n$ the operator given by $\mathcal{L}^n : \mathcal{A} \to \mathcal{A}$,*

$$\mathcal{L}^n f(z) = (1-\gamma)\mathcal{R}^n f(z) + \gamma\mathcal{D}^n_\lambda f(z), \quad z \in U.$$

***Remark* 4.** If $f \in \mathcal{A}$, then

$$\mathcal{L}^n f(z) = z + \sum_{k=2}^{\infty}\left\{\gamma[1+(k-1)\lambda]^n + (1-\gamma)\frac{(n+k-1)!}{n!(k-1)!}\right\}a_k z^k, \quad z \in U.$$

We consider the linear operator $\mathcal{L}^n_{\mathcal{H}} : \mathcal{H} \to \mathcal{H}$ defined for a function $f = h + \overline{g} \in \mathcal{H}$ by

$$\mathcal{L}^n_{\mathcal{H}} f := \mathcal{L}^n h + (-1)^n\overline{\mathcal{L}^n g}.$$

For a function $f \in \mathcal{H}$ of the form (3), we have

$$\mathcal{L}^n_{\mathcal{H}} f(z) = z + \sum_{k=2}^{\infty}[\gamma\eta(k,n,\lambda) + (1-\gamma)\mu(k,n)]a_k z^k +$$

$$+(-1)^n\sum_{k=2}^{\infty}[\gamma\eta(k,n,\lambda) + (1-\gamma)\mu(k,n)]\overline{b_k}\overline{z}^k, \quad z \in U,$$

where $\eta(k,n,\lambda) = [1+(k-1)\lambda]^n$ and $\mu(k,n) = \dfrac{(n+k-1)!}{n!(k-1)!}$.

Definition 4. *For* $-B \le A < B \le 1$ *and* $n \in \mathbb{N}$, *let* $\widetilde{\mathcal{S}}^n_{\mathcal{H}}(A,B)$ *denote the class of functions* $f \in \mathcal{H}$ *of the form* (3) *such that*

$$\left| \frac{\mathcal{L}^{n+1}_{\mathcal{H}} f(z) - \mathcal{L}^n_{\mathcal{H}} f(z)}{B\mathcal{L}^{n+1}_{\mathcal{H}} f(z) - A\mathcal{L}^n_{\mathcal{H}} f(z)} \right| < 1, \quad z \in U. \tag{6}$$

***Remark* 5.** Dziok et al. studied the case $\gamma = 0$ in [3], while the case $\gamma = 1$ and $\lambda = 1$ was studied in [4].

Note that the classes $\widetilde{\mathcal{S}}^0_{\mathcal{H}}(A,B)$ for the analytic case, i.e., $g \equiv 0$, were introduced by Janowski [8]. Jahangiri [6, 7] and Silverman [14] studied the classes $\mathcal{S}^*_{\mathcal{H}}(\alpha) = \widetilde{\mathcal{S}}^0_{\mathcal{H}}(2\alpha - 1, 1)$ and $\mathcal{S}^c_{\mathcal{H}}(\alpha) = \widetilde{\mathcal{S}}^1_{\mathcal{H}}(2\alpha - 1, 1)$ for the harmonic case.

2. Coefficient estimates.

Theorem 1. *A function* $f \in \mathcal{H}$ *of the form* (3) *belongs to the class* $\widetilde{\mathcal{S}}^n_{\mathcal{H}}(A,B)$ *if it satisfies the condition*

$$\sum_{k=2}^{\infty} (\alpha_k |a_k| + \beta_k |b_k|) \le B - A, \tag{7}$$

where

$$\alpha_k = \sigma(A,B,n,\gamma,\lambda,k) + \sigma(1,1,n,\gamma,\lambda,k),$$

$$\beta_k = \delta(A,B,n,\gamma,\lambda,k) + \delta(1,1,n,\gamma,\lambda,k),$$

$$\sigma(A,B,n,\gamma,\lambda,k) = \gamma\eta(k,n,\lambda)[(k-1)\lambda B + B - A] +$$

$$+ (1-\gamma)\mu(k,n)\frac{(B-A)n + Bk - A}{n+1},$$

$$\delta(A,B,n,\gamma,\lambda,k) = \gamma\eta(k,n,\lambda)[(k-1)\lambda B + B + A] +$$

$$+ (1-\gamma)\mu(k,n)\frac{(B+A)n + Bk + A}{n+1}.$$

Proof. We know from Definition 4 that $f \in \widetilde{\mathcal{S}}^n_{\mathcal{H}}(A,B)$ if and only if

$$\left| \frac{\mathcal{L}^{n+1}_{\mathcal{H}} f(z) - \mathcal{L}^n_{\mathcal{H}} f(z)}{B\mathcal{L}^{n+1}_{\mathcal{H}} f(z) - A\mathcal{L}^n_{\mathcal{H}} f(z)} \right| < 1, \quad z \in U.$$

