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COEFFICIENT BOUNDS FOR MULTIVALENT CLASSES OF STARLIKE AND CONVEX FUNCTIONS DEFINED BY HIGHER-ORDER DERIVATIVES AND COMPLEX ORDER

КОЕФІЦІЄНТНІ ОЦІНКИ ДЛЯ БАГАТОЗНАЧНИХ КЛАСІВ ЗІРКОПОДІБНИХ ТА ОПУКЛИХ ФУНКЦІЙ, ВИЗНАЧЕНИХ ПОХІДНИМИ ВИЩОГО ПОРЯДКУ ТА КОМПЛЕКСНИМ ПОРЯДКОМ

We determine coefficient bounds for functions from subclasses of p-valent starlike and p-valent convex functions defined by higher-order derivatives and complex order introduced with the help of a certain nonhomogeneous Cauchy – Euler differential equation for higher-order derivatives. Relevant connections of some of our results with the results obtained earlier are provided.

Знайдено оцінки для коефіцієнтів функцій, що належать до підкласів p-значних зіркоподібних і p-значних опуклих функцій, які визначаються похідними вищого порядку та комплексним порядком і вводяться за допомогою певного неоднорідного диференціального рівняння Коші – Ейлера для похідних вищого порядку. Наведено відповідні співвідношення між деякими нашими результатами та результатами, що були отримані раніше.

1. Introduction. Denote by $\mathcal{A}_n(p)$ the class of multivalent analytic functions of the form

$$f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k, \qquad z \in \mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}, \quad p, n \in \mathbb{N} := \{1, 2, \ldots\}, \tag{1.1}$$

and let $\mathcal{A}(p) := \mathcal{A}_1(p)$, $\mathcal{A}(n) := \mathcal{A}_n(1)$, and $\mathcal{A} := \mathcal{A}_1(1)$.

Definition 1.1. *For $p > q$, $p \in \mathbb{N}$, $q \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $0 \le \gamma < p - q$, we say that the function $f \in \mathcal{A}_n(p)$ belongs to the class $\mathbb{S}^*_p(q, n, \gamma)$ of (p, q)-valent starlike functions of order γ, if it satisfies the inequality*

$$\operatorname{Re} \frac{z f^{(1+q)}(z)}{f^{(q)}(z)} > \gamma, \quad z \in \mathbb{U},$$

and belongs to the class $\mathbb{K}_p(q, n, \gamma)$ of (p, q)-valent convex functions of order γ, if it satisfies

$$\operatorname{Re} \left(1 + \frac{z f^{(2+q)}(z)}{f^{(1+q)}(z)} \right) > \gamma, \quad z \in \mathbb{U}.$$

The classes $\mathbb{S}^*_p(q, 1, \gamma) =: \mathbb{S}^*_p(q, \gamma)$ and $\mathbb{K}_p(q, 1, \gamma) =: \mathbb{K}^*_p(q, \gamma)$ were introduced and studied by Aouf [7 – 9], and note that $\mathbb{S}^*_p(0, \gamma) =: \mathbb{S}^*_p(\gamma)$ and $\mathbb{K}_p(0, \gamma) =: \mathbb{K}_p(\gamma)$ are, respectively, the classes of p-valent starlike and convex functions of order γ with $0 \le \gamma < p$ (see Owa [18] and Aouf [3, 5, 6]). Also, we mention that $\mathbb{S}^*_1(\gamma) =: \mathbb{S}^*(\gamma)$ and $\mathbb{K}_1(\gamma) =: \mathbb{K}(\gamma)$, with $0 \le \gamma < 1$, were introduced and studied by Robertson [21] (see also [24, 25]), if $\mathbb{S}^*_1(0, n, \gamma) =: \mathbb{S}^*_n(\gamma)$, and $\mathbb{K}_1(0, n, \gamma) =: \mathbb{C}_n(\gamma)$, if $0 \le \gamma < 1$, were introduced and studied by Srivastava et al. [26].

