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LEFT AND RIGHT B-FREDHOLM OPERATORS
ЛІВИЙ ТА ПРАВИЙ B-ФРЕДГОЛЬМОВІ ОПЕРАТОРИ

We introduce the families of left and right B-Fredholm operators in Banach space, realize their stabilization with the help of finite-rank operators, and prove a spectral mapping theorem for the left and right B-Fredholm operators.

Уведено сім'ї лівих і правих B-фредгольмових операторів у банаховому просторі, проведено стабілізацію за допомогою операторів скінченного рангу та доведено теорему про спектральне відображення для лівих і правих B-фредгольмових операторів.

1. Introduction. M. Berkani in [1] studied the class of B-Fredholm operators on a Banach space. This class is defined by: If we have T a bounded linear operator acting on the Banach space X and for each integer n, then we define the restriction from T to $R(T^n)$ denoted by T_n viewed as $T_n = T|_{R(T^n)} \colon R(T^n) \to R(T^n)$ (for $n = 0,\ T_0 = T$). Now, we say that T is a B-Fredholm operator if for some integer n the range space $R(T^n)$ is closed and T_n is a Fredholm operator, in the sense of having null space $\mathcal{N}(T_n)$ of finite dimension $\alpha(T_n)$ and range $\mathcal{R}(T_n)$ of finite codimension $\beta(T_n)$, the difference $\mathrm{ind}(T_n) = \alpha(T_n) - \beta(T_n) = \mathrm{ind}(T)$ is known as the index of B-Fredholm operator T (see [1]). M. Berkani and M. Sarih extended in [2] this notion and they given the class of semi-B-Fredholm for which T_n is either upper or lower semi-Fredholm, in the sense that either $\mathcal{N}(T_n)$ is finite dimensional and $\mathcal{R}(T_n)$ closed, or $\mathcal{R}(T_n)$ is closed of finite codimension. In this paper, we extend our research to "left and right B-Fredholm operators". We say that T is a left Fredholm operator if $\mathcal{R}(T)$ is closed, $\alpha(T) < \infty$ and $\mathcal{R}(T)$ is a complemented subspace of X, and we call T a right Fredholm operator if $\beta(T) < \infty$ and $\mathcal{N}(T)$ is a complemented subspace of X. The notion of left and right Fredholm operators was introduced by the several mathematicians, for example, in [3] A. A. Boichuk, A. M. Samoilenko studied this notion. We shall see that the left B-Fredholm operator $\mathcal{BF}_l(X)$ on a Banach space X in general properly contain the left Fredholm operator $\Phi_l(X)$, and the right B-Fredholm operator $\mathcal{BF}_r(X)$ on a Banach space X contain the left Fredholm operator $\Phi_r(X)$. And we show that each a left B-Fredholm (resp., right B-Fredholm) operator is a quasi-Fredholm operator in the sense of M. Mbekhta and V. Muller in [7]. Conversely, a quasi-Fredholm operator such as there exists d such that $\mathcal{R}(T^n)$ is a closed subspace of X for each integer $n \geq d$ and $\mathcal{R}(T) + \mathcal{N}(T^d)$ is a closed subspace of X, is a left B-Fredholm (resp., right B-Fredholm) operator.

In Theorem 2.1 and in the case of operators acting on a Hilbert space H we prove that $T \in \mathcal{L}(H)$ is a left B-Fredholm (resp., right B-Fredholm) operator if and only if $T = Q \oplus F$, where Q is a nilpotent operator and F is a left Fredholm (resp., right Fredholm) operator. In Proposition 2.4, we

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prove that if T is a left B-Fredholm (resp., right B-Fredholm) operator and if F is a finite dimensional operator then $T+F$ is also a left B-Fredholm (resp., right B-Fredholm) operator.

In the third section, we prove the stability of a left and a right B-Fredholm operators, we show on Theorems 3.1 and 3.2 that if T and S are two left B-Fredholm (resp., right B-Fredholm) operators and the condition $TA+ST=I$ is satisfy, then TS is a left B-Fredholm (resp., right B-Fredholm) operators. Conversely, if TS is a left B-Fredholm (resp., right B-Fredholm) operator, then T and S are left B-Fredholm (resp., right B-Fredholm) operators such that $TA+ST=I$. Also, we prove a spectral mapping theorem for left and right B-Fredholm operators, more precisely in Theorem 3.3, for $T\in\mathcal{L}(X)$ and f an analytic function on the usual spectrum $\sigma(T)$ of T, we prove that $f\big(\sigma_{\mathcal{BF}_l}(T)\big) = \sigma_{\mathcal{BF}_l}(f(T))$, where $\sigma_{\mathcal{BF}_l}(T)=\{\lambda\in\mathbb{C}\,\text{such that}\,(T-\lambda I)\notin\mathcal{BF}_l(X)\}$, and $f(\sigma_{\mathcal{BF}_r}(T)) = \sigma_{\mathcal{BF}_r}(f(T))$, where $\sigma_{\mathcal{BF}_r}(T)=\{\lambda\in\mathbb{C}\,\text{such that}\,(T-\lambda I)\notin\mathcal{BF}_r(X)\}$.

In the sequel if E and F are two vector spaces, the notation $E\simeq F$ will mean that E and F are isomorphic. If E and F are vector subspaces of the same vector space H we shall write $E=_e F$ if there exist two finite dimensional vector subspaces G_1 and G_2 of H such that $E\subset F+G_1$ and $F\subset E+G_2$. Next, if $E\subset F$ then we denote the quotient space E modulo F by $\dfrac{E}{F}$ (see [4], Definition 1).

