DOI: 10.37863/umzh.v74i11.6083

UDC 517.9

Osman Tunç[1] (Van Yuzuncu Yil Univ., Turkey),
E. Korkmaz (Muş Alparslan Univ., Turkey)

NEW RESULTS ON THE QUALITATIVE ANALYSIS OF SOLUTIONS OF VIDEs BY THE LYAPUNOV – RAZUMIKHIN TECHNIQUE

НОВІ РЕЗУЛЬТАТИ З ЯКІСНОГО АНАЛІЗУ РОЗВ'ЯЗКІВ ІДРВ ЗА МЕТОДИКОЮ ЛЯПУНОВА – РАЗУМІХІНА

A new mathematical model described by a Volterra integro-differential equation (VIDE) with constant delay is examined. New agreeable conditions on the uniformly asymptotic stability, boundedness, and square integrability of solutions of the VIDE are obtained by using the Lyapunov – Razumikhin technique. The established conditions improve some former results and they are also nonlinear generalizations of these results. Moreover, they are weaker than some available results cited in the bibliography of this paper. Two examples are presented to demonstrate applications of these results and the introduced concepts. The use of the Lyapunov – Razumikhin technique leads to a significant difference and gives an advantage over the related methods used in the books and papers cited in the bibliography.

Розглянуто нову математичну модель, що описується інтегро-диференціальним рівнянням Вольтерра (ІДРВ) зі сталим запізненням. За допомогою техніки Ляпунова – Разуміхіна отримано нові прийнятні умови рівномірно асимптотичної стійкості, обмеженості та квадратичної інтегровності розв'язків цього ІДРВ. Встановлені умови покращують деякі попередні результати, а також є їх нелінійними узагальненнями. Крім того, вони слабші, ніж деякі результати, доступні в бібліографії цієї статті. Наведено два приклади, щоб продемонструвати можливе застосування цих результатів і введених концепцій. Використання техніки Ляпунова – Разуміхіна приводить до значних відмінностей та надає певну перевагу порівняно з відповідними методами, що використовуються в книгах і статтях, наведених у бібліографії.

1. Introduction. From the relevant literature, the effective methods or theories to investigate the properties of solutions such as stability of solutions, exponential decay of solutions, boundedness and convergence of solutions and so forth in different kinds of ordinary, functional or impulsive differential equations are known as fixed point and perturbation methods, integral inequalities, the second Lyapunov method, the Lyapunov – Razumikhin technique, the variations of parameters formulas, etc. However, it can be seen from relevant literature that the presented methods or theories are used intensively during discussion of the properties of solutions of that equations, except the Lyapunov – Razumikhin technique. According to the information available from the relevant literature, in generally, the Lyapunov – Razumikhin technique is very effectively used to discuss qualitative behaviors of solutions of impulsive DEs, however not in ODEs, IDEs or IDDEs (see Hale [8], Samoilenko and Perestyuk [20], Zhou and Egorov [34]). Probably, the possible reason could brought up from the nature of the problems in ODEs, IDEs or IDDEs for the mentioned concepts above. The proper details of this fact is not our matter here.

The goal of this article is to discuss some dynamical properties of solutions of a sample of nonlinear VIDEs with fixed delay by the Lyapunov – Razumikhin technique. The motivation of the results of this paper comes from the books and the papers referenced in [2, 8, 13, 15, 32] (see also [1, 3 – 7, 9 – 12, 14, 16 – 18, 21 – 31, 33 – 40] and the references therein). From this point, let us now summarize a very close result on the qualitative behavior of a linear VIDE.

[1] Corresponding author, e-mail: osmantunc89@gmail.com.

Xu [33] investigated the uniformly asymptotic stability of the solutions of the below scalar linear VIDE with infinite delay

$$\frac{dx}{dt} = a(t)x + \int_{-\infty}^{t} K(t,s)x(s)ds \tag{1}$$

by the second Lyapunov method.