It is sufficient to prove that

$$\left|\mathcal{L}^{n+1}_{\mathcal{H}} f(z) - \mathcal{L}^n_{\mathcal{H}} f(z)\right| - \left|B\mathcal{L}^{n+1}_{\mathcal{H}} f(z) - A\mathcal{L}^n_{\mathcal{H}} f(z)\right| < 0, \quad z \in U \setminus \{0\}.$$

Letting $|z| = r,\ 0 < r < 1$, we have

$$\left|\mathcal{L}^{n+1}_{\mathcal{H}} f(z) - \mathcal{L}^n_{\mathcal{H}} f(z)\right| - \left|B\mathcal{L}^{n+1}_{\mathcal{H}} f(z) - A\mathcal{L}^n_{\mathcal{H}} f(z)\right| \le$$

$$\le \sum_{k=2}^{\infty} \left[\gamma\eta(k,n,\lambda)(k-1)\lambda + (1-\gamma)\mu(k,n)\frac{k-1}{n+1}\right] |a_k| r^k +$$

$$+\sum_{k=2}^{\infty}\left[\gamma\eta(k,n,\lambda)[2+(k-1)\lambda]+(1-\gamma)\mu(k,n)\frac{2n+k+1}{n+1}\right]|b_k|r^k-(B-A)r+$$

$$+\sum_{k=2}^{\infty}\left[\gamma\eta(k,n,\lambda)[(k-1)\lambda B+B-A]+(1-\gamma)\mu(k,n)\left(B\frac{n+k}{n+1}-A\right)\right]|a_k|r^k+$$

$$+\sum_{k=2}^{\infty}\left[\gamma\eta(k,n,\lambda)[(k-1)\lambda B+B+A]+(1-\gamma)\mu(k,n)\left(B\frac{n+k}{n+1}+A\right)\right]|b_k|r^k\leq$$

$$\leq r\left\{\sum_{k=2}^{\infty}(\alpha_k|a_k|+\beta_k|b_k|)r^{k-1}-(B-A)\right\}<0,$$

whence $f\in\widetilde{\mathcal{S}}^n_{\mathcal{H}}(A,B)$.

Theorem 1 is proved.

Lemma 1. *If* $\lambda\geq 1,\ \gamma\in[0,1],\ n\geq 0,\ -B\leq A<B\leq 1,\ k\in\mathbb{N},\ k\geq 2,$ *then*

$$\alpha_k\geq k(B-A),\quad \beta_k\geq k(B-A),$$

where $\alpha_k,\ \beta_k$ *is defined in* (7).

Proof. It is known that

$$\eta(k,n,\lambda)=[1+(k-1)\lambda]^n\geq k^n. \tag{8}$$

First we prove that

$$\mu(k,n)=\frac{(n+k-1)!}{n!(k-1)!}\geq n+1. \tag{9}$$

For the proof we use the mathematical induction method.

1. Let $k\geq 2$ be fixed and $n=0$, then $\mu(k,0)=\dfrac{(k-1)!}{0!(k-1)!}=1$ is true.

Let $k\geq 2$ be fixed and $n=1$, then $\mu(k,1)=\dfrac{k!}{1!(k-1)!}\geq 2\Leftrightarrow k!\geq 2(k-1)!\Leftrightarrow k\geq 2$ is true.

2. Assume, for $n=l$, that the formula displayed below holds:

$$\mu(k,l)=\frac{(l+k-1)!}{l!(k-1)!}\geq l+1\Leftrightarrow (l+k-1)!\geq l!(k-1)!(l+1)=(l+1)!(k-1)!\,.$$

3. Let $n=l+1$, so we have to prove that

$$\mu(k,l+1)=\frac{(l+k)!}{(l+1)!(k-1)!}\geq l+2\Leftrightarrow (l+k)!\geq(l+1)!(k-1)!(l+2).$$

This holds using the previous item

$$(l+k)!=(l+k)(l+k-1)!\geq(l+k)(l+1)!(k-1)!\geq(l+2)(l+1)!(k-1)!.$$

Now, using (8) and (9), we prove that $\alpha_k \geq k(B-A)$:

$$\alpha_k = \sigma(A, B, n, \gamma, \lambda, k) + \sigma(1, 1, n, \gamma, \lambda, k) \geq$$

$$\geq \gamma k^n[(k-1)\lambda B + B - A] +$$

$$+(1-\gamma)[(B-A)n + Bk - A] + \gamma k^n(k-1)\lambda + (1-\gamma)(k-1).$$

But

$$k^n[(k-1)\lambda B + B - A] + k^n(k-1)\lambda = k^n[(B-A) + \underbrace{(k-1)\lambda(B+1)}_{>0}] >$$

$$> k^n(B-A) > k(B-A)$$

and

$$(B-A)n + Bk - A + (k-1) \geq B(k-1) + B - A + k - 1 =$$

$$= (k-1)(B+1) + B - A \geq (k-1)(B-A) + B - A = k(B-A).$$

So, $\alpha_k \geq \gamma(B-A)k + (1-\gamma)(B-A)k = k(B-A)$.