2. Definition and properties of left and right B-Fredholm operators.

Proposition 2.1. *Let $T\in\mathcal{L}(X)$. If there exists an integer $n\in\mathbb{N}$ such that $\mathcal{R}(T^n)$ is closed and the operator T_n is a left Fredholm operator, then $\mathcal{R}(T^m)$ is closed and the operator T_m is a left Fredholm operator and* $\operatorname{ind}(T_m)=\operatorname{ind}(T_n)$ *for each $m\ge n$.*

Proof. If $T_n\colon\mathcal{R}(T^n)\to\mathcal{R}(T^n)$ is a left Fredholm operator, then T_n is upper semi-Fredholm operator, so, for each $m\ge n$, the operator $T_n^{m-n}\colon\mathcal{R}(T^n)\to\mathcal{R}(T^n)$ is also an upper semi-Fredholm operator. Hence, $\mathcal{R}(T_n^{m-n})=\mathcal{R}(T^m)$ is closed in $\mathcal{R}(T^n)$. Since $\mathcal{R}(T^n)$ is closed in X, then $\mathcal{R}(T^m)$ is closed in X. Consider now the operator $T_m\colon\mathcal{R}(T^m)\to\mathcal{R}(T^m)$. We have $\mathcal{N}(T_m)=\mathcal{N}(T)\cap\mathcal{R}(T^m)\subset\mathcal{N}(T)\cap\mathcal{R}(T^n)=\mathcal{N}(T_n)$. So, $\alpha(T_m)<\infty$.

If the operator T_n is a left Fredholm operator, then $\mathcal{R}(T_n)$ is a complemented subspace of $\mathcal{R}(T^n)$. Since $\mathcal{N}(T_n^{m-n})$ is of finite dimension, then $\mathcal{R}(T_n)+\mathcal{N}(T_n^{m-n})$ is also a complemented subspace of $\mathcal{R}(T^n)$.

This means that there exists a finite dimensional subspace F_1 of $\mathcal{R}(T^n)$ such that

$$\mathcal{R}(T^n)=F_1\oplus\big(\mathcal{R}(T_n)+\mathcal{N}(T_n^{m-n})\big).$$

Then $\mathcal{R}(T^m)=T^{m-n}(F_1)+T^{m-n}\big(\mathcal{R}(T_n)\big)$.

First, it is known that the image of a closed subspace by an operator upper semi-Fredholm operator is closed, then $T^{m-n}(F_1)$ is a closed subspace of $\mathcal{R}(T^m)$. It remains to show that the sum is direct: Let $z\in T^{m-n}(F_1)\cap T^{m-n}\big(\mathcal{R}(T_n)\big)$. Then there exist $x\in F_1$ and $y\in\mathcal{R}(T_n)$ such that $z=T^{m-n}(x)=T^{m-n}(y)$. We obtain $x-y\in\mathcal{N}(T_n^{m-n})$, therefore, $x=y+(x-y)\in(\mathcal{R}(T_n)+\mathcal{N}(T_n^{m-n}))\cap F_1=\{0\}$. Hence, $x=0$ and therefore $z=0$, whence $\mathcal{R}(T^m)=T^{m-n}(F_1)+\mathcal{R}(T_m)$.

Thus, $\mathcal{R}(T_m)$ is a complemented subspace of $\mathcal{R}(T^m)$. Consequently, T_m is a left Fredholm operator.

Moreover, from [4] (Lemma 3.5), we have

$$\frac{\mathcal{N}(T)\cap\mathcal{R}(T^n)}{\mathcal{N}(T)\cap\mathcal{R}(T^{n+1})}\simeq\frac{\mathcal{N}(T^{n+1})+\mathcal{R}(T)}{\mathcal{N}(T^n)+\mathcal{R}(T)}.$$

Also, from [4] (Lemma 3.2), we get

$$\frac{\mathcal{R}(T^n)}{\mathcal{R}(T^{n+1})} \cong \frac{X}{\mathcal{R}(T)+\mathcal{N}(T^n)} \quad \text{and} \quad \frac{\mathcal{R}(T^{n+1})}{\mathcal{R}(T^{n+2})} \cong \frac{X}{\mathcal{R}(T)+\mathcal{N}(T^{n+1})}.$$

Hence, $\alpha(T_n) - \alpha(T_{n+1}) = \beta(T_n) - \beta(T_{n+1})$, which means that $\mathrm{ind}(T_n) = \mathrm{ind}(T_{n+1})$. It follows then that $\mathrm{ind}(T_m) = \mathrm{ind}(T_n)$ for each $m \geq n$.

Proposition 2.2. *Let $T \in \mathcal{L}(X)$. If there exists an integer $n \in \mathbb{N}$ such that $\mathcal{R}(T^n)$ is closed and the operator T_n is a right Fredholm operator, then $\mathcal{R}(T^m)$ is closed, the operator T_m is a right Fredholm operator and $\mathrm{ind}(T_m) = \mathrm{ind}(T_n)$ for each $m \geq n$.*

Proof. In the same way as the previous proposition we show that if $\mathcal{R}(T^n)$ is closed, then $\mathcal{R}(T^m)$ is closed. For $n \in \mathbb{N}$, suppose that T_n is a right Fredholm operator. We shall show that T_{n+1} is a right Fredholm operator.

T_n is a right Fredholm operator, then $\mathrm{codim}(\mathcal{R}(T_n)) < \infty$ in $\mathcal{R}(T^n)$. So, there exists F a finite dimensional subspace of $\mathcal{R}(T^n)$ such that $\mathcal{R}(T^n) = F \oplus \mathcal{R}(T_n) = F \oplus \mathcal{R}(T^{n+1})$. This implies that the injection $i_n : \mathcal{R}(T^{n+1}) \to \mathcal{R}(T^n)$ and the projection $p_n : \mathcal{R}(T^n) \to \mathcal{R}(T^{n+1})$ are both Fredholm operators. We can easily check that $T_{n+1} = p_n \circ T \circ i_n$. Hence, if T_n is a right Fredholm operator, then T_{n+1} is also right Fredholm operator. Consequently, if T_n is a right Fredholm operator, then T_m is likewise right Fredholm operator. We get the equality of the index by the same way as in the proof of the previous proposition.