In this study, we view the subsequent VIDE of first order

$$\frac{dx}{dt} = -a(t)g_1(t,x) + \int_{t-\tau}^{t} K(t,s)g_2(s,x(s))ds + f(t,x) \tag{2}$$

with

$$x(t_0+\theta) = \phi_0(\theta) \quad \forall\theta \in [-\tau, 0], \quad \tau > 0, \quad \tau \in \mathbb{R},$$

where $x \in \mathbb{R}$, $\mathbb{R} = (-\infty,\infty)$, $a(t) \in C(\mathbb{R}^+,(0,\infty))$, $\mathbb{R}^+ = [0,\infty)$, $g_1 \in C^1(\mathbb{R}^+ \times \mathbb{R},\mathbb{R})$, $g_1(t,0) = 0$, and $g_2,\ f \in C(\mathbb{R}^+ \times \mathbb{R},\mathbb{R})$ with $f(t,0) = g_2(s,0) = 0$, $K \in C(\mathbb{R}^+ \times \mathbb{R}^+,\mathbb{R})$ and $\phi_0 \in C([-\tau,0],\mathbb{R})$.

To go forward, we assume that there is a function $g_0 \in C^1(\mathbb{R}^+ \times \mathbb{R},\mathbb{R})$, which is represented by

$$g_0(t,x) = \begin{cases} \dfrac{g_1(t,x)}{x}, & x \neq 0, \\ \dfrac{\partial g_1(t,0)}{\partial x}, & x = 0. \end{cases}$$

From this point, VIDE (2) possesses a form as follows:

$$\frac{dx}{dt} = -a(t)g_0(t,x)x + \int_{t-\tau}^{t} K(t,s)g_2(s,x(s))ds + f(t,x).$$

For $\phi \in [t_0-\tau, t_0]$, where $t_0 \geq 0$ and ϕ is an initial function, let $x(t) = x(t,t_0,\phi)$ represent the solution of VIDE (2) on $(-\infty,\infty)$ such that $x(t) = \phi(t)$ on $[t_0-\tau, t_0]$. Through the paper, without mention, let x denote $x(t)$.

We now compare VIDEs (1), (2) and clarify the contribution of this paper:

1. It is seen that VIDE (1) is linear and has a very simple form. However, our equation, VIDE (2) is nonlinear and includes three nonlinear terms. This is an improvement from a linear IDE to a nonlinear IDE.

2. Let us take $t-\tau$ instead of $-\infty$ as the lower limit of the integral in VIDE (1). Then, from VIDEs (1) and (2), we observe that VIDE (2) involves VIDE (1). In fact, if $g_1(t,x) = -x$, $g_2(s,x(s)) = x(s)$ and $f(t,x) = 0$, then VIDE (2) reduces to VIDE (1) discussed by Xu [33]. This information yields an additional contribution to the results in the bibliography of this paper.

3. In [33], Xu studied the uniformly asymptotic stability of the trivial solution. However, here, together with the uniformly asymptotic stability of trivial solution, we investigate the boundedness of solutions at infinity and their square integrability by means of the Lyapunov – Razumikhin technique. This discussion shows improvements and extensions done here.

4. Through the results here, we employ the Lyapunov – Razumikhin technique to proceed the proofs. To the best of information from the relevant literature, we did not find any paper on the qualitative behaviors of IDEs and IDDEs of integer order, in which the Lyapunov – Razumikhin technique was used therein. Here, we also see that how the Lyapunov – Razumikhin technique is one of important and effective technique to do relative works on the qualitative properties of solutions. This is the next contribution and originality of this work.

5. Through the proofs, we do not also use the Gronwall inequality. Hence, we can delete some reasonless conditions through the past literature on IDEs, which are related to the topic of this paper. Then we can obtain some former results, which are available in the database of the relevant literature, under weaker conditions. Here, we would not like to compare and give some details on the behalf of briefness. These are the further contributions to the results in the bibliography of this paper.

6. In Xu [33], it was given an example to validate the result therein without plotting the graphs of the solutions. However, in this paper, we give an example to demonstrate the properties of the trajectories of the solutions of VIDE (2), in particular case. This is a desirable application for any paper on the concepts as they were introduced here.

2. Qualitative analysis of solutions. Let $f(t,x)=0$ in VIDE (2).

The basic assumptions for qualitative properties of VIDE (2) are given as follows.