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Definition 1.2. *For* $b \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$, $0 \le \lambda \le 1$, $0 < \beta \le 1$, $p \in \mathbb{N}$, $q \in \mathbb{N}_0$, *and* $p > q$, *we say that* $f \in \mathcal{A}_n(p)$ *belongs to the class* $\mathbb{H}_n(p, q, \lambda, \beta, b)$ *if it satisfies*

$$\operatorname{Re}\left\{1 + \frac{1}{b}\left(\frac{1}{p-q}\frac{zf^{(1+q)}(z) + \lambda z^2 f^{(2+q)}(z)}{(1-\lambda)f^{(q)}(z) + \lambda z f^{(1+q)}(z)} - 1\right)\right\} > \beta, \quad z \in \mathbb{U}. \tag{1.2}$$

***Remark* 1.1.** For different values of $p,\ q,\ n,\ b, \lambda$ we obtain the following subclasses:

(i) $\mathbb{H}_n(p, q, 0, \beta, b) =: \mathbb{S}_n(p, q, \beta, b) =$

$$= \left\{f \in \mathcal{A}_n(p) : \operatorname{Re}\left[1 + \frac{1}{b}\left(\frac{1}{p-q}\frac{zf^{(1+q)}(z)}{f^{(q)}(z)} - 1\right)\right] > \beta,\ z \in \mathbb{U}\right\}.$$

(ii) $\mathbb{H}_n(p, q, 1, \beta, b) =: C_n(p, q, \beta, b) =$

$$= \left\{f \in \mathcal{A}_n(p) :\ \operatorname{Re}\left[1 + \frac{1}{b}\left(\frac{1}{p-q}\left(1 + \frac{zf^{(2+q)}(z)}{f^{(1+q)}(z)}\right) - 1\right)\right] > \beta,\ z \in \mathbb{U}\right\}.$$

(iii) $\mathbb{H}_n(p, 0, \lambda, \beta, b) =: \mathbb{G}_n(p, \lambda, \beta, b) =$

$$= \left\{f \in \mathcal{A}_n(p) :\ \operatorname{Re}\left[1 + \frac{1}{b}\left(\frac{1}{p}\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda z f'(z)} - 1\right)\right] > \beta,\ z \in \mathbb{U}\right\}.$$

Moreover, $\mathbb{G}_1(p, 0, \beta, b) =: \mathbb{S}_p(\gamma, b),\ \gamma = p\beta,\ 0 \le \beta < 1$ (see [13] with $A = 1$ and $B = -1$ and [11] with $m = 0$), that is,

$$\mathbb{S}_p(\gamma, b) = \left\{f \in \mathcal{A}(p) :\ \operatorname{Re}\left[p + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - p\right)\right] > \gamma,\ z \in \mathbb{U},\ 0 \le \gamma < p\right\},$$

and $\mathbb{G}_1(p, 1, \beta, b) =: C_p(\gamma, b),\ \gamma = p\beta,\ 0 \le \beta < 1$ $\Big($see [10] with $B = -1,\ A = 1 - \dfrac{2\gamma}{p},\ 0 \le \gamma < p$ and [11] with $m = 0\Big)$, that is,

$$C_p(\gamma, b) = \left\{f \in \mathcal{A}(p) :\ \operatorname{Re}\left[p + \frac{1}{b}\left(1 + \frac{zf''(z)}{f'(z)} - p\right)\right] > \gamma,\ z \in \mathbb{U},\ 0 \le \gamma < p\right\}.$$

(iv) $\mathbb{H}_n(1, 0, \lambda, \beta, b) =: \mathbb{SC}_n(\lambda, \beta, b)$ (see [1]), that is,

$$\mathbb{SC}_n(\lambda, \beta, b) = \left\{f \in \mathcal{A}(n) :\ \operatorname{Re}\left[1 + \frac{1}{b}\left(\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda z f'(z)} - 1\right)\right] > \beta,\ z \in \mathbb{U}\right\},$$

and, also, $\mathbb{H}_1(1, 0, \lambda, \beta, b) =: \mathbb{S}(\lambda, \beta, b),\ 0 \le \beta < 1$ (see [22]).