Definition 2.1. *Let $T \in \mathcal{L}(X)$.*

(i) *If, for some integer $n \in \mathbb{N}$, $\mathcal{R}(T^n)$ is closed and the operator T_n is a left Fredholm operator, then T is called a left B-Fredholm operator.*

(ii) *If, for some integer $n \in \mathbb{N}$, $\mathcal{R}(T^n)$ is closed and the operator T_n is a right Fredholm operator, then T is called a right B-Fredholm operator.*

Observe from the definition of left and right B-Fredholm operators all nilpotent operators and all bounded linear projections are left and right B-Fredholm operators. Hence the class $\mathcal{BF}_l(X)$ (resp., $\mathcal{BF}_r(X)$) of left B-Fredholm (resp., right B-Fredholm) operators contains the class of left Fredholm (resp., right Fredholm) operators as a proper subclass. Note also that obviously every B-Fredholm operator is a left B-Fredholm (resp., right B-Fredholm) and every left B-Fredholm (resp., right B-Fredholm) operator is upper semi B-Fredholm (resp., lower semi-B-Fredholm).

Definition 2.2. *Let $T \in \mathcal{L}(X)$ be a left (resp., right) B-Fredholm operator and n any integer such that $\mathcal{R}(T^n)$ is closed and T_n is a left (resp., right) Fredholm operator. Then we define the index of T denote by $\mathrm{ind}(T)$ as the index of the left (resp., right) Fredholm operator T_n. From Propositions 2.1 and 2.2, this definition is independent of the choice of the integer n. Furthermore, if T is a Fredholm operator, this reduces to the usual definition of the index.*

Definition 2.3 [5]. *Let $T \in \mathcal{L}(X)$ and*

$$\Delta(T) = \big\{n \in \mathbb{N}; \forall m \in \mathbb{N},\, m \geq n \Rightarrow (\mathcal{R}(T^n) \cap \mathcal{N}(T)) \subset (\mathcal{R}(T^m) \cap \mathcal{N}(T))\big\}.$$

Then the degree of stable iteration $\mathrm{dis}(T)$ of T is defined as $\mathrm{dis}(T) = \inf(\Delta(T))$. If $\Delta(T) = \varnothing$ then $\mathrm{dis}(T) = \infty$.

Definition 2.4. *Let $T \in \mathcal{L}(X)$. Then T is called a quasi-Fredholm operator of degree d if there is an integer $d \in \mathbb{N}$ such that:*

(i) $\mathrm{dis}(T) = d$,

(ii) *$\mathcal{R}(T^n)$ is a closed subspace of X for each integer $n \geq d$,*

(iii) $\mathcal{R}(T)+\mathcal{N}(T^d)$ *is a closed subspace of* X.

In the sequel $\mathcal{QF}(d)$ *will denote the set of quasi-Fredholm operators of degree* d.

Proposition 2.3. *Let* $T\in\mathcal{L}(X)$. *Then* T *is a left* (*resp.*, *right*) *B-Fredholm operator if and only if there exists an integer* $d\in\mathbb{N}$ *such that* $T\in\mathcal{QF}(d)$ *and*:

(i) $\dim(\mathcal{R}(T^d)\cap\mathcal{N}(T))<\infty$ (*resp.*, $\operatorname{codim}(\mathcal{R}(T)+\mathcal{N}(T^d))<\infty$),

(ii) $\mathcal{R}(T)+\mathcal{N}(T^d)$ (*resp.*, $\mathcal{R}(T^d)\cap\mathcal{N}(T)$) *is a complemented subspace of* $\mathcal{R}(T^d)$.

Proof. Suppose that $T\in\mathcal{BF}_l(X)$. Then, there exists $n\in\mathbb{N}$ such that $\mathcal{R}(T^n)$ is closed and T_n is a left Fredholm operator in $\mathcal{R}(T^n)$. Then $\dim(\mathcal{R}(T^n)\cap\mathcal{N}(T))<\infty$ and $\mathcal{R}(T)+\mathcal{N}(T^n)$ is a complemented subspace of $\mathcal{R}(T^n)$.

Let $m\geq n$, then $\mathcal{R}(T^m)\cap\mathcal{N}(T)\subset\mathcal{R}(T^n)\cap\mathcal{N}(T)$. Since $\dim(\mathcal{R}(T^n)\cap\mathcal{N}(T))<\infty$, the sequence $(\mathcal{R}(T^p)\cap\mathcal{N}(T))_p$ is a stationary sequence for p large enough. Therefore,

$$d=\operatorname{dis}(T)\in\mathbb{N}\quad\text{and}\quad\dim\big(\mathcal{R}(T^d)\cap\mathcal{N}(T)\big)<\infty.$$

If $\mathcal{R}(T)+\mathcal{N}(T^n)$ is a complemented subspace of $\mathcal{R}(T^n)$, then there exists $F\in\mathcal{R}(T^n)$ such that $\mathcal{R}(T^n)=F\oplus\mathcal{R}(T)+\mathcal{N}(T^n)$.