A. Assumptions.

(A1) For some positive constant α_0, we have

$$g_2(t,0)=0,\quad g_2(t,x)\neq 0\quad \text{if}\quad x\neq 0,\quad \frac{|g_2(t,x)|}{|x|}\leq \alpha_0\quad (x\neq 0)\quad \forall t\in\mathbb{R}^+\quad \forall x\in\mathbb{R}.$$

(A2) We have some positive constant k_1 such that

$$2a(t)g_0(t,x)-(1+\alpha_0^2)\int\limits_{t-\tau}^{t}|K(t,s)|ds\geq k_1\quad \forall\mathbb{R}^+\quad \forall x\in\mathbb{R}$$

with

$$\int\limits_{t-\tau}^{t}|K(t,s)|ds<\infty.$$

Firstly, we state a theorem on the boundedness, uniformly asymptotic stability and integrability for the solutions of nonlinear VIDE (2).

Theorem 1. *We have the following results for VIDE* (2) *if assumptions* (A1) *and* (A2) *hold*:

1) *the zero solution of VIDE* (2) *is uniformly asymptotically stable*;

2) *the solutions of VIDE* (2) *are bounded when* $t\to\infty$ *and square integrable.*

Proof. Define a LF $\Lambda=\Lambda(t,x)$ with

$$\Lambda(t,x)=x^2.$$

Next, it is derived

$$\Lambda(t,0)=0$$

and

$$\Lambda(t,x) \geq k_0 x^2, \quad 0 < k_0 \leq 1, \quad k_0 \in \mathbb{R}. \tag{3}$$

Differentiating the function Λ along VIDE (2) results in

$$\frac{d\Lambda}{dt} = 2xx' = 2x\left[-a(t)g_0(t,x)x + \int\limits_{t-\tau}^{t} K(t,s)g_2(s,x(s))ds\right] =$$

$$= -2a(t)g_0(t,x)x^2 + 2x(t)\int\limits_{t-\tau}^{t} K(t,s)g_2(s,x(s))ds \leq$$

$$\leq -2a(t)g_0(t,x)x^2 + 2|x|\int\limits_{t-\tau}^{t} |K(t,s)||g_2(s,x(s))|ds \leq$$

$$\leq -2a(t)g_0(t,x)x^2 + \int\limits_{t-\tau}^{t} |K(t,s)|[x^2(t)) + g_2^2(s,x(s))]ds =$$

$$= -2a(t)g_0(t,x)x^2 + x^2(t)\int\limits_{t-\tau}^{t} |K(t,s)|ds + \int\limits_{t-\tau}^{t} |K(t,s)|g_2^2(s,x(s))ds.$$

Employing assumption (A1), that is,

$$|g_2(t,x)| \leq \alpha_0 |x|,$$

we derive

$$\frac{d\Lambda}{dt} \leq -\left[2a(t)g_0(t,x) - \int\limits_{t-\tau}^{t} |K(t,s)|ds\right]x^2(t) + \alpha_0^2 \int\limits_{t-\tau}^{t} |K(t,s)|x^2(s)ds. \tag{4}$$

For the next step, we benefit from the Lyapunov – Razumikhin technique (see [8, 20, 34]).

Hence, let

$$\Lambda(t+s, x(t+s)) < \Lambda(t, x(t)) \quad \forall s \in [-\tau, 0).$$

By this inequality, we have

$$[x(t+s)]^2 < [x(t)]^2 \quad \forall s \in [-\tau, 0).$$

Note the following term:

$$\alpha_0^2 \int\limits_{t-\tau}^{t} |K(t,s)|x^2(s)ds,$$

which is included in the inequality (4).

We now apply this integration the transformation $s - t = \xi$. Then it follows that $ds = d\xi$. Hence, if $s = t - \tau$, then $\xi = -\tau$. Similarly, if $s = t$, then $\xi = 0$. For this reason, it is followed that

$$\alpha_0^2 \int\limits_{t-\tau}^{t} x^2(s)|K(t,s)|ds = \alpha_0^2 \int\limits_{-\tau}^{0} x^2(t+\xi)|K(t,t+\xi)|d\xi \leq$$

$$\leq \alpha_0^2 \int\limits_{-\tau}^{0} x^2(t)|K(t,t+\xi)|d\xi = \alpha_0^2 x^2(t) \int\limits_{-\tau}^{0} |K(t,t+\xi)|d\xi =$$

$$= \alpha_0^2 x^2(t) \int\limits_{t-\tau}^{t} |K(t,s)|ds. \quad (5)$$

Hence, from (4) and (5), we observe that

$$\frac{d\Lambda}{dt} \leq -\left[2a(t)g_0(t,x) - (1+\alpha_0^2) \int\limits_{t-\tau}^{t} |K(t,s)|ds\right] x^2. \quad (6)$$

By the inequality (6) and assumption (A2), it is seen that

$$\frac{d\Lambda}{dt} \leq -k_1 x^2.$$

The above analysis allows that the trivial solution of VIDE (2) is stable.