(v) $\mathbb{S}_1^*(p, q, \beta, \cos\alpha e^{-i\alpha}) =: \mathbb{S}_p^\alpha(q, \gamma),\ \gamma = (p-q)\beta,\ 0 \le \beta < 1$, that is,

$$\mathbb{S}_p^\alpha(q, \gamma) = \left\{f \in \mathcal{A}(p) :\ \operatorname{Re}\left[e^{i\alpha}\frac{zf^{(1+q)}(z)}{f^{(q)}(z)}\right] > \gamma\cos\alpha, |\alpha| < \frac{\pi}{2},\ z \in \mathbb{U},\ 0 \le \gamma < p - q\right\}.$$

Also, $\mathbb{S}_p^\alpha(0, \gamma) = \mathbb{S}_p^\alpha(\gamma)$ (see [4, 19, 23]).

(vi) $C_1(p, q, \beta, \cos\alpha e^{-i\alpha}) =: C_p^\alpha(q, \gamma),\ \gamma = (p-q)\beta,\ 0 \le \beta < 1$, that is,

$$C_p^\alpha(q,\gamma)=$$

$$=\left\{f\in\mathcal{A}(p)\colon\ \operatorname{Re}\left[e^{i\alpha}\left(1+\frac{zf^{(2+q)}(z)}{f^{(1+q)}(z)}\right)\right]>\gamma\cos\alpha,\ z\in\mathbb{U},\ |\alpha|<\frac{\pi}{2},\ 0\le\gamma<p-q\right\},$$

and $C_p^\alpha(0,\gamma)=:C_p^\alpha(\gamma)$ (see [10, 23]).

(vii) $\mathbb{H}_1(1,0,0,0,b)=:S(b)$ and $\mathbb{H}_1(1,0,1,0,b)=:C(b)$ (see [15–17]).

Definition 1.3. *A function $f\in\mathcal{A}_n(p)$ belongs to the class $\mathbb{B}_n(p,q,\lambda,\beta,b)$ if $w=f(z)$ satisfies the following nonhomogeneous Cauchy–Euler differential equation* (*see* [14])

$$z^2\frac{d^{(2+q)}w}{dz^{(2+q)}}+2(1+\mu)z\frac{d^{(1+q)}w}{dz^{(1+q)}}+\mu(1+\mu)\frac{d^{(q)}w}{dz^{(q)}}=(p-q+\mu)(p-q+\mu+1)\frac{d^{(q)}g}{dz^{(q)}},\qquad(1.3)$$

where $g\in\mathbb{H}_n(p,q,\lambda,\beta,b)$, $\mu\in\mathbb{R}$ with $\mu>q-p$, and $p\in\mathbb{N}$, $q\in\mathbb{N}_0$.

Note that $\mathbb{B}_1(1,0,\lambda,\beta,b)=:\mathcal{H}(\lambda,\beta,\mu,b)$ (see [1, 12]) and $\mathbb{B}_n(p,0,\lambda,\beta,\mu,b)=:\mathcal{G}_n(p,\lambda,\beta,\mu,b)$ (see [11] with $m=0$).

2. Coefficient estimates for the function class $\mathbb{H}_n(p,q,\lambda,\beta,b)$. Unless otherwise stated we assume that $b\in\mathbb{C}^*$, $0\le\lambda\le1$, $0<\beta\le1$, $p\in\mathbb{N}$, $q\in\mathbb{N}_0$, $p>q$, $\mu>q-p$, and $\mu\in\mathbb{R}$. Let Γ denotes the well-known *Euler integral of the second kind*, that is,

$$\Gamma(z):=\int_0^\infty t^{z-1}e^{-t}\,dt,$$

that converges absolutely on $\mathcal{D}:=\{z\in\mathbb{C}\colon\ \operatorname{Re}z>0\}$, therefore $\Gamma(1)=1$ and $\Gamma(m+1)=m!$ for all $m\in\mathbb{N}$.