We have, for each $n\geq d$, $\mathcal{R}(T)\subset\mathcal{N}(T^d)+\mathcal{R}(T)$ such that $\mathcal{R}(T)+\mathcal{N}(T^n)\subset\mathcal{R}(T)+$ $+\mathcal{N}(T^d)$. Since $\mathcal{N}(T^d)\subset\mathcal{N}(T^n)$, then $\mathcal{R}(T)+\mathcal{N}(T^d)\subset\mathcal{R}(T)+\mathcal{N}(T^n)$. This shows that $\mathcal{R}(T)+$ $+\mathcal{N}(T^n)=\mathcal{R}(T)+\mathcal{N}(T^d)$. If $\mathcal{R}(T^n)\subset\mathcal{R}(T^d)$, then there exists $F\in\mathcal{R}(T^d)$ such that $\mathcal{R}(T^d)=$ $=F\oplus\mathcal{R}(T)+\mathcal{N}(T^d)$. Therefore, $\mathcal{R}(T)+\mathcal{N}(T^d)$ is a complemented subspace of $\mathcal{R}(T^d)$.

As though, $\mathcal{R}(T^m)$ is closed for each $m\geq n$, we deduced that $\mathcal{R}(T^m)$ is closed for each $m\geq d$. Moreover, we have $\mathcal{R}(T)+\mathcal{N}(T^d)=(T^d)^{-1}\big(\mathcal{R}(T^{d+1})\big)$. Hence, $\mathcal{R}(T)+\mathcal{N}(T^d)$ is a closed subspace of X. Consequently, $T\in\mathcal{QF}(d)$ such that $\dim(\mathcal{R}(T^d)\cap\mathcal{N}(T))<\infty$ and $\mathcal{R}(T)+\mathcal{N}(T^d)$ is a complemented subspace of $\mathcal{R}(T^d)$.

Conversely, suppose that $T\in\mathcal{QF}(d)$ such that $\dim(\mathcal{R}(T^d)\cap\mathcal{N}(T))<\infty$ and $\mathcal{R}(T)+\mathcal{N}(T^d)$ is a complemented subspace of $\mathcal{R}(T^d)$. Thus, $\mathcal{R}(T^n)$ is closed for each $n\geq d$, since, $\dim(\mathcal{R}(T^d)\cap$ $\cap\,\mathcal{N}(T))<\infty$ and $\mathcal{R}(T)+\mathcal{N}(T^d)$ is a complemented subspace of $\mathcal{R}(T^d)$. Hence, T_d is a left Fredholm operator. Finally, $T\in\mathcal{BF}_l(X)$.

Suppose that $T\in\mathcal{BF}_r(X)$. Then there exists $n\in\mathbb{N}$ such that $\mathcal{R}(T^n)$ is closed and T_n is a right Fredholm operator of $\mathcal{R}(T^n)$. Then $\operatorname{codim}(\mathcal{R}(T)+\mathcal{N}(T^n))<\infty$ and $\mathcal{R}(T)\cap\mathcal{N}(T^n)$ is a complemented subspace of $\mathcal{R}(T^n)$.

Let $m\geq n$, then $\mathcal{R}(T)+\mathcal{N}(T^n)\subset\mathcal{R}(T)+\mathcal{N}(T^m)$. Since $\operatorname{codim}(\mathcal{R}(T)+\mathcal{N}(T^n))<\infty$, thus, the sequence $(\mathcal{R}(T)+\mathcal{N}(T^p))_p$ is a stationary sequence for p large enough. This shows that

$$d=\operatorname{dis}(T)\in\mathbb{N}\quad\text{and}\quad\operatorname{codim}\big(\mathcal{R}(T)+\mathcal{N}(T^d)\big)<\infty.$$

If $\mathcal{R}(T)\cap\mathcal{N}(T^n)$ is a complemented subspace of $\mathcal{R}(T^n)$, then there exists $F\in\mathcal{R}(T^n)$ such that $\mathcal{R}(T^n)=F\oplus\mathcal{R}(T)\cap\mathcal{N}(T^n)$. Hence $\mathcal{R}(T)\cap\mathcal{N}(T^d)$ is a complemented subspace of $\mathcal{R}(T^d)$.

We have $\operatorname{codim}(\mathcal{R}(T)+\mathcal{N}(T^n))<\infty$. Thus, $\mathcal{R}(T^m)$ is closed for each $m\geq n$, and then $\mathcal{R}(T^m)$ is a closed for each $m\geq d$. Therefore, $\mathcal{R}(T)+\mathcal{N}(T^d)$ is a closed subspace of X.

Hence, $T\in\mathcal{QF}(d)$ such that $\operatorname{codim}(\mathcal{R}(T)+\mathcal{N}(T^d))<\infty$ and $\mathcal{R}(T)\cap\mathcal{N}(T^d)$ is a complemented subspace of $\mathcal{R}(T^d)$.

Conversely, we suppose that $T\in\mathcal{QF}(d)$ such that $\operatorname{codim}(\mathcal{R}(T)+\mathcal{N}(T^d))<\infty$ and $\mathcal{R}(T)\cap$ $\cap\,\mathcal{N}(T^d)$ is a complemented subspace of $\mathcal{R}(T^d)$. Thus, $\mathcal{R}(T^n)$ is closed for each $n\geq d$, as though, $\operatorname{codim}(\mathcal{R}(T)+\mathcal{N}(T^d))<\infty$ and $\mathcal{R}(T)\cap\mathcal{N}(T^d)$ is a complemented subspace of $\mathcal{R}(T^d)$. Therefore, T_d is a right Fredholm operator. Consequently, $T\in\mathcal{BF}_r(X)$.

Theorem 2.1. *Let X be an Hilbert space and $T \in \mathcal{L}(X)$. Then T is a left B-Fredholm (resp., right B-Fredholm) operator if and only if there exist two closed subspaces M and N of X and an integer $d \in \mathbb{N}$ such that*:

(i) $X = M \oplus N$,

(ii) $T(N) \subset N$ *and* $T|_N$ *is a nilpotent operator*,

(iii) $T(M) \subset M$ *and* $T|_M$ *is a left Fredholm (resp., right Fredholm) operator.*

Proof. Since H is a Hilbert space, then each subspace of H admits a complemented. So according to [2] (Theorem 2.6) we obtain the result.