Now we prove that $\beta_k \geq k(B-A)$:

$$\beta_k = \delta(A, B, n, \gamma, \lambda, k) + \delta(1, 1, n, \gamma, \lambda, k) \geq$$

$$\geq \gamma k^n[(k-1)\lambda B + B + A] + (1-\gamma)[(B+A)n + Bk + A] +$$

$$+\gamma k^n[(k-1)\lambda + 2] + (1-\gamma)[2n + k + 1] >$$

$$> \gamma k^n[(k-1)(B+1) + B + A + 2] + (1-\gamma)[(B+A)n + 2n + Bk + k + A + 1].$$

But

$$(k-1)(B+1) + B + A + 2 = kB + k + 1 + A \geq$$

$$\geq k(B-A), \quad B \geq -1, \quad A \geq -1,$$

$$k + 1 + A \geq -kA \Leftrightarrow k(A+1) + A + 1 \geq 0 \Leftrightarrow (k+1)(A+1) \geq 0$$

and

$$(B+A)n + 2n + Bk + k + A + 1 \geq Bk + k + A + 1 \geq Bk - Ak,$$

because

$$k + A + 1 \geq -Ak \Leftrightarrow k(A+1) + A + 1 \geq 0 \Leftrightarrow (k+1)(A+1) \geq 0.$$

So, $\beta_k \geq \gamma(B-A)k + (1-\gamma)(B-A)k = k(B-A)$.

Lemma 1 is proved.

Lemma 2. *If* $\lambda \geq 1,\ \gamma > 1,\ n \geq 0,\ -B \leq A < B \leq 1,\ k \in \mathbb{N},\ k \geq 2$, *then*

$$\alpha_k \geq k(B-A), \quad \beta_k \geq k(B-A),$$

where $\alpha_k,\ \beta_k$ *is defined in* (7).

Proof. First we note that

$$\mu(k,n)=\frac{(n+k-1)!}{n!(k-1)!}\le k^n,\quad k,n\in\mathbb{N},\quad k\ge 2. \tag{10}$$

Let k be fixed. If $n=0$ then (10) holds true.

Suppose that, for n, (10) is true, then, for $n+1$, we have

$$(n+k)!=(n+k)(n+k-1)!\le(n+k)k^n n!(k-1)!\le$$

$$\le(n+1)kk^n n!(k-1)!=k^n(n+1)!(k-1)!.$$

Now

$$\alpha_k\ge\gamma k^n[(k-1)(B+1)+B-A]-(\gamma-1)k^n\frac{(B-A)n+Bk-A}{n+1}$$

by (8) and (10).

But

$$\frac{(B-A)n+Bk-A+k-1}{n+1}<(B-A)+(k-1)(B+1)$$

and so

$$\alpha_k\ge[\gamma-(\gamma-1)][B-A+(k-1)(B+1)]k^n\ge k(B-A),$$

$$\beta_k\ge\gamma k^n[(k-1)(B+1)+B+A+2]+$$

$$+(1-\gamma)k^n\frac{(B+A)n+2n+Bk+k+A+1}{n+1}\ge$$

$$\ge k^n[(k-1)(B+1)+B+A+2]\ge k(B-A),$$

because $(B+A)n+2n+Bk+k+A+1<(n+1)[(k-1)(B+1)+B+A+2]$.

Lemma 2 is proved.

Theorem 2. *If $f\in\mathcal{H}$ of the form* (3) *and f satisfies the condition* (7), *then $f\in\mathcal{S}_{\mathcal{H}}$.*

Proof. The theorem is true for the function $f(z)\equiv z$. Let $f\in\mathcal{H}$ be a function of the form (3) and let us assume that exists $k\in\{2,3,\ldots\}$ such that $a_k\ne0$ or $b_k\ne0$. Since $\dfrac{\alpha_k}{B-A}\ge k$, $\dfrac{\beta_k}{B-A}\ge k,\ k=2,3,\ldots$, proved in Lemma 1 and 2, then by (7) we have

$$\sum_{k=2}^{\infty}(k|a_k|+k|b_k|)\le 1 \tag{11}$$

and

$$\left|h'(z)\right|-\left|g'(z)\right|\ge 1-\sum_{k=2}^{\infty}k|a_k||z|^k-\sum_{k=2}^{\infty}k|b_k||z|^k\ge 1-|z|\sum_{k=2}^{\infty}(k|a_k|+k|b_k|)\ge$$

$$\ge 1-\frac{|z|}{B-A}\sum_{k=2}^{\infty}(\alpha_k|a_k|+\beta_k|b_k|)\ge 1-|z|>0,\quad z\in U.$$