Theorem 2.1. *Let the function $f\in\mathcal{A}_n(p)$ defined by* (1.1) *belongs to the class $\mathbb{H}_n(p,q,\lambda,\beta,b)$. Then*

$$|a_{p+n}|\le\frac{2[1+\lambda(p-q-1)]\delta(p,q+1)(1-\beta)|b|}{n[1+\lambda(p+n-q-1)]\delta(p+n,q)}$$

and

$$|a_k|\le\frac{2\Gamma(n)[1+\lambda(p-q-1)]\delta(p,q+1)(1-\beta)|b|}{\Gamma(k-p+1)[1+\lambda(k-q-1)]\delta(k,q)}\prod_{j=0}^{k-(p+n+1)}\big[n+j+2p(1-\beta)|b|\big],$$

$$k\ge p+n+1,$$

where $\delta(p,q):=\dfrac{p!}{(p-q)!}$, $p>q$.

Proof. For $f\in\mathcal{A}_n(p)$ given by (1.1) we define the function $F_{\lambda,p,q}$ by

$$F_{\lambda,p,q}(z):=\frac{(1-\lambda)f^{(q)}(z)+\lambda zf^{(1+q)}(z)}{\big[1+\lambda(p-q-1)\big]\delta(p,q)},\ z\in\mathbb{U},\qquad(2.1)$$

that is,

$$F_{\lambda,p,q}(z) = z^{p-q} + \sum_{k=n+p}^{\infty} A_{k,q} z^{k-q}, \quad z \in \mathbb{U},$$

where

$$A_{k,q} = \frac{\big[1+\lambda(k-q-1)\big]\delta(k,q)}{\big[1+\lambda(p-q-1)\big]\delta(p,q)} a_k, \quad k \ge n+p. \tag{2.2}$$

From (1.2) and (2.1) we have

$$\operatorname{Re}\left[1 + \frac{1}{b}\left(\frac{1}{p-q}\frac{zF'_{\lambda,p,q}(z)}{F_{\lambda,p,q}(z)} - 1\right)\right] > \beta, \quad z \in \mathbb{U}.$$

If we define the function g by

$$g(z) := \frac{1 + \dfrac{1}{b}\left(\dfrac{1}{p-q}\dfrac{zF'_{\lambda,p,q}(z)}{F_{\lambda,p,q}(z)} - 1\right) - \beta}{1-\beta}, \quad z \in \mathbb{U},$$

then g is analytic in $\mathbb{U}$ with $g(0) = 1$ and $\operatorname{Re} g(z) > 0,\ z \in \mathbb{U}$. Since the above relation is equivalent to

$$\frac{1}{b}\left(\frac{1}{p-q}\frac{zF'_{\lambda,p,q}(z)}{F_{\lambda,p,q}(z)} - 1\right) = (1-\beta)\big[g(z)-1\big], \quad z \in \mathbb{U},$$

and $F_{\lambda,p,q} \in \mathcal{A}_n(p-q)$, it follows that

$$g(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + \ldots, \quad z \in \mathbb{U}.$$

Therefore, we obtain

$$\frac{1}{b}\left(\frac{1}{p-q}\frac{zF'_{\lambda,p,q}(z)}{F_{\lambda,p,q}(z)} - 1\right) = (1-\beta)(c_n z^n + c_{n+1} z^{n+1} + \ldots), \quad z \in \mathbb{U},$$

or, equivalently,

$$zF'_{\lambda,p,q}(z) - (p-q)F_{\lambda,p,q}(z) = (p-q)b(1-\beta)(c_n z^n + c_{n+1} z^{n+1} + \ldots)F_{\lambda,p,q}(z), \quad z \in \mathbb{U}.$$

The last equality implies that

$$(k-p)A_{k,q} = (p-q)b(1-\beta)\big(c_{k-p} + c_{k-p-n}A_{p+n,q} + \ldots + c_n A_{k-n,q}\big),$$

and putting $k = p+n+r,\ r \in \mathbb{N}_0$, we have

$$(n+r)A_{p+n+r,q} = (p-q)b(1-\beta)\big(c_{n+r} + c_r A_{p+n,q} + \ldots + c_n A_{p+r,q}\big).$$

Applying the coefficient estimates $|c_k| \le 2,\ k \ge 1$, for the Carathéodory functions (see [20]), we obtain

$$|A_{p+n+r,q}| \le \frac{2(p-q)(1-\beta)|b|}{n+r}\big(1 + |A_{p+n,q}| + \ldots + |A_{p+r,q}|\big).$$

