DOI: 10.37863/umzh.v74i11.6523

UDC 517.5

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ON A CLASS OF ANALYTIC FUNCTIONS CLOSELY RELATED TO A CLASS DEFINED BY SILVERMAN AND SILVIA[2]

ПРО КЛАС АНАЛІТИЧНИХ ФУНКЦІЙ, ТІСНО ПОВ'ЯЗАНИЙ ІЗ КЛАСОМ, ВИЗНАЧЕНИМ СІЛЬВЕРМАНОМ І СІЛЬВІА

We define and study a class of analytic functions in the unit disc by using the modification of the well-known Silverman and Silvia's analytic formula for starlike functions with respect to a boundary point. The representation theorem, as well as growth and distortion theorems are established for the new class of functions. Further, early coefficients of the new class of functions are also estimated.

Визначено та вивчено клас аналітичних функцій в одиничному крузі шляхом модифікації відомої аналітичної формули Сільвермана та Сільвіа для зіркоподібних функцій щодо граничної точки. Для нового класу функцій доведено теорему зображення, а також теореми зростання та спотворення. Крім того, оцінено ранні коефіцієнти для нового класу функцій.

1. Introduction. Let $\mathcal{H}$ be the class of all holomorphic functions in the open unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. By $\mathcal{A}$ we denote the subclass of $\mathcal{H}$ consisting of functions h that are normalized by $h(0) = 0$ and $h'(0) = 1$. Hence, the functions belonging to $\mathcal{A}$ will be of the form

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D},$$

and, by $\mathcal{S}$, the subclass of $\mathcal{A}$ consisting functions that are univalent in $\mathbb{D}$. A function $f \in \mathcal{H}$ is subordinate to another function $g \in \mathcal{H}$ if there exists a function $\omega \in \mathcal{H}$ such that $\omega(0) = 0$, $\omega(\mathbb{D}) \subset \mathbb{D}$ and $f(z) = g(\omega(z))$ for every $z \in \mathbb{D}$. We write this subordination as $f \prec g$. Further, if g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$. Let

$$\mathcal{S}^*(\beta) = \left\{ h \in \mathcal{A} : \Re\left(\frac{zh'(z)}{h(z)} \right) > \beta, \ 0 \le \beta < 1, \ z \in \mathbb{D} \right\}$$

and

$$\mathcal{C}(\beta) = \left\{ h \in \mathcal{A} : \Re\left(1 + \frac{zh''(z)}{h'(z)} \right) > \beta, \ 0 \le \beta < 1, \ z \in \mathbb{D} \right\}$$

respectively denote the well-known classes of starlike functions of order β and convex functions of order β. Note that $\mathcal{S}^* = \mathcal{S}^*(0)$ is called the class of starlike functions (with respect to the origin). The classes $\mathcal{S}^*(\beta)$ and $\mathcal{C}(\beta)$ were introduced by Robertson [**?**]. In view of Alexander's relation, it is

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[2] The work of the first author was supported by minor research grant from SDNB Vaishnav College for Women, Chennai while the work of the second author was supported by UKM Grant FRGS/1/2019/STG06/UKM/01/1. The work of the third author was supported by a grant from the Science and Engineering Research Board, Government of India under Mathematical Research Impact Centric Support of Department of Science and Technology (DST) (vide ref: MTR/2017/000607).

a known fact that $h \in \mathcal{C}(\beta)$ if and only if $zh'(z) \in \mathcal{S}^*(\beta)$ for $0 \leq \beta < 1$. If h is an analytic function in the $\mathbb{D}$, we say that h is close-to-convex in $\mathbb{D}$ if and only if there exists a convex function Φ such that $\Re\left(\dfrac{h'(z)}{\Phi'(z)}\right) > 0,\ z \in \mathbb{D}$. We note that the class of normalized close-to-convex functions was introduced by Kaplan [**?**]. Further, let us denote by $\mathcal{P}$ the class of functions p holomorphic in $\mathbb{D}$ with $p(0) = 1$ and $\Re(p(z)) > 0$ for $z \in \mathbb{D}$. This class $\mathcal{P}$ of functions with positive real part is known as the class of Carathéodory functions. Robertson [**?**] introduced a class $\mathcal{G}^*$ of functions $G(z)$ analytic in $\mathbb{D}$, normalized by $G(0) = 1,\ G(1) = \lim_{r\to 1^-} G(r) = 0$ and for some real $\alpha,\ \Re\big(e^{i\alpha}G(z)\big) > 0$, $z \in \mathbb{D}$. This can be considered as a breakthrough since not many work were initiated in this direction. In addition, $G(z)$ maps $\mathbb{D}$ univalently onto a domain starlike with respect to $G(1)$. Let the constant function 1 also belongs to the class $\mathcal{G}^*$. He conjectured that the class $\mathcal{G}$ of functions g of the form