In this case the function f is locally univalent and sense-preserving in U. Moreover, if $z_1, z_2 \in U$, $z_1 \neq z_2$, then

$$\left|\frac{z_1^k - z_2^k}{z_1 - z_2}\right| = \left|\sum_{l=1}^{k} z_1^{l-1} z_2^{k-l}\right| \leq \sum_{l=1}^{k} |z_1|^{l-1} |z_2|^{k-1} < k, \quad k = 2, 3, \ldots .$$

Therefore, by (11), we have

$$|f(z_1) - f(z_2)| \geq |h(z_1) - h(z_2)| - |g(z_1) - g(z_2)| \geq$$

$$\geq \left|z_1 - z_2 - \sum_{k=2}^{\infty} a_k \left(z_1^k - z_2^k\right)\right| - \left|\sum_{k=2}^{\infty} \overline{b_k \left(z_1^k - z_2^k\right)}\right| \geq$$

$$\geq |z_1 - z_2| \left(1 - \sum_{k=2}^{\infty} |a_k| \left|\frac{z_1^k - z_2^k}{z_1 - z_2}\right| - \sum_{k=2}^{\infty} |b_k| \left|\frac{z_1^k - z_2^k}{z_1 - z_2}\right|\right) >$$

$$> |z_1 - z_2| \left(1 - \sum_{k=2}^{\infty} k|a_k| - \sum_{k=2}^{\infty} k|b_k|\right) \geq 0.$$

This leads to the univalence of f, so $f \in \mathcal{S}_{\mathcal{H}}$.

Theorem 2 is proved.

Let $\mathcal{N}$ denote the class of functions $f = h + \overline{g} \in \mathcal{H}$ of the form (see [14])

$$f(z) = z - \sum_{k=2}^{\infty} |a_k| z^k + (-1)^n \sum_{k=2}^{\infty} |b_k| \overline{z}^k, \tag{12}$$

and denote by $\widetilde{\mathcal{S}}_{\mathcal{HN}}^n(A, B)$ the class $\mathcal{N} \cap \widetilde{\mathcal{S}}_{\mathcal{H}}^n(A, B)$.

Theorem 3. *Let $f = h + \overline{g}$ be defined by* (12). *Then $f \in \widetilde{\mathcal{S}}_{\mathcal{HN}}^n(A, B)$ if and only if the condition* (7) *holds true.*

Proof. For the 'if' part see Theorem 1. For the 'only if' part, assume that $f \in \widetilde{\mathcal{S}}_{\mathcal{HN}}^n(A, B)$, then, by (6), we have

$$\left|\frac{\sum_{k=2}^{\infty}\left[\sigma(1,1,n,\gamma,\lambda,k)|a_k|z^{k-1} + \delta(1,1,n,\gamma,\lambda,k)|b_k|\overline{z}^{k-1}\right]}{(B-A) - \sum_{k=2}^{\infty}\left[\sigma(A,B,n,\gamma,\lambda,k)|a_k|z^{k-1} + \delta(A,B,n,\gamma,\lambda,k)|b_k|\overline{z}^{k-1}\right]}\right| < 1, z \in U.$$

For $z = r < 1$, we obtain

$$\frac{\sum_{k=2}^{\infty}[\sigma(1,1,n,\gamma,\lambda,k)|a_k| + \delta(1,1,n,\gamma,\lambda,k)|b_k|]r^{k-1}}{(B-A) - \sum_{k=2}^{\infty}[\sigma(A,B,n,\gamma,\lambda,k)|a_k| + \delta(A,B,n,\gamma,\lambda,k)|b_k|]r^{k-1}} < 1.$$

The denominator of the left-hand side can not vanish for $r \in [0, 1)$ and it is positive. So $\sum_{k=2}^{\infty}(\alpha_k|a_k| + \beta_k|b_k|)r^{k-1} \leq B - A$, which, upon letting $r \to 1^-$, yields to assertion (7).

Theorem 3 is proved.

3. Extreme points.

Definition 5. *We say that a class* $\mathcal{F}$ *is convex if* $\eta f + (1-\eta)g \in \mathcal{F}$ *for all* f *and* g *in* $\mathcal{F}$ *and* $0 \le \eta \le 1$. *The closed convex hull of* $\mathcal{F}$, *denoted by* $\overline{\mathrm{co}}\mathcal{F}$, *is the intersection of all closed convex subsets of* $\mathcal{H}$ *(with respect to the topology of locally uniform convergence) that contain* $\mathcal{F}$.