But, for $r=0,1,2$, we get

$$|A_{p+n,q}|\le\frac{2(p-q)(1-\beta)|b|}{n},$$

$$|A_{p+n+1,q}|\le\frac{2(p-q)(1-\beta)|b|}{n+1}\big(1+|A_{p+n,q}|\big)\le$$

$$\le\frac{2(p-q)(1-\beta)|b|\big[n+2(p-q)(1-\beta)|b|\big]}{n(n+1)}, \tag{2.3}$$

$$|A_{p+n+2,q}|\le\frac{2(p-q)(1-\beta)|b|}{n+2}\big(1+|A_{p+n,q}|+|A_{p+n+1,q}|\big)\le$$

$$\le\frac{2(p-q)(1-\beta)|b|\big[n+2(p-q)(1-\beta)|b|\big]\big[n+1+2(p-q)(1-\beta)|b|\big]}{n(n+1)(n+2)},$$

respectively. By mathematical induction we have

$$|A_{p+n+r,q}|\le\frac{2(p-q)(1-\beta)|b|}{n(n+1)\ldots(n+r)}\prod_{j=0}^{r-1}\big[n+j+2(p-q)(1-\beta)|b|\big]=$$

$$=\frac{2(p-q)(1-\beta)|b|\Gamma(n)}{\Gamma(n+r+1)}\prod_{j=0}^{r-1}\big[n+j+2(p-q)(1-\beta)|b|\big],\quad r\ge1. \tag{2.4}$$

Then, from (2.3), (2.4) and $k=p+n+r$, we get

$$|A_{k,q}|\le\frac{2(p-q)(1-\beta)|b|\Gamma(n)}{\Gamma(k-p+1)}\prod_{j=0}^{k-p-n-1}\big[n+j+2(p-q)(1-\beta)|b|\big],\quad k\ge p+n+1.$$

From (2.2) we obtain

$$a_k=\frac{[1+\lambda(p-q-1)]\delta(p,q)}{[1+\lambda(k-q-1)]\delta(k,q)}A_{k,q},\quad k\ge n+p,$$

so, we have

$$|a_{p+n}|\le\frac{2(1-\beta)|b|\big[1+\lambda(p-q-1)\big]\delta(p,q+1)}{n\big[1+\lambda(p+n-q-1)\big]\delta(p+n,q)},$$

$$|a_k|\le\frac{2\Gamma(n)(1-\beta)|b|\big[1+\lambda(p-q-1)\big]\delta(p,q+1)}{\Gamma(k-p+1)\big[1+\lambda(k-q-1)\big]\delta(k,q)}\prod_{j=0}^{k-(p+n+1)}\big[n+j+2(p-q)(1-\beta)|b|\big].$$

Theorem 2.1 is proved.

Putting $\lambda=0$ and $\lambda=1$ in Theorem 2.1, we have the next two results, respectively.

***Corollary* 2.1.** *If* $f\in\mathbb{S}_n(p,q,\beta,b)$ (*see Remark* 1.1 (i)), *then*

$$|a_{p+n}|\le\frac{2(1-\beta)|b|\delta(p,q+1)}{n\delta(p+n,q)}$$

and

$$|a_k| \le \frac{2(1-\beta)|b|\Gamma(n)\delta(p,q+1)}{\Gamma(k-p+1)\delta(k,q)} \prod_{j=0}^{k-(p+n+1)} \big[n+j+2(p-q)(1-\beta)|b|\big],$$

$$k \ge p+n+1.$$

***Corollary* 2.2.** *If* $f \in C_n(p,q,\beta,b)$ (*see Remark* 1.1 (ii)), *then*

$$|a_{p+n}| \le \frac{2(1-\beta)|b|(p-q)\delta(p,q+1)}{n\delta(p+n+1,q)}$$

and

$$|a_k| \le \frac{2(1-\beta)|b|\Gamma(n)(p-q)\delta(p,q+1)}{\Gamma(k-p+1)\delta(k,q+1)} \prod_{j=0}^{k-(p+n+1)} \big[n+j+2(p-q)(1-\beta)|b|\big],$$

$$k \ge p+n+1.$$

Considering $\beta = q = 0$ in Corollaries 2.1 and 2.2, we obtain the next two special cases, respectively.