We now view a set represented by

$$S := \left\{ x : \frac{d\Lambda}{dt}(x) = 0 \right\}.$$

Then

$$\frac{d\Lambda}{dt}(x) = 0 \quad \text{if and only if} \quad x = 0.$$

For this step, we put $x = 0$ in VIDE (2) when $f(t,x) = 0$ This leads $g_2(t,0) = 0$. Since $g_2(t,x) \neq 0$ if $x \neq 0$, then $0 \in S$. Therefore, clearly, it is derived that the largest invariant set of S is zero by the LaSalle invariance principle (see [19]). Hence, it is confirmed that the trivial solution of VIDE (2) is uniformly asymptotically stable. This result finishes the proof of Theorem 1.

We now observe that $\dfrac{d\Lambda}{dt} \leq 0$. If calculate the integration of this inequality, we find

$$\Lambda(t,x) \leq \Lambda(t_0, x(t_0)) \quad \forall t \geq t_0.$$

It now is evident that

$$\Lambda(t,0) = 0, \qquad \Lambda(t_0, x(t_0)) = x^2(t_0) = \theta_0 > 0, \quad \theta_0 \in \mathbb{R},$$

in which $x(t_0) \neq 0$.

On the account of the discussion above, we can write

$$k_0 x^2 \leq \Lambda(t_0, x(t_0)) = \theta_0.$$

Next,

$$|x(t)| \leq \sqrt{k_0^{-1}\theta_0} \quad \forall t \geq t_0.$$

From the inequalities above, we confirm that

$$\lim_{t\to\infty} |x(t)| \leq \lim_{t\to\infty} \sqrt{k_0^{-1}\theta_0} = \sqrt{k_0^{-1}\theta_0},$$

which verifies that the solutions of nonlinear VIDE (2) are bounded when $t \to \infty$.

On the other hand, by the integration of $\dfrac{d\Lambda}{dt} \leq -k_1 x^2$, we get

$$\Lambda(t,x) \leq \Lambda(t_0, x(t_0)) - k_1 \int_{t_0}^{t} x^2(s)ds.$$

Hence, we see that

$$k_1 \int_{t_0}^{t} x^2(s)ds \leq \Lambda(t,x) + k_1 \int_{t_0}^{t} x^2(s)ds \leq \Lambda(t_0, x(t_0)) = \theta_0.$$

Thus, we can observe

$$\int_{t_0}^{\infty} x^2(s)ds \leq k_1^{-1}\theta_0 < \infty,$$

that is, $x \in L^2[0,\infty)$. The inequality above completes the proof of Theorem 1.

Let $f(t,x) \neq 0$.

In our coming result, Theorem 2, we will give a boundedness result related to the nonlinear VIDE (2), when $f(t,x) \neq 0$.

B. Assumptions.

(A3) Let $h \in C(\mathbb{R}^+, \mathbb{R})$ such that

$$|f(t,x)| \leq 2^{-1}|h(t)||x|,$$

where the function $|h(t)|$ is integrable for $t \geq t_0$, that is,

$$\int_{t_0}^{\infty} |h(s)|ds < \infty.$$

Theorem 2. *The solutions of nonlinear VIDE* (2) *are bounded when* $t \to \infty$ *if assumptions* (A1) – (A3) *hold*.

Proof. We can easily obtain

$$\frac{d\Lambda}{dt}(t,x) \leq 2f(t,x)x \leq |h(t)|x^2 \leq |h(t)|\Lambda(t,x)$$

if assumptions (A1) – (A3) of Theorem 2 hold.