Proposition 2.4. *Let $T \in \mathcal{L}(X)$ be a left B-Fredholm (resp., right B-Fredholm) operator and $F \in \mathcal{L}(X)$ be a finite rank operator. Then $T+F$ is also a left B-Fredholm (resp., right B-Fredholm) operator.*

Proof. If T is a left B-Fredholm (resp., right B-Fredholm) operator, then T is an upper semi-B-Fredholm (resp., lower semi-B-Fredholm) operator. Hence, from [2] (Proposition 2.7) we obtain that $T+F$ is an upper semi-B-Fredholm.

Moreover, we have $\mathcal{R}\big((T+F)_n\big) = \mathcal{R}((T+F)^{n+1})$. Since $\mathcal{R}((T+F)^{n+1}) =_e \mathcal{R}(T^{n+1}) = \mathcal{R}(T_n)$ and $\mathcal{R}((T+F)^n) =_e \mathcal{R}(T^n)$.

If T is left B-Fredholm, then $\mathcal{R}(T_n)$ is a complemented subspace of $\mathcal{R}(T^n)$ for some $n \in \mathbb{N}$. Thus, $\mathcal{R}\big((T+F)_n\big)$ is a complemented subspace of $\mathcal{R}((T+F)^n)$. Consequently, $T+F$ is a left B-Fredholm operator. Now suppose that T is a right B-Fredholm.

Let us show that $\mathcal{N}\big((T+F)_n\big)$ is a complemented subspace of $\mathcal{R}((T+F)^n)$. We have $\mathcal{N}\big((T+F)_n\big) = \mathcal{N}((T+F)) \cap \mathcal{R}((T+F)^n) =_e \mathcal{N}(T) \cap \mathcal{R}(T^n) = \mathcal{N}(T_n)$.

As T is a right B-Fredholm operator, then $\mathcal{N}(T_n)$ is a complemented subspace of $\mathcal{R}(T^n)$. Hence, $\mathcal{N}\big((T+F)_n\big)$ is a complemented subspace of $\mathcal{R}((T+F)^n)$. Therefore, $T+F$ is a right B-Fredholm operator.

Proposition 2.5. *Let $T \in \mathcal{L}(X)$. The following properties are equivalent*:

(i) $T \in \mathcal{BF}_l(X)$,

(ii) $T^m \in \mathcal{BF}_l(X)$ *for each* $m > 0$,

(iii) $T^m \in \mathcal{BF}_l(X)$ *for some* $m > 0$.

Proof. (i)⇒(ii). Suppose that $T \in \mathcal{BF}_l(X)$ and let $d = \operatorname{dis}(T)$. From Proposition 2.1 we obtain that $\mathcal{R}(T^{md})$ is a closed subspace of X and T_{md} is a left Fredholm operator. Since $(T_{md})^m = (T^m)_d$, then the operator $(T^m)_d$ is a left Fredholm operator. Consequently, T^m is a left B-Fredholm operator.

(ii)⇒(iii). This is obvious.

(iii)⇒(i). Suppose that T^m is a left B-Fredholm for some $m > 0$. Then there exists an integer n such that $\mathcal{R}(T^{mn})$ is a closed subspace of X and $(T^m)_n$ is a left Fredholm operator. Since $(T_{mn})^m = (T^m)_n$, hence, $(T_{mn})^m$ is a left Fredholm operator. Therefore, if the operator (T_{mn}) is a left Fredholm operator, then T is a left B-Fredholm operator.

Proposition 2.6. *Let $T \in \mathcal{L}(X)$. The following properties are equivalent*:

(i) $T \in \mathcal{BF}_r(X)$,

(ii) $T^m \in \mathcal{BF}_r(X)$ *for each* $m > 0$,

(iii) $T^m \in \mathcal{BF}_r(X)$ *for some* $m > 0$.

Proof. (i)$\Rightarrow$(ii). Suppose that $T \in \mathcal{BF}_r(X)$ and let $d = \operatorname{dis}(T)$. From Proposition 2.2 we obtain that $\mathcal{R}(T^{md})$ is a closed subspace of X and T_{md} is a right Fredholm operator. Since $(T_{md})^m = (T^m)_d$, then the operator $(T^m)_d$ is a right Fredholm operator. Consequently, T^m is a right B-Fredholm operator.

(ii)$\Rightarrow$(iii). This is obvious.

(iii)$\Rightarrow$(i). Suppose that T^m is a right B-Fredholm for some $m > 0$. Then there exists an integer n such that $\mathcal{R}(T^{mn})$ is a closed subspace of X and $(T^m)_n$ is a right Fredholm operator. Since $(T_{mn})^m = (T^m)_n$, then $(T_{mn})^m$ is a right Fredholm operator. Therefore, the operator (T_{mn}) is a right Fredholm operator, which means that T is a right B-Fredholm operator.