***Example* 2.1.** If $f \in \mathbb{S}_n(p,0,0,b)$, then $|a_{p+n}| \le \dfrac{2p|b|}{n}$ and

$$|a_k| \le \frac{2p|b|\Gamma(n)}{\Gamma(k-p+1)} \prod_{j=0}^{k-(p+n+1)} \big[n+j+2p|b|\big], \quad k \ge p+n+1.$$

***Example* 2.2.** If $f \in C_n(p,0,0,b)$, then $|a_{p+n}| \le \dfrac{2p^2|b|}{n(p+n)}$ and

$$|a_k| \le \frac{2p^2|b|\Gamma(n)}{k\Gamma(k-p+1)} \prod_{j=0}^{k-(p+n+1)} \big[n+j+2p|b|\big], \quad k \ge p+n+1.$$

Putting $n = 1$ in Example 2.1, we get the following example.

***Example* 2.3.** If $f \in \mathbb{S}_1(p,0,0,b)$, then

$$|a_{p+1}| \le 2p|b|$$

and

$$|a_k| \le \frac{2p|b|}{\Gamma(k-p+1)} \prod_{j=0}^{k-(p+2)} \big[j+1+2p|b|\big], \quad k \ge p+2.$$

Taking $n = 1$ in Example 2.2, we have (see [2], Corollary 1 and Theorem 4, and [10], Corollary 1 and Theorem 3 with $A = 1$ and $B = -1$) the following example.

***Example* 2.4.** If $f \in C_1(p,0,0,b)$, then

$$|a_{p+1}| \leq \frac{2p^2|b|}{p+1}$$

and

$$|a_k| \leq \frac{2p^2|b|}{k\Gamma(k-p+1)} \prod_{j=0}^{k-(p+2)} \big[j+1+2p|b|\big], \quad k \geq p+2.$$

Putting $p = n = 1$ and $q = 0$ in Theorem 2.1, we have the next corollary (see [1], Theorem 1, and Deng [12], Theorem 1 with $n = 0$).

***Corollary* 2.3.** *If* $f \in \mathbb{S}(\lambda,\beta,b)$ (*see Remark* 1.1 (iv)), *then*

$$|a_k| \leq \frac{2(1-\beta)|b|}{\Gamma(k)[1+\lambda(k-1)]} \prod_{j=0}^{k-3} \big[j+1+2(1-\beta)|b|\big] =$$

$$= \frac{1}{(k-1)![1+\lambda(k-1)]} \prod_{j=0}^{k-2} \big[j+2(1-\beta)|b|\big], \quad k \geq 2.$$

***Remark* 2.1.** Putting $\lambda = 0$ in Corollary 2.3, we get the result obtained by Deng [12] (Corollary 2 with $n = 0$).

If we take $\beta = 0$ in Corollary 2.3 we get the next result (see also [1], Corollary 1).

***Example* 2.5.** If $f \in \mathbb{S}(\lambda,0,b)$, then

$$|a_k| \leq \frac{1}{(k-1)![1+\lambda(k-1)]} \prod_{j=0}^{k-2} \big[j+2|b|\big], \quad k \geq 2.$$

Putting $\lambda = 0$ in Example 2.5, we obtain the result of [17] (Theorems 2 and 3).

***Example* 2.6.** If $f \in \mathbb{S}(0,0,b)$, then

$$|a_k| \leq \frac{1}{(k-1)!} \prod_{j=0}^{k-2} \big[j+2|b|\big], \quad k \geq 2.$$

***Remark* 2.2.** For the special case $\lambda = 1$, Corollary 2.3 reduces to the result of [15] (Theorem 2).

3. Coefficient bounds for the function class $\mathbb{B}_n(p,q,\lambda,\beta,\mu,b)$.

Theorem 3.1. *Let the function* $f \in \mathcal{A}_n(p)$ *defined by* (1.1) *belongs to the class* $\mathbb{B}_n(p,q,\lambda,\beta,\mu,b)$. *Then*