$$g(z) = 1 + \sum_{n=1}^{\infty} d_n z^n, \tag{1.1}$$

holomorphic and nonvanishing in $\mathbb{D}$, such that

$$\Re\left\{\frac{2zg'(z)}{g(z)} + \frac{1+z}{1-z}\right\} > 0, \quad z \in \mathbb{D}, \tag{1.2}$$

coincides with $\mathcal{G}^*$. The above hypothesis was confirmed by Lyzzaik [**?**]. Robertson [**?**] also showed, if $g \in \mathcal{G}$ and $g \neq 1$, then g is close-to-convex and univalent in $\mathbb{D}$. It is worth to be mentioned here that the analytic condition f was known to Styer [**?**] earlier. An association of the class $\mathcal{G}$ with a functional $\dfrac{f(z)}{1-z}$ was considered by Todorov [**?**]. In the same article, Todorov [**?**] obtained a structured formula and coefficient estimates for the class $\dfrac{f(z)}{1-z}$. Silverman and Silvia [**?**] introduced the class $\mathcal{G}_\alpha$ consisting of functions g of the form (**??**), holomorphic and nonvanishing in $\mathbb{D}$, and such that, for each $0 \leq \alpha < 1$,

$$\Re\left(\frac{zg'(z)}{g(z)} + (1-\alpha)\frac{1+z}{1-z}\right) > 0, \quad z \in \mathbb{D}. \tag{1.3}$$

Silverman and Silvia [**?**] observed that for each $\alpha \in [0,1)$ the class $\mathcal{G}_\alpha$ is the subclass of $\mathcal{G}^*$. Note that, for the choice of $\alpha = \dfrac{1}{2}$, $\mathcal{G}_\alpha = \mathcal{G}$. They also gave a full description of the class of univalent functions on $\mathbb{D}$, the image of which is star-shaped, with respect to a boundary point. Few properties and some inequalities relating to the functional coefficients associated with the class $\mathcal{G}$ were obtained in [**?**]. In another notable work, Obradovic and Owa [**?**] investigated the class $\mathcal{G}(\alpha),\ 0 \leq \alpha < 1$, of functions g of the form (**??**), holomorphic in the disc $\mathbb{D}$, $g(z) \neq 0$ for $z \in \mathbb{D}$ and satisfying the condition $\Re\left\{\dfrac{zg'(z)}{g(z)} + (1-\alpha)\dfrac{1+z}{1-z}\right\} > 0,\ z \in \mathbb{D}$. However, the works in [**?**, **?**] were completely independent. In [**?**], a class closely related to $\mathcal{G}$ denoted by $\mathcal{G}(M),\ M > 1$, of functions g of the form (**??**), holomorphic and nonvanishing in $\mathbb{D}$ was introduced and such that

$$\Re\left\{\frac{2zg'(z)}{g(z)} + z\frac{P'(z;M)}{P(z;M)}\right\} > 0, \quad z \in \mathbb{D},$$

where $P(z;M)$ denotes the Pick function. The class

$$\mathcal{G}(1) = \left\{ g \ \text{ of the form } (\textbf{??})\colon g(z) \neq 0 \ \text{ and } \Re\left\{ 2z\frac{g'(z)}{g(z)} + 1 \right\} > 0, \ \ z \in \mathbb{D} \right\}$$

was also considered in [**?**]. Since then, this class of starlike functions with respect to a boundary point has gained a few notable interest among geometric function theorist and also other researchers. The distortion results for starlike functions with respect to a boundary point were obtained in [**?**, **?**]. The dynamical characterizations of functions starlike with respect to a boundary point can be seen in [**?**]. Lecko [**?**] gave another representation of starlike functions with respect to a boundary point and Lecko and Lyzzaik [**?**] obtained different characterizations of the class $\mathcal{G}$. In [**?**], Jakubowski and Włodarczyk defined the class $\mathcal{G}(A, B)$ for $-1 < A \leq 1$ and $-A < B \leq 1$ for all $g \in \mathcal{H}$ of the form (**??**) such that