Definition 6. *Let* $\mathcal{F}$ *be a convex set. A function* $f \in \mathcal{F} \subset \mathcal{H}$ *is called an extreme point of* $\mathcal{F}$ *if* $f = \eta f_1 + (1-\eta) f_2$ *implies* $f_1 = f_2 = f$ *for all* f_1 *and* f_2 *in* $\mathcal{F}$ *and* $0 < \eta < 1$. *We shall use the notation* $E\mathcal{F}$ *to denote the set of all extreme points of* $\mathcal{F}$. *It is clear that* $E\mathcal{F} \subset \mathcal{F}$.

For the extreme points we use the Krein – Milman theorem (see [3, 4, 9]) which implies.

Lemma 3 [3, 4]**.** *Let* $\mathcal{F}$ *be a non-empty compact convex subclass of the class* $\mathcal{H}$ *and* $\mathcal{J} : \mathcal{H} \to \mathbb{R}$ *be a real-valued, continuous, and convex functional on* $\mathcal{F}$. *Then*

$$\max\{\mathcal{J}(f) \colon f \in \mathcal{F}\} = \max\{\mathcal{J}(f) \colon f \in E\mathcal{F}\}.$$

Since $\mathcal{H}$ is a complete metric space, we can use Montel's theorem [10].

Lemma 4 [3, 4]**.** *A class* $\mathcal{F} \subset \mathcal{H}$ *is compact if and only if* $\mathcal{F}$ *is closed and locally uniformly bounded.*

Theorem 4. *The class* $\widetilde{\mathcal{S}}^n_{\mathcal{HN}}(A,B)$ *is a convex and compact subset of* $\mathcal{H}$.

Proof. For $0 \le \eta \le 1$, let $f_1, f_2 \in \widetilde{\mathcal{S}}^n_{\mathcal{HN}}(A,B)$ be defined by (2). Then

$$\eta f_1(z) + (1-\eta) f_2(z) = z - \sum_{k=2}^{\infty} (\eta |a_{1,k}| + (1-\eta)|a_{2,k}|) z^k +$$

$$+(-1)^n \sum_{k=2}^{\infty} (\eta |b_{1,k}| + (1-\eta)|b_{2,k}| \overline{z}^k)$$

and

$$\sum_{k=2}^{\infty} \Big\{ \alpha_k |\eta |a_{1,k}| + (1-\eta)|a_{2,k}|| + \beta_k \Big| \eta |b_{1,k}| + (1-\eta)|b_{2,k}| z^k \Big| \Big\} =$$

$$= \eta \sum_{k=2}^{\infty} \{\alpha_k |a_{1,k}| + \beta_k |b_{1,k}|\} + (1-\eta) \sum_{k=2}^{\infty} \alpha_k |a_{2,k}| + \beta_k |b_{2,k}| \le$$

$$\le \eta(B-A) + (1-\eta)(B-A).$$

Therefore, the function $\phi = \eta f_1 + (1-\eta) f_2$ belongs to the class $\widetilde{\mathcal{S}}^n_{\mathcal{HN}}(A,B)$, so $\widetilde{\mathcal{S}}^n_{\mathcal{HN}}(A,B)$ is convex.

On the other hand, for $f \in \widetilde{\mathcal{S}}^n_{\mathcal{HN}}(A,B)$, $|z| \le r$ and $0 < r < 1$, we have

$$|f(z)| \le r + \sum_{k=2}^{\infty} (|a_k| + |b_k|) r^n \le r + \sum_{k=2}^{\infty} (\alpha_k |a_k| + \beta_k |b_k|) \le r + (B-A).$$

From this comes that $\widetilde{\mathcal{S}}^n_{\mathcal{HN}}(A,B)$ is locally uniformly bounded. Let

$$f_e(z) = z + \sum_{k=2}^{\infty} a_{e,k} z^k + \overline{\sum_{k=1}^{\infty} b_{e,k} z^k}, \quad z \in U, \quad k \in \mathbb{N},$$

and $f \in \mathcal{H}$. Using Theorem 3, we have

$$\sum_{k=2}^{\infty}(\alpha_k|a_{e,k}| + \beta_k|b_{e,k}|) \leq B - A, \quad k \in \mathbb{N}.$$

If $f_e \to f$, then $|a_{e,k}| \to |a_k|$ and $|b_{e,k}| \to |b_k|$ when $k \to \infty$, $k \in \mathbb{N}$. This gives condition (7). Therefore, $f \in \widetilde{\mathcal{S}}^n_{\mathcal{HN}}(A,B)$ and the class $\widetilde{\mathcal{S}}^n_{\mathcal{HN}}(A,B)$ is closed. We can now say, by Lemma 3, that the class $\widetilde{\mathcal{S}}^n_{\mathcal{HN}}(A,B)$ is compact subset of $\mathcal{H}$.