$$|a_{p+n}| \leq \frac{2(p-q+\mu)(p-q+\mu+1)\big[1+\lambda(p-q-1)\big]\delta(p,q+1)(1-\beta)|b|}{n(p+n-q+\mu)(p+n-q+\mu+1)\big[1+\lambda(p+n-q-1)\big]\delta(p+n,q)}$$

and

$$|a_k| \leq \frac{2\Gamma(n)(p-q+\mu)(p-q+\mu+1)[1+\lambda(p-q-1)]\delta(p,q+1)(1-\beta)|b|}{(k-q+\mu)(k-q+\mu+1)\Gamma(k-p+1)[1+\lambda(k-q-1)]\delta(k,q)} \times$$

$$\times \prod_{j=0}^{k-(p+n+1)} \big[n+j+2p(1-\beta)|b|\big],$$

where $k \geq p+n+1$ *and* $\mu \in \mathbb{R}$ *with* $\mu > q-p$.

Proof. For $f \in \mathcal{A}_n(p)$ given by (1.1), since $f \in \mathbb{B}_n(p,q,\lambda,\beta,\mu,b)$, there exists a function $g \in \mathbb{H}_n(p,q,\lambda,\beta,b)$ of the form

$$g(z) = z^p + \sum_{k=p+n}^{\infty} b_k z^k, \quad z \in \mathbb{U},$$

that satisfies the Cauchy – Euler differential equation (1.3). Equating the coefficients of both sides of this differential equation it follows that

$$b_k = \frac{(k-q+\mu)(k-q+\mu+1)}{(p-q+\mu)(p-q+\mu+1)}\, a_k, \qquad k \geq p+n, \quad \mu > q-p.$$

Thus, by using the conclusions of Theorem 2.1 for the function g we obtain the required inequalities.

Theorem 3.1 is proved.

Putting $q=0$ in Theorem 3.1, we get the next special case (see [11], Theorem 3.1 with $m=0$).

***Corollary* 3.1.** *If* $f \in \mathcal{G}_n(p,\lambda,\beta,\mu,b) := \mathbb{B}_n(p,0,\lambda,\beta,\mu,b)$, *then*

$$|a_{p+n}| \leq \frac{2(p+\mu)(p+\mu+1)[1+\lambda(p-1)](1-\beta)|b|}{n(p+n+\mu)(p+n+\mu+1)\big[1+\lambda(p+n-1)\big]}$$

and

$$|a_k| \leq \frac{2\Gamma(n)(p+\mu)(p+\mu+1)\big[1+\lambda(p-1)\big](1-\beta)|b|}{(k+\mu)(k+\mu+1)\Gamma(k-p+1)\big[1+\lambda(k-1)\big]} \prod_{j=0}^{k-(p+n+1)} \big[n+j+2p(1-\beta)|b|\big],$$

where $k \geq p+n+1$ *and* $\mu \in \mathbb{R}$ *with* $\mu > q-p$.

Putting $p=n=1$ in the last corollary, we have the following result (see also [1], Theorem 2, [26], Corollary 4, and [12], Theorem 2 with $n=0$).

***Corollary* 3.2.** *If* $f \in \mathcal{H}(\lambda,\beta,\mu,b) := \mathbb{B}_1(1,0,\lambda,\beta,b)$, *then*

$$|a_k| \leq \frac{(1+\mu)(2+\mu)\prod_{j=0}^{k-2}\big[j+2(1-\beta)|b|\big]}{(k+\mu)(k+\mu+1)[1+\lambda(k-1)](k-1)!}, \qquad k \geq 2, \quad \mu > -1.$$

References

1. O. Altintas, H. Irmak, S. Owa, H. M. Srivastava, *Coefficient bounds for some families of starlike and convex functions of complex order*, Appl. Math. Lett., **20**, № 12, 1218 – 1222 (2007).
2. M. K. Aouf, *p-Valent classes related to convex functions of complex order*, Rocky Mountain J. Math., **15**, № 4, 853 – 863 (1985).
3. M. K. Aouf, *On a class of p-valent starlike functions of order α*, Int. J. Math. and Math. Sci., **10**, № 4, 733 – 744 (1987).
4. M. K. Aouf, *On coefficient bounds of a certain class of p-valent λ-spiral functions of order α*, Int. J. Math. and Math. Sci., **10**, № 2, 259 – 266 (1987).
5. M. K. Aouf, *A generalization of multivalent functions with negative coefficients*, J. Korean Math. Soc., **25**, № 1, 53 – 66 (1988).
6. M. K. Aouf, H. M. Hossen, H. M. Srivastava, *Some families of multivalent functions*, Comput. Math. Appl., **39**, № 7-8, 39 – 48 (2000).