$$\Re\left(\frac{2zg'(z)}{g(z)} + J(z; A, B) \right) > 0, \quad z \in \mathbb{D},$$

where

$$J(z; A, B) := \frac{1 + Az}{1 - Bz}, \quad z \in \mathbb{D}. \tag{1.4}$$

Aharonov et al. [**?**] gave a comprehensive definition for spiral-shaped domains with respect to a boundary point which is defined as follows: Let $\mathcal{G}_\mu$ denote the class of functions $f \in H(\mathbb{D})$, non vanishing in $\mathbb{D}$ with $g(0) = 1$, and for $\mu \in \mathbb{C}$, $\left| \dfrac{\mu}{\pi} - 1 \right| \leq 1$ satisfying

$$\Re\left\{ \frac{2\pi zg'(z)}{\mu g(z)} + \frac{1 + z}{1 - z} \right\} > 0, \quad z \in \mathbb{D}.$$

Basic properties and a number of equivalent characterizations of the class $\mathcal{G}_\mu$ are formulated in [**?**] (see also [**?**]). For the choice $\mu = \pi$, the class $\mathcal{G}_\mu$ coincides with the class introduced by Robertson [**?**] who has generated interest on this class, and its associated classes (see the interesting works of Elin and Shoikhet [**?**] and a book by the same authors [**?**]). It is also known that functions in $\mathcal{G}_\mu$ are either close-to-convex or just the constant 1. A recent work on this direction was considered in [**?**]. The purpose of this paper is to introduce and investigate a new class of functions that are closely related to the class considered by Silverman and Silvia [**?**]. Representation theorem, growth and distortion theorems are established for the new class of functions. Further, early coefficients of the new class of functions are also estimated.

Definition 1.1. *For $0 \leq \alpha < 1$, $-1 < A \leq 1$, $-A < B \leq 1$, let $\mathcal{G}(\alpha, A, B)$ denote the class of functions g of the form* (**??**)*, holomorphic and nonvanishing in $\mathbb{D}$ and such that*

$$\Re\left\{ \frac{zg'(z)}{g(z)} + (1 - \alpha)\frac{1 + Az}{1 - Bz} \right\} > 0, \quad z \in \mathbb{D}. \tag{1.5}$$

The above analytic criteria can be put in the form as

$$\Re\left\{ \frac{zg'(z)}{g(z)} + (1 - \alpha)J(z; A, B) \right\} > 0, \quad z \in \mathbb{D}, \tag{1.6}$$

with $J(z; A, B) = \dfrac{1 + Az}{1 - Bz}$, $z \in \mathbb{D}$.

We note that the class $\mathcal{G}\left(\frac{1}{2},1,1\right)$ is identical to the known class $\mathcal{G}$ and $\mathcal{G}(\alpha,1,1)$ is identical to the class $\mathcal{G}_\alpha$. If $B=-A$ then the function $J(z;A,-A)\equiv 1$ and therefore again we have the class $G(1)$. It is worth reminding at this place, that the function (**??**) was used in many articles, where different classes generated by the appropriate Carathéodory functions were considered. It is to be observed that the function J of the form (**??**), maps conformally the disc $\mathbb{D}$ onto a disc situated on the right in the half-plane, for $B<1$. However if $B=1,\ J(\mathbb{D};A,B)$ is the half-plane $\left\{w:\Re(w)>\frac{1-A}{2}\right\}$, where $0\le\frac{1-A}{2}<1$.

Let us first construct few examples for the new class of functions to show that the class is non-empty.

***Example* 1.1.** The functions

$$g_0(z;\alpha,A,B)=\begin{cases}(1-Bz)^{\frac{(A+B)(1-\alpha)}{B}}, & z\in\mathbb{D}, \quad \text{for} \quad B\ne 0,\\ \exp\big(-(1-\alpha)Az\big), & z\in\mathbb{D}, \quad \text{for} \quad B=0,\end{cases}$$

and

$$g_1(z;\alpha,A,B)=\begin{cases}\dfrac{(1-Bz)^{\frac{(A+B)(1-\alpha)}{B}}}{1-z}, & z\in\mathbb{D}, \quad \text{for} \quad B\ne 0,\\ \dfrac{\exp\big(-(1-\alpha)Az\big)}{1-z}, & z\in\mathbb{D}, \quad \text{for} \quad B=0,\end{cases}$$

belong to the class $\mathcal{G}(\alpha,A,B)$.