3. A spectral mapping theorem for left and right B-Fredholm operators.

Definition 3.1. (i) *Let $T \in \mathcal{L}(X)$. We call right B-Fredholm resolving set of T and we write $\rho_{\mathcal{BF}_r}(T)$ the set $\rho_{\mathcal{BF}_r}(T) = \{\lambda \in \mathbb{C}, (T - \lambda I) \in \mathcal{BF}_r(X)\}$. We call right B-Fredholm spectrum of T, denoted $\sigma\mathcal{BF}_r(T)$, the set $\sigma_{\mathcal{BF}_r}(T) = \{\lambda \in \mathbb{C}, (T - \lambda I) \notin \mathcal{BF}_r(X)\}$.*

(ii) *Let $T \in \mathcal{L}(X)$. We call left B-Fredholm resolving set of T and we write $\rho_{\mathcal{BF}_l}(T)$ the set $\rho_{\mathcal{BF}_l}(T) = \{\lambda \in \mathbb{C}, (T - \lambda I) \in \mathcal{BF}_l(X)\}$. We call left B-Fredholm spectrum of T, denoted $\sigma\mathcal{BF}_l(T)$, the set $\sigma_{\mathcal{BF}_l}(T) = \{\lambda \in \mathbb{C}, (T - \lambda I) \notin \mathcal{BF}_l(X)\}$.*

Proposition 3.1. *Let $T \in \mathcal{L}(X)$, then the left B-Fredholm (resp., right B-Fredholm) spectrum of T is a closed subset of $\mathbb{C}$ such that*

$$\sigma_{\mathcal{BF}_l}(T) \subset \sigma(T) \quad \textit{and} \quad \sigma_{\mathcal{BF}_r}(T) \subset \sigma(T).$$

Proof. If $\lambda \notin \sigma(T)$ then $T - \lambda I \in \mathcal{BF}(X)$. Thus, $T - \lambda I$ is a left B-Fredholm (resp., right B-Fredholm) operator. So, $\sigma_{\mathcal{BF}_l}(T) \subset \sigma(T)$ and $\sigma_{\mathcal{BF}_r}(T) \subset \sigma(T)$. If $\alpha \notin \sigma_{\mathcal{BF}_l}(T)$ then $S = T - \alpha I \in \mathcal{BF}_l(X)$. If ϵ is small and not equal to zero, by [7, p. 144] (Table 2) $S - \epsilon I$ is a quasi-Fredholm operator. From [2] (Theorem 3.1) we have $\dim(\mathcal{N}(S - \epsilon I)_n) = \dim(\mathcal{N}(S_n)) < \infty$. Furthermore, $\mathcal{R}(S_n)$ is a complemented subspace of $\mathcal{R}(S^n)$. So, there exists $F \subset \mathcal{R}(S^n)$ such that $\mathcal{R}(S^n) = F \oplus \mathcal{R}(S_n)$. Hence, $\mathcal{R}((S - \epsilon I)^n) =_e \mathcal{R}(S^n) = F \oplus \mathcal{R}(S_n) =_e F \oplus \mathcal{R}((S - \epsilon I)_n)$. Then $S - \epsilon I$ is a left B-Fredholm operator. So, $\rho_{\mathcal{BF}_l}(T)$ is open in $\mathbb{C}$. Consequently, $\sigma_{\mathcal{BF}_l}(T)$ is a closed subset of $\mathbb{C}$.

On the same way, let $\beta \notin \sigma_{\mathcal{BF}_r}(T)$, then $A = T - \alpha I \in \mathcal{BF}_r(X)$. If ϵ is small and not equal to zero, by [7, p. 144] (Table 2), $A - \epsilon I$ is a quasi-Fredholm operator. From [2] (Theorem 3.1), $\operatorname{codim}(\mathcal{R}(S - \epsilon I)_n) = \operatorname{codim}(\mathcal{R}(S_n)) < \infty$. Furthermore, $\mathcal{N}(S_n)$ is a complemented subspace of $\mathcal{R}(S^n)$. So, there exists $F_1 \subset \mathcal{R}(S^n)$ such that $\mathcal{R}(S^n) = F_1 \oplus \mathcal{N}(S_n)$. If $\mathcal{R}\big((S - \epsilon I)^n\big) =_e \mathcal{R}(S^n)$, then, $\mathcal{R}((S - \epsilon I)^n) = F \oplus \mathcal{N}(S_n) =_e F \oplus \mathcal{N}\big((S - \epsilon I)_n\big)$. Then $S - \epsilon I$ is a right B-Fredholm operator. So, $\rho_{\mathcal{BF}_r}(T)$ is open in $\mathbb{C}$. Finally, $\sigma_{\mathcal{BF}_r}(T)$ is a closed subset of $\mathbb{C}$.

Proposition 3.2. *Let $T \in \mathcal{L}(X)$, then the right B-Fredholm spectrum $\sigma_{\mathcal{BF}_r}(T)$ of T is a closed subset of $\mathbb{C}$ contained in the usual spectrum $\sigma(T)$ of T.*

Proof. If $\lambda \notin \sigma(T)$ then $T - \lambda I \in \mathcal{BF}(X)$. Thus, $T - \lambda I \in \mathcal{BF}_r(X)$ and $\lambda \notin \sigma_{\mathcal{BF}_r}(T)$. So, $\sigma_{\mathcal{BF}_r}(T) \subset \sigma(T)$.

Theorem 3.1. *Let S, T, A, B be mutually commuting operators in $\mathcal{L}(X)$, satisfying $TA + BS = I$. $T, S \in \mathcal{BF}_l(X)$ if and only if $TS \in \mathcal{BF}_l(X)$.*