Theorem 4 is proved.

Theorem 5. *The set of extreme points of the class* $\widetilde{\mathcal{S}}^n_{\mathcal{HN}}(A,B)$ *is* $E\widetilde{\mathcal{S}}^n_{\mathcal{HN}}(A,B) = \{h_k : k \in \mathbb{N}\} \cup \cup \{g_k : k \in \{2,3,\ldots\}\}$, *where*

$$h_1 = z, \quad h_k(z) = z - \frac{B-A}{\alpha_k} z^k,$$

$$g_k(z) = z + (-1)^n \frac{B-A}{\beta_k} \overline{z}^k, \quad z \in U, \quad k \in \{2,3,\ldots\}. \tag{13}$$

Proof. If we use (7), we can see that the functions of the above form are the extreme points of the class $\widetilde{\mathcal{S}}^n_{\mathcal{HN}}(A,B)$. Supposing that $f \in E\widetilde{\mathcal{S}}^n_{\mathcal{HN}}(A,B)$ and f is not of the form seen above, there exists $m \in \{2,3,\ldots\}$ such that $0 < |a_m| < \dfrac{B-A}{\alpha_m}$ or $0 < |b_m| < \dfrac{B-A}{\beta_m}$. If $0 < |a_m| < \dfrac{B-A}{\alpha_m}$, then putting $\gamma = \dfrac{|a_m|\alpha_m}{B-A}, \varphi = \dfrac{1}{1-\eta}(f - \eta h_m)$, we have $0 < \eta < 1, h_m, \varphi \in \widetilde{\mathcal{S}}^*_{\mathcal{HN}}(A,B)$, $h_m \neq \varphi$ and $f = \eta h_m + (1-\eta)\varphi$. Thus, $f \notin E\widetilde{\mathcal{S}}^n_{\mathcal{HN}}(A,B)$. We get the same result for $0 < |b_m| < \dfrac{B-A}{\beta_m}$.

Theorem 5 is proved.

If the class $\mathcal{F} = \{f_k \in \mathcal{H} : k \in \mathbb{N}\}$ is locally uniformly bounded, then its closed convex hull is

$$\overline{\text{co}}\mathcal{F} = \left\{\sum_{k=1}^{\infty} \eta_k f_k : \sum_{k=1}^{\infty} \eta_k = 1, \ \eta_k \geq 0, \ k \in \mathbb{N}\right\}.$$

***Corollary* 1.** *Let* h_k, g_k *be defined by* (13), *then*

$$\widetilde{\mathcal{S}}^n_{\mathcal{HN}}(A,B) = \left\{\sum_{k=1}^{\infty}(\eta_k h_k + \delta_k g_k) : \sum_{k=1}^{\infty}(\eta_k + \delta_k) = 1, \delta_1 = 0, \eta_k, \delta_k \geq 0, \ k \in \mathbb{N}\right\}.$$

For each fixed value of $k \in \mathbb{N}$, $z \in U$, the following real-valued functionals are continuous and convex on $\mathcal{H}$:

$$\mathcal{J}(f) = |a_k|, \quad \mathcal{J}(f) = |b_k|, \quad \mathcal{J}(f) = |f(z)|, \quad \mathcal{J}(f) = \left|\mathcal{L}^k_{\mathcal{H}} f(z)\right|, \quad f \in \mathcal{H}.$$

The real-valued functional

$$\mathcal{J}(f) = \left(\frac{1}{2\pi}\int_0^{2\pi}\left|f\left(re^{i\theta}\right)\right|^{\gamma} d\theta\right)^{1/\gamma}, \quad f \in \mathcal{H}, \quad \gamma \geq 1, \quad 0 < r < 1,$$

is continuous on $\mathcal{H}$. For $\gamma \geq 1$ it is also convex on $\mathcal{H}$ (Minkowski's inequality).

***Corollary* 2.** *Let* $f \in \widetilde{\mathcal{S}}^n_{\mathcal{HN}}(A,B)$ *be a function of the form* (12). *Then*

$$|a_k| \le \frac{B-A}{\alpha_k}, \quad |b_k| \le \frac{B-A}{\beta_k}, \quad k = 2,3,\ldots,$$

where α_k, β_k *are defined by* (7). *The result is sharp. The extremal functions are* h_k, g_k *of the form* (13).

Theorem 6. *Let* $f \in \widetilde{\mathcal{S}}^n_{\mathcal{HN}}(A,B)$ *and* $|z| = r < 1$. *Then*

$$r - \frac{B-A}{\alpha_2} r^2 \le |f(z)| \le r + \frac{B-A}{\alpha_2} r^2,$$

$$r - \frac{(B-A)[\gamma(1+\lambda)^n + (1-\gamma)(n+1)]}{\alpha_2} r^2 \le |\mathcal{L}^n_{\mathcal{H}} f(z)| \le$$

$$\le r + \frac{(B-A)[\gamma(1+\lambda)^n + (1-\gamma)(n+1)]}{\alpha_2} r^2.$$

The result is sharp. The extremal functions are h_2 *of the form* (13).