A straight forward computations will show that both the functions $g_0(z;\alpha,A,B)$ and $g_1(z;\alpha,A,B)$ belong to the class $\mathcal{G}(\alpha,A,B)$. Furthermore, for $-1<A\le 1,\ -A<B\le 1$, we have, for $z\in\mathbb{D}$,

$$g_1(z;\alpha,A,B)=1+\left(\alpha-\frac{(1-\alpha)A}{B}\pi\right)z+$$
$$+\left(\frac{\big((\alpha-1)A+\alpha B\big)\big((\alpha-1)A+(\alpha+1)B\big)}{2B^2}\right)z^2+\ldots.$$

2. Main results. We start this section with the following theorem.

Theorem 2.1. *Let g be a holomorphic function in $\mathbb{D}$ with $g(0)=1$. Then $g\in\mathcal{G}(\alpha,A,B)$ if and only if there exists a function $h\in\mathcal{S}^*(\alpha)$ such that*

$$g(z)=\left(\frac{h(z)}{z}\right)(1-Bz)^{\frac{(A+B)(1-\alpha)}{B}}, \quad z\in\mathbb{D}, \quad \textit{for} \quad B\ne 0, \tag{2.1}$$

$$g(z)=\left(\frac{h(z)}{z}\right)\exp\big(-(1-\alpha)Az\big), \quad z\in\mathbb{D}, \quad \textit{for} \quad B=0. \tag{2.2}$$

Proof. First, we will prove the theorem for the case $B\ne 0$. Suppose that $h\in\mathcal{S}^*(\alpha)$ and $g(z)$ be given as in (**??**). Then g is analytic in $\mathbb{D}$ and $g(0)=1$. We have

$$\frac{zg'(z)}{g(z)}+(1-\alpha)\frac{1+Az}{1-Bz}=\frac{zh'(z)}{h(z)}-1-\frac{(A+B)(1-\alpha)z}{1-Bz}+(1-\alpha)\frac{1+Az}{1-Bz}=\frac{zh'(z)}{h(z)}-\alpha.$$

Therefore,

$$\Re\left(\frac{zg'(z)}{g(z)}+(1-\alpha)\frac{1+Az}{1-Bz}\right)>0.$$

Conversely, suppose that $g\in\mathcal{G}(\alpha,A,B)$ and $h(z)=\dfrac{zg(z)}{(1-Bz)^{\frac{(A+B)(1-\alpha)}{B}}}$. Then $h(0)=0,\ h'(0)=$ $=1$. Further,

$$\frac{zh'(z)}{h(z)}=\alpha+\frac{zg'(z)}{g(z)}+(1-\alpha)\frac{1+Az}{1-Bz}.$$

Hence, $h\in\mathcal{S}^*(\alpha)$. For the case $B=0$, the theorem can be proved in similar lines and hence the proof is omitted.

Since, for the choices of $A=1$ and $B=1$, $\mathcal{G}(\alpha,A,B)$ reduces to $\mathcal{G}_\alpha$ it is worthy to investigate the relation between $\mathcal{G}(\alpha,A,B)$ and $\mathcal{G}_\alpha$, which is given in the remark below.

***Remark* 2.1.** Let us consider the function $g_3,\ g_3(0)=1$, satisfying the equation

$$\frac{zg_3'(z)}{g_3(z)}+(1-\alpha)\frac{1+Az}{1-Bz}=\frac{1+z^2}{1-z^2},\quad z\in\mathbb{D}.$$

In view of (**??**) and (**??**) it is obvious that $g_3\in\mathcal{G}(\alpha,A,B)$. Further, it is easy to see that if $B<1$ then there exists a point $z_0\in\mathbb{D}$ such that $g_3'(z)=0$. In other words, g_3 is not a univalent function in $\mathbb{D}$. Therefore, $g_3\notin\mathcal{G}_\alpha$ and hence $g_3\notin\mathcal{G}$.