7. M. K. Aouf, *Some families of p-valent functions with negative coefficients*, Acta Math. Univ. Comenian. (N.S.), **78**, № 1, 121 – 135 (2009).
8. M. K. Aouf, *On certain multivalent functions with negative coefficients defined by using a differential operator*, Mat. Vesnik, **62**, № 1, 23 – 35 (2010).
9. M. K. Aouf, *Bounded p-valent Robertson functions defined by using a differential operator*, J. Franklin Inst., **347**, № 10, 1927 – 1941 (2010).
10. M. K. Aouf, H. E. Darwish, A. E. Alhosseny, *A generalization of p-valent classes related to convex functions*, Demonstr. Math., **33**, № 3, 467 – 479 (2000).
11. S. Bulut, *The generalization of the generalized Al-Oboudi differential operator*, Appl. Math. and Comput., **215**, № 4, 1448 – 1455 (2009).
12. Q. Deng, *Certain subclass of analytic functions with complex order*, Appl. Math. and Comput., **208**, № 2, 359 – 362 (2009).
13. L. Dileep, S. Latha, *On p-valent functions of complex order*, Demonstr. Math., **45**, № 3, 541 – 547 (2012).
14. R. M. El-Ashwah, M. K. Aouf, S. M. El-Deeb, *Inclusion and neighborhood properties of certain subclasses of p-valent functions of complex order defined by convolution*, Ann. Univ. Mariae Curie-Skłodowska Sect. A, **65**, № 1, 33 – 48 (2011).
15. M. A. Nasr, M. K. Aouf, *On convex functions of complex order*, Bull. Fac. Sci. Mansoura Univ., **9**, 565 – 582 (1982).
16. M. A. Nasr, M. K. Aouf, *Bounded starlike functions of complex order*, Proc. Indian Acad. Sci. Math. Sci., **92**, № 2, 97 – 102 (1983).
17. M. A. Nasr, M. K. Aouf, *Starlike function of complex order*, J. Natur. Sci. Math., **25**, № 1, 1 – 12 (1985).
18. S. Owa, *On certain classes of p-valent functions with negative coefficients*, Bull. Belg. Math. Soc. Simon Stevin, **59**, 385 – 402 (1985).
19. D. A. Patil, N. K. Thakare, *On coefficient bounds of p-valent λ-spiral functions of order α*, Indian J. Pure and Appl. Math., **10**, № 7, 842 – 853 (1979).
20. Ch. Pommerenke, *On univalent functions, Bloch functions and VMOA*, Math. Ann., **26**, № 3, 199 – 208 (1978).
21. M. S. Robertson, *On the theory of univalent functions*, Ann. Math., **37**, 374 – 408 (1936).
22. H. M. Srivastava, O. Altintas, S. K. Serenbay, *Coefficient bounds for certain subclasses of starlike functions of complex order*, Appl. Math. Lett., **24**, № 8, 1359 – 1363 (2011).
23. H. M. Srivastava, M. K. Aouf, S. Owa, *Certain classes of multivalent functions of order α and type β*, Bull. Soc. Math. Belg., Tijdschr. Belg. Wisk. Gen., **42**, Ser B, № 1, 31 – 66 (1990).
24. H. M. Srivastava, S. Owa (editors), *Univalent functions, fractional calculus, and their applications*, Halsted Press (Ellis Harwood Limited, Chichester), John Wiley and Sons, New York (1989).
25. H. M. Srivastava, S. Owa (editors), *Current topics in analytic function theory*, World Sci. Publ. Co., Singapore etc. (1992).
26. H. M. Srivastava, S. Owa, S. K. Chatterjea, *A note on certain classes of starlike functions*, Rend. Semin. Mat. Univ. Padova, **77**, 115 – 124 (1987).

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