Proof. Suppose that T and S are a left B-Fredholm operator. Then there exist $A_n \in \mathcal{L}(\mathcal{R}(T^n))$, $B_n \in \mathcal{L}(\mathcal{R}(S^n))$ and $K_n \in \mathcal{K}(\mathcal{R}(T^n)), C_n \in \mathcal{K}(\mathcal{R}(S^n))$ such that

$$A_n T_n = I + K_n, \qquad B_n S_n = I + C_n.$$

Let

$$G_n : \mathcal{R}((TS)^n) \longrightarrow \mathcal{R}((TS)^n),$$

$$x \longmapsto (B^n A_n + A^n B_n)x.$$

If $TA + BS = I$, then, according to [6] (Lemma 2.6), we obtain $\mathcal{R}((TS)^n) = \mathcal{R}(T^n) \cap \mathcal{R}(S^n)$. Then G_n becomes

$$G_n : \mathcal{R}(T^n) \cap \mathcal{R}(S^n) \longrightarrow \mathcal{R}(T^n) \cap \mathcal{R}(S^n),$$

$$x \longmapsto (B^n A_n + A^n B_n)x.$$

If $T^n(\mathcal{R}(S^n)) \subset \mathcal{R}(S^n)$, in fact, if $x \in T^n\big(\mathcal{R}(S^n)\big)$, then there exists $x' \in X$ such that $S^n(x') = x$. Thus, $(ST)^n x = (TS)^n x = T^n x \subset \mathcal{R}(S^n)$. Therefore, the operator G_n is well defined. Since

$$G_n T_n S_n = (B^n A_n + A^n B_n) T_n S_n =$$

$$= B^n A_n T_n S_n + A^n B_n T_n S_n =$$

$$= B^n (I + K_n) S_n + A^n (I + C_n) T_n =$$

$$= B^n S_n + B^n K_n S_n + A^n T_n + A^n C_n T_n =$$

$$= B^n S_n + A^n T_n + (B^n K_n S_n + A^n C_n T_n) = I + \tilde{K}.$$

It is clear that $B^n K_n S_n + A^n C_n T_n \in \mathcal{K}(\mathcal{R}(T^n) \cap \mathcal{R}(S^n))$. Consequently, $TS \in \mathcal{BF}_l(X)$.

Conversely, if $TS \in \mathcal{BF}_l(X)$, then $(TS)_n$ and $(ST)_n$ are a left Fredholm operators.

Let $\tilde{T} = T|_{\mathcal{R}((TS)^n)}$ and $\tilde{S} = S|_{\mathcal{R}((TS)^n)}$. Then $(TS)_n = \tilde{T}\tilde{S}$ and $(ST)_n = \tilde{S}\tilde{T}$. Thus, $\tilde{T}\tilde{S}$ and $\tilde{S}\tilde{T}$ are a left Fredholm operators. Therefore, $\tilde{T}$ and $\tilde{S}$ are a left Fredholm operators in $\mathcal{R}((TS)^n)$. If $TA + BS = I$, then, according to [6] (Lemma 2.6), we obtain $\mathcal{R}((TS)^n) = \mathcal{R}(T^n) \cap \mathcal{R}(S^n)$. Since S_n and T_n are a left Fredholm operators. Consequently, T and S are a left B-Fredholm operators.

Theorem 3.2. *Let S, T, A, B be mutually commuting operators in $\mathcal{L}(X)$, satisfying $TA + {} + BS = I$. $T, S \in \mathcal{BF}_r(X)$ if and only if $TS \in \mathcal{BF}_r(X)$.*

Proof. Suppose that T and S are a right B-Fredholm operators. Then there exist $A_n \in {} \in \mathcal{L}(\mathcal{R}(T^n))$, $B_n \in \mathcal{L}(\mathcal{R}(S^n))$ and $K_n \in \mathcal{K}(\mathcal{R}(T^n)), C_n \in \mathcal{K}(\mathcal{R}(S^n))$ such that

$$T_n A_n = I + K_n \quad \text{and} \quad S_n B_n = I + C_n.$$

Let

$$G_n : \mathcal{R}((TS)^n) \longrightarrow \mathcal{R}((TS)^n),$$

$$x \longmapsto (B_n A^n + A_n B^n)x.$$

If $S^n(\mathcal{R}(T^n)) \subset \mathcal{R}(T^n)$, in fact, if $x \in S^n(\mathcal{R}(T^n))$, then there exists $x' \in X$ such that $T^n(x') = x$. Thus, $(TS)^n x = (ST)^n x = S^n x \subset \mathcal{R}(T^n)$. Therefore, the operator G_n is well defined. Since

$$T_n S_n G_n = T_n S_n (B_n A^n + A_n B^n) =$$

$$= T_n S_n B_n A^n + T_n S_n A_n B^n =$$

$$= T_n(I + C_n)A^n + S_n(I + K_n)B^n =$$

$$= T_nA^n + T_nC_nA^n + S_nB^n + S_nK_nB_n =$$

$$= T_nA^n + S_nB^n + (T_nC_nA^n + S_nK_nB_n) = I + \tilde{C}.$$

It is clear that $T_nC_nA^n + S_nK_nB_n \in \mathcal{K}(\mathcal{R}(T^n) \cap \mathcal{R}(S^n))$. Consequently, $TS \in \mathcal{BF}_r(X)$.

Conversely, if $TS \in \mathcal{BF}_r(X)$, then $(TS)_n$ and $(ST)_n$ are a right Fredholm operators.

Let $\tilde{T} = T|_{\mathcal{R}((TS)^n)}$ and $\tilde{S} = S|_{\mathcal{R}((TS)^n)}$. Then $(TS)_n = \tilde{T}\tilde{S}$ and $(ST)_n = \tilde{S}\tilde{T}$, thus, $\tilde{T}\tilde{S}$ and $\tilde{S}\tilde{T}$ are a right Fredholm operators. Therefore, $\tilde{T}$ and $\tilde{S}$ are a right Fredholm operators in $\mathcal{R}((TS)^n)$. If $TA + BS = I$, then, according to [6] (Lemma 2.6), we obtain $\mathcal{R}((TS)^n) = \mathcal{R}(T^n) \cap \mathcal{R}(S^n)$. Since, S_n and T_n are a right Fredholm operators. Consequently, T and S are a right B-Fredholm operators.