***Remark* 2.2.** It follows immediately from the Herglotz representation that for $\mathcal{S}^*(\alpha)$ that $g\in$ $\in\mathcal{G}(\alpha,A,B)$ if and only if

$$g(z)=(1-Bz)^{\frac{(A+B)(1-\alpha)}{B}}\exp\left(-2(1-\alpha)\int\limits_{-\pi}^{\pi}\log\left(\frac{1}{1-ze^{-it}}\right)d\mu(t)\right),\quad z\in\mathbb{D},\quad\text{for}\quad B\neq 0,$$

$$g(z)=\exp(-(1-\alpha)Az)\exp\left(-2(1-\alpha)\int\limits_{-\pi}^{\pi}\log\left(\frac{1}{1-ze^{-it}}\right)d\mu(t)\right),\quad z\in\mathbb{D},\quad\text{for}\quad B=0.$$

where $\mu(t)$ is a probability measure on $[-\pi,\pi]$.

For the choice of $\alpha=\dfrac{1}{2}$, we have the following theorem as a consequence of Theorem **??**.

Theorem 2.2. *Let g be a holomorphic function in $\mathbb{D}$ such that $g(0)=1$. Then $g\in\mathcal{G}(\alpha,A,B)$ if and only if there exists a function $f\in\mathcal{S}^*\left(\dfrac{1}{2}\right)$ such that*

$$g(z)=\frac{f(z)}{z}(1-Bz)^{\frac{A+B}{2B}},\quad z\in\mathbb{D},\quad \textit{for}\quad B\neq 0, \tag{2.3}$$

$$g(z)=\frac{f(z)}{z}\exp\left(-\frac{Az}{2}\right),\quad z\in\mathbb{D},\quad \textit{for}\quad B=0. \tag{2.4}$$

***Remark* 2.3.** It follows immediately from the Herglotz representation that for $\mathcal{S}^*\left(\frac{1}{2}\right)$, $g \in$ $\in \mathcal{G}(\alpha, A, B)$ if and only if

$$g(z) = (1 - Bz)^{\frac{A+B}{2B}} \exp\left(-\int_{-\pi}^{\pi} \log\left(\frac{1}{1 - ze^{-it}}\right) d\mu(t)\right), \quad z \in \mathbb{D}, \quad \text{for} \quad B \neq 0,$$

$$g(z) = \exp\left(-\frac{Az}{2}\right)\left(-\int_{-\pi}^{\pi} \log\left(\frac{1}{1 - ze^{-it}}\right) d\mu(t)\right), \quad z \in \mathbb{D}, \quad \text{for} \quad B = 0,$$

where $\mu(t)$ is a probability measure on $[-\pi, \pi]$.

Let $0 \leq \alpha_1 \leq \alpha_2 < 1$ and $f \in \mathcal{G}(\alpha_2, A, B)$. Then

$$\Re\left(\frac{zg'(z)}{g(z)} + (1 - \alpha_1)\frac{1 + Az}{1 - Bz}\right) = \Re\left(\frac{zg'(z)}{g(z)} + (1 - \alpha_2)\frac{1 + Az}{1 - Bz}\right) + (\alpha_2 - \alpha_1)\Re\left(\frac{1 + Az}{1 - Bz}\right) > 0$$

as $\Re\left(\frac{1 + Az}{1 - Bz}\right) > 0$ for $-1 < A \leq 1$, $-A < B \leq 1$.

Hence, we have the following theorem.

Theorem 2.3. *Let $-1 < A \leq 1$, $-A < B \leq 1$ and $0 \leq \alpha_1 \leq \alpha_2 < 1$. Then*

$$\mathcal{G}(\alpha_2, A, B) \subseteq \mathcal{G}(\alpha_1, A, B).$$

From Theorem **??** and from the known estimates of the respective functionals in the class $\mathcal{S}^*$ we have the following theorem.

Theorem 2.4. *If $g \in \mathcal{G}(\alpha, A, B)$, $-1 < A \leq 1$, $-A < B \leq 1$, $B \neq 0$, then the sharp estimates*

$$\frac{1}{1 + |z|}\left|(1 - Bz)^{\frac{(A+B)(1-\alpha)}{B}}\right| \leq |g(z)| \leq \frac{1}{1 - |z|}\left|(1 - Bz)^{\frac{(A+B)(1-\alpha)}{B}}\right|, \quad |z| = r < 1, \tag{2.5}$$

hold. The upper estimate for (**??**) *is attained for the function g of the form*

$$g(z) = (1 - Bz)^{\frac{(A+B)(1-\alpha)}{B}} \sqrt{\frac{k(z)}{z}},$$

where $k(z) = \frac{z}{(1 - z)^2}$. The upper bound is achieved at $z = r$ and the lower at $z = -r$.