Next, we derive

$$\frac{d\Lambda(t,x)}{\Lambda(t,x)} \leq |h(t)|dt.$$

Later, by integration on the interval $[t_0, t]$, we have

$$\int_{t_0}^{t} \frac{d\Lambda(s,x(s))}{\Lambda(s,x(s))} ds \leq \int_{t_0}^{t} |h(s)|ds.$$

From this point, it is noted that

$$\Lambda(t,x) \leq \Lambda(t_0, x(t_0)) \exp\left[\int_{t_0}^{t} h(s)ds\right].$$

Next, it clear that

$$\Lambda(t,x) \leq x^2(t_0) \exp\left(\int_{t_0}^{\infty} |h(s)|ds\right).$$

Since

$$\int_{t_0}^{\infty} |h(s)|ds < \infty,$$

let

$$\int_{t_0}^{\infty} |h(s)|ds = h_0 < \infty, \quad h_0 > 0, \quad h_0 \in \mathbb{R}.$$

Then

$$x^2(t_0) \exp\left(\int_{t_0}^{\infty} |h(s)|ds\right) = \theta_0 \exp(h_0).$$

Let $M = \theta_0 \exp(h_0) > 0$. For the coming step, consequently, we derive

$$x^2(t) = \Lambda(t,x) \leq \Lambda(t_0, x(t_0)) \exp\left[\int_{t_0}^{\infty} h(s)ds\right] = M.$$

Thus,

$$|x(t)| \leq \sqrt{M} \quad \text{as} \quad t \to \infty.$$

The inequality above proves the claim of Theorem 2.

3. Numerical applications.

***Example* 1.** We have the following scalar nonlinear VIDE of first order when $f(t,x) = 0$:

$$\frac{dx}{dt} = -101x - \sin x + \frac{1}{100} \int_{t-1}^{t} \exp(-(t-s)) \frac{\sin x(s)}{20 + s^2 + x^2(s)} ds, \quad t \geq 1. \tag{7}$$

Hence, by comparison VIDE (7) with VIDE (2), we can observe the following formulae:

$$\tau = 1, \quad a(t) = 1, \quad g_1(t,x) = 101x + \sin x,$$

$$g_0(t,x) = 101 + \frac{\sin x}{x} \geq 100, \quad x \neq 0,$$

$$K(t,s) = \frac{1}{100}\exp(-(t-s)), \quad t \geq s,$$

$$\int\limits_{t-\tau}^{t} |K(t,s)|ds = \frac{1}{100}\int\limits_{t-1}^{t} \exp(-(t-s))ds = \frac{1}{100}\left(1 - \frac{1}{e}\right) < 1 < \infty,$$

$$g_2(t,0) = 0, \qquad |g_2(t,x)| = \frac{|\sin x|}{20 + s^2 + x^2} \leq \frac{|x|}{20 + s^2 + x^2} \leq \frac{1}{20}|x|, \quad \alpha_0 = \frac{1}{20},$$

and

$$2a(t)g_0(t,x) - (1 + \alpha_0^2)\int\limits_{t-\tau}^{t} |K(t,s)|ds =$$

$$= 200 - \left(1 + \frac{1}{400}\right)\frac{1}{100}\int\limits_{t-1}^{t} \exp(-(t-s))ds \geq$$

$$\geq 200 - \left(1 + \frac{1}{400}\right)\frac{1}{100}\left(1 - \frac{1}{e}\right) =$$

$$= 200 - \frac{401}{40000}\left(1 - \frac{1}{e}\right) \cong 199.9 = \rho > 0.$$

Define a Lyapunov function as follows:

$$\Lambda_1(t,x) = \frac{1}{2}x^2.$$

For this step, the time derivative of $\Lambda_1(t,x)$ along nonlinear VIDE (7) gives

$$\frac{d}{dt}\Lambda_1(t,x) = -101x^2 - x\sin x + \frac{1}{100}x(t)\int\limits_{t-1}^{t} \exp(-(t-s))\frac{\sin x(s)}{20 + s^2 + x^2(s)}ds.$$