***Corollary* 3.1.** *Let $P(X) = (X - \lambda_1 I)^{m_1} \ldots (X - \lambda_n I)^{m_n}$ be a polynomial with complex coefficients. Then $P(T) = (T - \lambda_1 I)^{m_1} \ldots (T - \lambda_n I)^{m_n}$ is a left B-Fredholm (resp., right B-Fredholm) operator if and only if, for some $1 \le i \le n$, $(T - \lambda_i I)$ is a left B-Fredholm (resp., right B-Fredholm) operator.*

Theorem 3.3. *Let $T \in \mathcal{L}(X)$ and f an analytic function in a neighborhood of $\sigma(T)$ of T. Then*

$$f(\sigma_{\mathcal{BF}_l}(T)) = \sigma_{\mathcal{BF}_l}(f(T)) \quad \textit{and} \quad f(\sigma_{\mathcal{BF}_r}(T)) = \sigma_{\mathcal{BF}_r}(f(T)).$$

Proof. Let $\mu \in \sigma_{\mathcal{BF}_l}(T)$ and f an analytic function in a neighborhood of $\sigma(T)$. If $\sigma(T)$ is a compact subset of $\mathbb{C}$, then the function $f(z) - f(\mu)$ possesses at most a finite number of zeros in $\sigma(T)$. So,

$$f(z) - f(\mu) = (z - \mu)^{m_0}(z - \lambda_1)^{m_1} \ldots (z - \lambda_n)^{m_n} g(z),$$

where $g(z)$ is a non-vanishing analytic function on $\sigma(T)$. Thus,

$$f(T) - f(\mu)I = (T - \mu I)^{m_0}(T - \lambda_1 I)^{m_1} \ldots (T - \lambda_n I)^{m_n} g(T),$$

where $g(T)$ an invertible operator. So, $[g(T)]^{-1}$ is a B-Fredholm operator, then $[g(T)]^{-1}$ is a left B-Fredholm operator. If $f(T) - f(\mu)I \in \mathcal{BF}_l(X)$, then by Theorem 3.1 we obtain $(f(T) - - f(\mu)I)[g(T)]^{-1} \in \mathcal{BF}_l(X)$. Thus, $(T - \mu I)^{m_0}(T - \lambda_1 I)^{m_1} \ldots (T - \lambda_n I)^{m_n} \in \mathcal{BF}_l(X)$. So, from Corollary 3.1, we have $(T - \mu I) \in \mathcal{BF}_l(X)$, a fact which contradicts our assumption. So, $\mu \in \sigma_{\mathcal{BF}_l}(T)$. Hence, $f(\mu) \in \sigma_{\mathcal{BF}_l}(f(T))$. Consequently, $f(\sigma_{\mathcal{BF}_l}(T)) \subset \sigma_{\mathcal{BF}_l}(f(T))$.

Conversely, let $\alpha \in \sigma_{\mathcal{BF}_l}(f(T))$, then $\alpha \in \sigma(f(T))$. Hence, there exists $\mu \in \sigma(T)$ such that $f(\mu) = \alpha$. We have

$$f(z) - f(\mu) = (z - \mu)^{m_0}(z - \mu_1)^{m_1} \ldots (z - \mu_n)^{m_n} g(z),$$

where $g(z)$ is a non-vanishing analytic function on $\sigma(T)$. Thus,

$$f(T) - f(\mu)I = (T - \mu I)^{m_0}(T - \mu_1 I)^{m_1} \ldots (T - \mu_n I)^{m_n} g(T) = f(T) - \alpha I,$$

where $g(T)$ is an invertible operator. If $f(T) - \alpha I \notin \mathcal{BF}_l(X)$, then $(f(T) - \alpha I)[g(T)]^{-1} \notin \notin \mathcal{BF}_l(X)$. From Corollary 3.1, there exists $\beta = \{\mu, \mu_1, \ldots, \mu_n\}$ such that $T - \beta I \notin \mathcal{BF}_l(X)$. Hence, $\beta \in \sigma_{\mathcal{BF}_l}(T)$ and $f(\beta) = \alpha$. Thus, $\alpha = f(\beta) \in f(\sigma_{\mathcal{BF}_l}(T))$. Consequently, $\sigma_{\mathcal{BF}_l}(f(T)) \subset \subset f(\sigma_{\mathcal{BF}_l}(T))$.

We do the same steps that we applied for the first equality and get $\sigma_{\mathcal{BF}_r}(f(T)) \subset f(\sigma_{\mathcal{BF}_r}(T))$.

References

1. M. Berkani, *On a class of quasi-Fredholm operators*, Int. Equat. Oper. Theory, **34**, 244 – 249 (1999).
2. M. Berkani, M. Sarih, *On semi B-Fredholm operators*, Glasgow Math. J., **43**, 457 – 465 (2001).
3. A. A. Boichuk, A. M. Samoilenko, *Generalized inverse operators and Fredholm boundary value problems*, 2nd ed., Inverse and Ill-Posed Probl. Ser., **59**, De Gruyter, Berlin (2016).
4. M. A. Kaashoek, *Ascent, descent, nullity and defect, a note on a paper by A. E. Taylor*, Math. Ann., **172**, 105 – 115 (1967).
5. J. P. Labrousse, *Les Opérateurs quasi-Fredholm: Une généralisation des opérateurs semi-Fredholm*, Rend. Circ. Mat. Palermo (2), **29**, 161 – 258 (1980).
6. V. Kordula, V. Muller, *On the axiomatic theory of the spectrum*, Stud. Math., **119**, № 2, 109 – 128 (1996).
7. M. Mbekhta, V. Muller, *On the axiomatic theory of the spectrum, II*, Stud. Math., **119**, № 2, 129 – 147 (1996).

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