If $g \in \mathcal{G}(\alpha, A, 0)$, then

$$\frac{1}{1 + |z|} \exp\big(-(1 - \alpha)A\Re(z)\big) \leq |g(z)| \leq \frac{1}{1 - |z|} \exp\big(-(1 - \alpha)A\Re(z)\big). \tag{2.6}$$

The extremal function for the upper estimate (**??**) *is the function g^*_ε of the form*

$$g^*_\varepsilon(z) = \exp\big(-(1 - \alpha)Az\big)\sqrt{\frac{k_\varepsilon(z)}{z}},$$

where $\varepsilon = e^{-i\varphi}$, while for the lower estimate is the function g^*_ε for $\varepsilon = -e^{-i\varphi}$.

Theorem 2.5. *If* $g(z) = 1 + \sum_{n=1}^{\infty} d_n z^n \in \mathcal{G}(\alpha, A, B)$, *then the coefficients* d_n *satisfy the sharp coefficient inequalities*

$$\big|d_1 + (1-\alpha)(A+B)\big| \le 2(1-\alpha), \tag{2.7}$$

$$\big|2d_2 - d_1^2 + (1-\alpha)(A+B)B\big| \le 2(1-\alpha), \tag{2.8}$$

$$\big|3d_3 - 3d_1d_2 + d_1^3 + (1-\alpha)(A+B)B^2\big| \le 2(1-\alpha), \tag{2.9}$$

$$|\mathcal{H}(\alpha, A, B, \gamma)| \le \begin{cases} 2(1-\alpha) - \gamma|d_1 + (1-\alpha)(A+B)|^2, & \gamma \le \dfrac{1}{2}, \\ 2(1-\alpha) - (1-\gamma)|d_1 + (1-\alpha)(A+B)|^2, & \gamma \ge \dfrac{1}{2}, \end{cases} \tag{2.10}$$

where

$$\mathcal{H}(\alpha, \gamma) = 2d_2 - d_1^2(1-\gamma) + (1-\alpha)(A+B)\big\{B - \gamma(1-\alpha)(A+B) - 2\gamma d_1\big\},$$

and finally

$$\big|3d_3 - 5d_1d_2 + 2d_1^3 - (1-\alpha)(A+B)\big\{B^2 - 2d_2 + d_1^2 - Bd_1 - (1-\alpha)(A+B)B\big\}\big| \le 2(1-\alpha). \tag{2.11}$$

Proof. Let $d_0 = 1$ and $p(z) = \dfrac{zg'(z)}{f(z)} + (1-\alpha)\dfrac{1+Az}{1-Bz}$. Then

$$\frac{zg'(z)}{g(z)} + (1-\alpha)\frac{1+Az}{1-Bz} = 1 - \alpha + \big(d_1 + (1-\alpha)(A+B)\big)z +$$
$$+\big[2d_2 - d_1^2 + (1-\alpha)(A+B)B\big]z^2 +$$
$$+\big[3d_3 - 3d_1d_2 + d_1^3 + (1-\alpha)(A+B)B^2\big]z^3 + \ldots, \quad z \in \mathbb{D}.$$

The above expansion shows that, if $f \in \mathcal{G}(\alpha, A, B)$, it can be compared to $p(z) = 1 + p_1z + p_2z + \ldots$ $\ldots + \in \mathcal{P}(\alpha), 0 \le \alpha < 1$. It is a known fact that if $f \in \mathcal{G}(\alpha, A, B)$, then $|p_i| \le 2(1-\alpha)$, $i = 1, 2, \ldots$. By virtue of this inequality one may easily get (**??**), (**??**) and (**??**). Inequality (**??**) follows from the fact that

$$\big|p_2 - \gamma p_1^2\big| \le \begin{cases} 2(1-\alpha) - \gamma|p_1|^2, & \gamma \le \dfrac{1}{2}, \\ 2(1-\alpha) - (1-\gamma)|p_1|^2, & \gamma \ge \dfrac{1}{2}. \end{cases}$$

By applying a less unknown familiar inequality $|p_3 - p_1p_2| \le 2$ and performing a computation yields the inequality (**??**).

Theorem 2.5 is proved.