Proof. We only prove the right-hand side inequality. The proof for the left-hand side inequality is similar and will be omitted. We have

$$|f(z)| \le r + \sum_{k=2}^{\infty} (|a_k| + |b_k|) r^k \le r + \sum_{k=2}^{\infty} (|a_k| + |b_k|) r^2 \le$$

$$\le r + \left(\frac{1}{\alpha_2} \sum_{k=2}^{\infty} \alpha_2 |a_k| + \frac{1}{\beta_2} \sum_{k=2}^{\infty} \beta_2 |b_k| \right) r^2 \le$$

$$\le r + \frac{1}{\alpha_2} \sum_{k=2}^{\infty} (\alpha_k |a_k| + \beta_k |b_k|) r^2 \le$$

$$\le r + \frac{B-A}{\alpha_2} r^2, \quad \alpha_2 \le \alpha_k, \quad \alpha_2 \le \beta_2, \quad \beta_2 \le \beta_k \quad \text{for all} \quad k \ge 2.$$

An other proof can be made using the Lemma 3 with extreme points.

Theorem 6 is proved.

***Corollary* 3.** *If* $f \in \widetilde{\mathcal{S}}^n_{\mathcal{HN}}(A,B)$, *then* $U(r) \subset f(U(r))$, *where*

$$r = 1 - \frac{B-A}{\alpha_2}$$

and

$$U(r) := \{z \in \mathbb{C} \colon |z| < r \le 1\}.$$

***Corollary* 4.** *Let* $0 < r < 1$ *and* $\xi \ge 1$. *If* $f \in \widetilde{\mathcal{S}}^n_{\mathcal{HN}}(A,B)$, *then*

$$\frac{1}{2\pi} \int_0^{2\pi} \left| f\left(re^{i\theta}\right) \right|^{\xi} d\theta \le \frac{1}{2\pi} \int_0^{2\pi} \left| h_2\left(re^{i\theta}\right) \right|^{\xi} d\theta,$$

$$\frac{1}{2\pi} \int_0^{2\pi} \left| \mathcal{L}^k_{\mathcal{H}} f\left(re^{i\theta}\right) \right|^{\xi} d\theta \le \frac{1}{2\pi} \int_0^{2\pi} \left| \mathcal{L}^k_{\mathcal{H}} h_2\left(re^{i\theta}\right) \right|^{\xi} d\theta, \quad \xi = 1,2,\ldots.$$

4. Radii of starlikeness and convexity. We note that a harmonic function $f \in \mathcal{S}^*_{\mathcal{H}}(\alpha)$ if and only if

$$\Re \frac{\mathcal{L}_{\mathcal{H}} f(z)}{f(z)} > \alpha, \quad |z| = r < 1,$$

where $\mathcal{L}_{\mathcal{H}} f(z) = zh'(z) - \overline{zg'(z)}$. For $0 \le \alpha < 1$, $f \in \mathcal{S}^c_{\mathcal{H}}(\alpha)$ is equivalent with $\mathcal{L}_{\mathcal{H}} f(z) \in \mathcal{S}^*_{\mathcal{H}}(\alpha)$.

Let $\mathcal{B} \subseteq \mathcal{H}$. We define the radius of starlikeness and the radius of convexity of the class $\mathcal{B}$:

$$R^*_\alpha(\mathcal{B}) := \inf_{f \in \mathcal{B}} (\sup\{r \in (0,1] : f \text{ is starlike of order } \alpha \in U(r)\}),$$

$$R^c_\alpha(\mathcal{B}) := \inf_{f \in \mathcal{B}} (\sup\{r \in (0,1] : f \text{ is convex of order } \alpha \in U(r)\}).$$

Theorem 7. *Let* $0 \le \alpha < 1$ *and* $\alpha_k,\ \beta_k$ *be defined by* (7). *Then*

$$R^*_\alpha\left(\widetilde{\mathcal{S}}^n_{\mathcal{HN}}(A,B)\right) = \inf_{k \ge 2} \left(\frac{1-\alpha}{B-A} \min\left\{\frac{\alpha_k}{k-\alpha}, \frac{\beta_k}{k+\alpha}\right\}\right)^{\frac{1}{k-1}}.$$

Proof. Let $f \in \widetilde{\mathcal{S}}^n_{\mathcal{HN}}(A,B)$ be of the form (12).