Taking into consideration the above estimates and utilizing a simple inequality, we get

$$\frac{d}{dt}\Lambda_1(t,x) \leq -100x^2 + \frac{1}{100}|x(t)|\int\limits_{t-1}^{t} \exp(-(t-s))\frac{|\sin x(s)|}{20 + s^2 + x^2(s)}ds \leq$$

$$\leq -100x^2 + \frac{1}{2000}|x(t)|\int\limits_{t-1}^{t} \exp(-(t-s))|\sin x(s)|ds \leq$$

$$\leq -100x^2 + \frac{1}{2000}|x(t)| \int\limits_{t-1}^{t} \exp(-(t-s))|x(s)|ds \leq$$

$$\leq -100x^2 + \frac{1}{1000} \int\limits_{t-1}^{t} 2|x(t)||x(s)|ds \leq$$

$$\leq -100x^2 + \frac{1}{1000} \int\limits_{t-1}^{t} \left[x^2(t) + x^2(s)\right] ds \leq$$

$$\leq -\left[100 - \frac{1}{1000}\right] x^2(t) + \frac{1}{1000} \int\limits_{t-1}^{t} x^2(s)ds \leq$$

$$\leq -(99.999)x^2(t) + \frac{1}{1000} \int\limits_{t-1}^{t} x^2(s)ds. \tag{8}$$

By the Lyapunov – Razumikhin technique [8, 20, 34], we observe

$$\Lambda_1(t, x(t)) > \Lambda_1(t+s, x(t+s)) \quad \forall s \in [-1, 0).$$

Consequently,

$$\frac{1}{2}[x(t)]^2 > \frac{1}{2}[x(t+s)]^2 \quad \forall s \in [-1, 0).$$

We now note the last term in (8) as follows:

$$\int\limits_{t-1}^{t} x^2(s)ds.$$

Applying this integration the transformation $s - t = \xi$. Then, it follows that $ds = d\xi$. Hence, if $s = t-1$, then $\xi = -1$. As before, if $s = t$, then $\xi = 0$. Hence, from the inequality (8), we deduce the following relations:

$$\frac{d}{dt}\Lambda_1(t, x) \leq -(99.999)x^2(t) + \frac{1}{1000} \int\limits_{-1}^{0} x^2(t+\xi)d\xi \leq$$

$$\leq -(99.999)x^2(t) + \frac{1}{1000} \int\limits_{-1}^{0} x^2(t)d\xi =$$

$$= -\left[(99.999) - \frac{1}{1000}\right] x^2(t) = -(98.999)x^2 \leq 0. \tag{9}$$

By the above estimates, we can reach to a decision that the zero solution of nonlinear VIDE (7) is asymptotically stable. On the same time, it is also uniformly asymptotically stable.

Indeed, from the last inequality, we have

$$\frac{d}{dt}\Lambda_1(t,x) \leq -(98.999)x^2 = -\frac{1}{2}(98.999)\Lambda_1(t,x).$$

Integrating this inequality, we get

$$\Lambda_1(t,x) \leq \Lambda_1(t_0,\phi(t_0))\exp\left[-\frac{1}{2}(98.999)(t-t_0)\right], \quad t \geq t_0.$$

Let $0 < k < \frac{1}{2}$ and take

$$\Lambda_1(t_0,\phi(t_0)) = k_0, \quad k_0 > 0, \quad k_0 \in \mathbb{R}.$$

Then

$$k\big|x(t,t_0,\phi)\big|^2 \leq \Lambda_1(t,x) \leq \Lambda_1(t_0,\phi(t_0))\exp\left[-\frac{1}{2}(98.999)(t-t_0)\right], \quad t \geq t_0,$$

so that

$$\big|x(t,t_0,\phi)\big| \leq \sqrt{k_0k^{-1}\exp\big[-2^{-1}(98.999)(t-t_0)\big]}, \quad t \geq t_0.$$

Taking the limit of the former inequality when $t \to \infty$, we obtain

$$\lim_{t\to\infty}\big|x(t,t_0,\phi)\big| \leq \lim_{t\to\infty}\sqrt{k_0k^{-1}k_0k^{-1}\exp(-\rho(t-t_0))} = 0.$$

By the above reason, we have

$$\lim_{t\to\infty}\big|x(t,t_0,\phi)\big| = 0.$$

This result and the discussion done above therefore complete the proof of the asymptotic stability of the trivial solution of VIDE (2). Next, we can say that the trivial solution of nonlinear VIDE (7) is also uniformly asymptotically stable.