It is known that for each function $h \in \mathcal{S}^*$ the functions

$$z \to \frac{1}{\rho}h(\rho z), \qquad z \to e^{i\varphi}h(e^{-i\varphi}z), \quad 0 < \rho < 1, \quad \varphi \in \mathbb{R}, \quad z \in \mathbb{D},$$

also belong to $\mathcal{S}^*$. From Theorem **??** and estimation (**??**) we obtain the following theorem.

Theorem 2.6. *The region of values of the coefficient* d_1, *i.e.*, $\{d_1 : g \in \mathcal{G}(\alpha, A, B),\ g(z) = 1 + d_1 z + \ldots\}$ *has the form*

$$\big\{w \in \mathbb{C} \colon |w + (1-\alpha)(A+B)| \le 2(1-\alpha)\big\}.$$

***Remark* 2.4.** If $f \in \mathcal{S}^*\left(\frac{1}{2}\right)$, then the function Φ, defined by the formula

$$\Phi(z,\xi) = \frac{\xi}{f(\xi)}\frac{f(z)-f(\xi)}{z-\xi}, \quad z,\xi \in \mathbb{D}, \tag{2.12}$$

satisfies the condition $\Re(\Phi(z,\xi)) > \frac{1}{2}$ (see [**?**, p. 121]). Further, if $g \in \mathcal{G}(\alpha, A, B),\ B \neq 0$, then from (**??**) the function

$$f(z) = zg(z)(1-Bz)^{-\frac{(A+B)}{2B}}, \quad z \in \mathbb{D},$$

belongs to the class $\mathcal{S}^*\left(\frac{1}{2}\right)$. We denote

$$d_0 = 1 = P_0(A,B), \qquad (1-Bz)^{-\frac{A+B}{2B}} = 1 + \sum_{k=1}^{\infty} P_k(A,B) z^k, \quad z \in \mathbb{D},$$

where

$$P_k(A,B) = \frac{(A+B)(A+3B)\ldots(A+(2k-1)B)}{2^k k!}, \quad k = 1,2,\ldots.$$

Next, we apply the classical Clunie method to obtain an inequality to compute the coefficients of the function $\mathcal{G}(\alpha, A, B)$ without using the function (**??**).

Theorem 2.7. *If the function* g *of the form* (**??**) *belongs to the class* $\mathcal{G}(\alpha, A, B)$, *then*, *for* $n = 2, 3, \ldots$, *the estimates*

$$\big|((1-\alpha)(A+B) - B(n-1))d_{n-1} + nd_n\big|^2 \le$$

$$\le 4(1-\alpha)^2 + \sum_{k=1}^{n-1} \big|(2 - 2\alpha + k)d_k + ((A-B)(1-\alpha) - B(k-1))d_{k-1}\big|^2$$

hold.

Proof. Let the function g of the form (**??**) belong to the class $\mathcal{G}(\alpha, A, B)$. Thus the conditions (**??**) and (**??**) are satisfied. It follows that there exists a function $p \in \mathcal{P}(\alpha)$ such that

$$p(z) = \frac{zg'(z)}{g(z)} + (1-\alpha)\frac{1+z}{1-z} + \alpha, \quad z \in \mathbb{D}. \tag{2.13}$$

It is known that if $p \in \mathcal{P}(\alpha)$ then the function ω of the form

$$\omega(z) = \frac{p(z)-1}{p(z)+1-2\alpha}, \quad z \in \mathbb{D},$$

belongs to the known class Ω (ω holomorphic in $\mathbb{D}$, $\omega(0) = 0$, $|\omega(z)| < 1$ for $z \in \mathbb{D}$). From this fact and (**??**), we get

$$\big\{(1-\alpha)(2+(A-B)z)g(z)+zg'(z)(1-Bz)\big\}\omega(z)=$$
$$=(1-\alpha)(A+B)zg(z)+zg'(z)(1-\alpha)(1-Bz),\quad z\in\mathbb{D}.$$

Let $\omega(z)=\sum_{n=1}^{\infty}\omega_n z^n$. By expanding the function g in power series, we get

$$\left(2(1-\alpha)+\sum_{n=1}^{\infty}(2-2\alpha+n)d_n z^n+\right.$$
$$\left.+\sum_{n=1}^{\infty}((A-B)(1-\alpha)-B(n-1))d_{n-1}z^n\right)\left(\sum_{n=1}^{\infty}\omega_n z^n\right)=$$
$$=\sum_{n=1}^{\infty}\big((1-\alpha)(A+B)-B(n-1)\big)d_{n-1}z^n+\sum_{n=1}^{\infty}nd_n z^n,\quad z\in\mathbb{D}.$$