We note that f is starlike of order α in $U(r)$ if and only if (see [7])

$$\sum_{k=2}^{\infty} \left(\frac{k-\alpha}{1-\alpha}|a_k| + \frac{k+\alpha}{1-\alpha}|b_k|\right) r^{k-1} \le 1. \tag{14}$$

Also, we have, from Theorem 3, that

$$\sum_{k=2}^{\infty} \left(\frac{\alpha_k}{B-A}|a_k| + \frac{\beta_k}{B-A}|b_k|\right) \le 1.$$

Since $\alpha_k < \beta_k$, $k = 2, 3, \ldots$, the condition (14) is true if

$$\frac{k-\alpha}{1-\alpha} r^{k-1} \le \frac{\alpha_k}{B-A} \quad \text{and} \quad \frac{k+\alpha}{1-\alpha} r^{k-1} \le \frac{\beta_k}{B-A}, \quad k = 2, 3, \ldots,$$

or

$$r \le \left(\frac{1-\alpha}{B-A} \min\left\{\frac{\alpha_k}{k-\alpha}, \frac{\beta_k}{k+\alpha}\right\}\right)^{\frac{1}{k-1}}, \quad k = 2, 3, \ldots.$$

So, the function f is starlike of order α in the disk $U(r^*)$, where

$$r^* := \inf_{k \ge 2} \left(\frac{1-\alpha}{B-A} \min\left\{\frac{\alpha_k}{k-\alpha}, \frac{\beta_k}{k+\alpha}\right\}\right)^{\frac{1}{k-1}}.$$

From the function

$$f_k = h_k(z) + \overline{g_k(z)} = z - \frac{B-A}{\alpha_k} z^k + (-1)^n \frac{B-A}{\beta_k} \overline{z}^k$$

comes that the radius r^* cannot be any larger.

Theorem 7 is proved.

Similarly, we get the following theorem.

Theorem 8. *Let* $0 \leq \alpha < 1$ *and* α_k *and* β_k *be defined by* (7)*. Then*

$$R_\alpha^c\left(\widetilde{\mathcal{S}}^n_{\mathcal{HN}}(A,B)\right) = \inf_{k\geq 2}\left(\frac{1-\alpha}{B-A}\,\min\left\{\frac{\alpha_k}{k(k-\alpha)},\frac{\beta_k}{k(k+\alpha)}\right\}\right)^{\frac{1}{k-1}}.$$

Now, we will examine the closure properties of the class $\widetilde{\mathcal{S}}^n_{\mathcal{H}}(A,B)$ under the generalized Bernardi – Libera – Livingston integral operator $\mathcal{L}_c(f)$, $c > -1$, which is defined by $\mathcal{L}_c(f) = \mathcal{L}_c(h) + \overline{\mathcal{L}_c(g)}$, where

$$\mathcal{L}_c(h)(z) = \frac{c+1}{z^c}\int_0^z t^{c-1}h(t)dt \quad \text{and} \quad \mathcal{L}_c(g)(z) = \frac{c+1}{z^c}\int_0^z t^{c-1}g(t)dt.$$

Theorem 9. *Let* $f \in \widetilde{\mathcal{S}}^n_{\mathcal{H}}(A,B)$*. Then* $\mathcal{L}_c(f) \in \widetilde{\mathcal{S}}^n_{\mathcal{H}}(A,B)$*.*

Proof. From the representation of $\mathcal{L}_c(f(z))$, it follows that

$$\mathcal{L}_c(f)(z) = \frac{c+1}{z^c}\int_0^z t^{c-1}\left[h(t)+\overline{g(t)}\right]dt =$$

$$= \frac{c+1}{z^c}\left[\int_0^z t^{c-1}\left(t-\sum_{k=2}^\infty a_k t^k\right)dt + \overline{\int_0^z t^{c-1}\left(t+(-1)^n\sum_{k=2}^\infty b_k t^k\right)dt}\right] =$$

$$= z - \sum_{k=2}^\infty A_k z^k + (-1)^n\sum_{k=2}^\infty B_k z^k,$$

where

$$A_k = \frac{c+1}{c+k}\,a_k, \quad B_k = \frac{c+1}{c+k}\,b_k.$$

Therefore,

$$\sum_{k=2}^\infty(\alpha_k|A_k| + \beta_k|B_k|) \leq \sum_{k=2}^\infty\left(\alpha_k\,\frac{c+1}{c+k}\,|a_k| + \beta_k\,\frac{c+1}{c+k}\,|b_k|\right) \leq$$

$$\leq \sum_{k=2}^\infty(\alpha_k|a_k| + \beta_k|b_k|) \leq B-A.$$

Since $f \in \widetilde{\mathcal{S}}^n_{\mathcal{H}}(A,B)$, therefore by Theorem 1, $\mathcal{L}_c(f) \in \widetilde{\mathcal{S}}^n_{\mathcal{H}}(A,B)$.

Theorem 9 is proved.

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