After further simplication of the above equation, it can be easily seen that

$$\left(2(1-\alpha)+\sum_{n=1}^{\infty}\Big\{(2-2\alpha+n)d_n+\right.$$
$$\left.+\big((A-B)(1-\alpha)-B(n-1)\big)d_{n-1}\Big\}z^n\right)\left(\sum_{n=1}^{\infty}\omega_n z^n\right)=$$
$$=\sum_{n=1}^{\infty}\Big\{((1-\alpha)(A+B)-B(n-1))d_{n-1}+nd_n\Big\}z^n,\quad z\in\mathbb{D}. \tag{2.14}$$

Let

$$p_n(\alpha,A,B)=(2-2\alpha+n)d_n+\big((A-B)(1-\alpha)-B(n-1)\big)d_{n-1},\quad n=1,2,\ldots, \tag{2.15}$$

and

$$s_n(\alpha,A,B)=\big((1-\alpha)(A+B)-B(n-1)\big)d_{n-1}+nd_n,\quad n=1,2,\ldots. \tag{2.16}$$

Then we obtain

$$2(1-\alpha)\sum_{n=1}^{\infty}\omega_n z^n+\sum_{n=2}^{\infty}\Big(p_1(\alpha,A,B)\omega_{n-1}+\ldots+p_{n-1}(\alpha,A,B)\omega_1\Big)z^n=$$
$$=\sum_{n=1}^{\infty}s_n(\alpha,A,B)z^n,\quad z\in\mathbb{D}.$$

Equating coefficients on both sides of the above identity, we have

$$2(1-\alpha)\omega_1=(1-\alpha)(A+B)+d_1, \tag{2.17}$$

$$2(1-\alpha)\omega_n+p_1(\alpha,A,B)\omega_{n-1}+\ldots+p_{n-1}(\alpha,A,B)\omega_1=s_n(\alpha,A,B)\quad\text{for}\quad n=2,3,\ldots. \tag{2.18}$$

Since $|\omega_1| \leq 1,$ from (**??**) we obtain

$$\left|d_1 + (1-\alpha)(A+B)\right| \leq 2(1-\alpha),$$

which is identical to the estimate (**??**).

Next by making use of equations (**??**), (**??**) and (**??**), we get

$$\left(2(1-\alpha) + \sum_{k=1}^{n-1} p_k(\alpha, A, B) z^k\right)\left(\sum_{k=1}^{\infty} \omega_k z^k\right) = \sum_{k=1}^{n} s_k(\alpha, A, B) z^k + \sum_{k=n+1}^{\infty} E_k z^k,$$

where E_k are the appropriate coefficients. Since $|\omega(z)| < 1$ for $z \in \mathbb{D}$, then

$$\left|\sum_{k=1}^{n} s_k(\alpha, A, B) z^k + \sum_{k=n+1}^{\infty} E_k z^k\right|^2 < \left|2(1-\alpha) + \sum_{k=1}^{n-1} p_k(\alpha, A, B) z^k\right|^2, \quad z \in \mathbb{D}.$$

Upon simplification, we have

$$\sum_{k=1}^{n} |s_k(\alpha, A, B)|^2 \leq 4(1-\alpha)^2 + \sum_{k=1}^{n-1} |p_k(\alpha, A, B)|^2, \quad n = 2, 3, \ldots. \tag{2.19}$$

Since $\left|s_k(\alpha, A, B)\right|^2 \geq 0$ for $k = 1, \ldots, n-1$, then

$$\left|s_n(\alpha, A, B)\right|^2 \leq 4(1-\alpha)^2 + \sum_{k=1}^{n-1} \left|p_k(\alpha, A, B)\right|^2, \quad n = 2, 3, \ldots.$$

Theorem 2.7 is proved.

***Remark* 2.5.** If we put $n = 2$ in (**??**), then we have

$$|(1-\alpha)(A+B) + d_1|^2 + |((1-\alpha)(A+B) - B)d_1 + 2d_2|^2 \leq$$
$$\leq 4(1-\alpha)^2 + |(3-2\alpha)d_1 + (1-\alpha)(A-B)|^2.$$

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Received 13.01.21