Moreover, in addition to these results, we know that $\Lambda_1(t,x)$ is a decreasing function. An integration of the inequality (9) from t_0 to t yields

$$\Lambda_1(t,x) - \Lambda_1(t_0,\phi(t_0)) \leq -(98.999)\int_{t_0}^{t} x^2(s)ds.$$

Hence,

$$\int_{t_0}^{t} x^2(s)ds \leq (98.999)^{-1}\Lambda_1(t_0,\phi(t_0)) - (98.999)^{-1}\Lambda_1(t,x) \leq$$

$$\leq (98.999)^{-1}\Lambda_1(t_0,\phi(t_0)) = (98.999)^{-1}k_0.$$

In this case, we can conclude that

$$\int_{t_0}^{\infty} x^2(s)ds \leq (98.999)^{-1}k_0 < \infty.$$

Thus, we can say that the solutions of nonlinear VIDE (7) are square integrable. Figure 1 shows the behaviors of the trajectories of the solutions of VIDE (7).

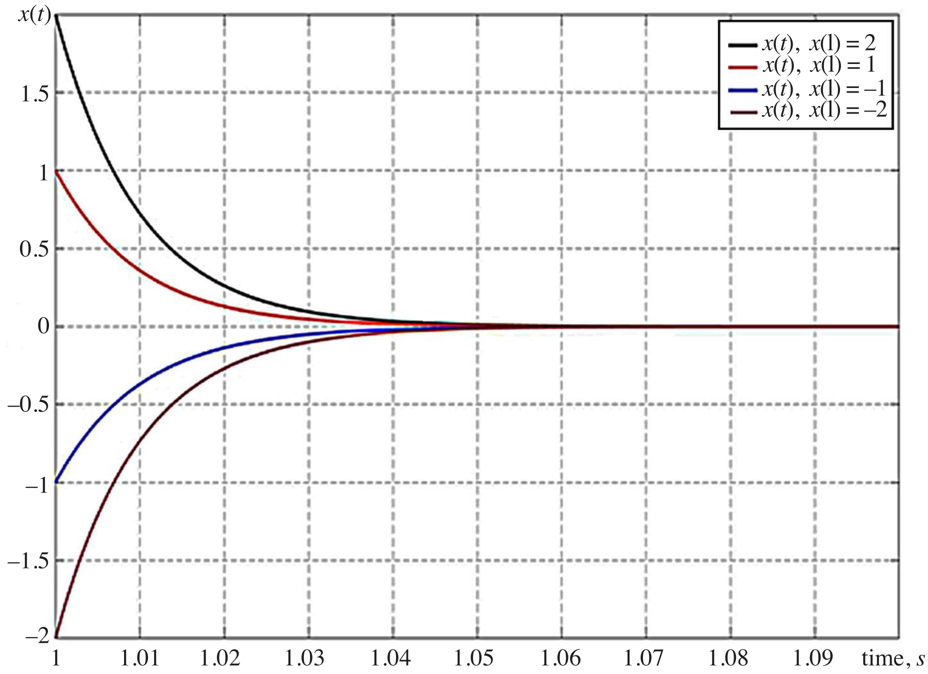

Fig. 1. Behaviors of the trajectories of the solution $x(t)$ of VIDE (7) for $\tau = 1$ and different initial values.

***Example* 2.** We now modify the scalar nonlinear VIDE (7) when $f(t,x) \neq 0$ as follows:

$$\frac{dx}{dt} = -101x - \sin x + \frac{1}{100}\int\limits_{t-1}^{t} \exp(-(t-s))\frac{\sin x(s)}{20+s^2+x^2(s)}ds+$$

$$+\frac{1}{2}\left(\frac{\sin x}{1+t^2+x^2}\right), \quad t \geq 1. \tag{10}$$

Then it is clear that

$$f(t,x) = \frac{1}{2}\left(\frac{\sin x}{1+t^2+x^2}\right).$$

From this point, we know that the discussion for the functions in Example 1 is valid for VIDE (10) except the term $f(t,x) = \dfrac{1}{2}\dfrac{\sin x}{1+t^2+x^2}$.

Then, for this case differentiating $\Lambda_1(t,x)$ along nonlinear VIDE (10), we can obtain

$$\frac{d}{dt}\Lambda_1(t,x) \leq -(98.999)x^2 + \frac{x\sin x}{1+t^2+x^2} \leq$$

$$\leq \frac{|x||\sin x|}{1+t^2+x^2} \leq \frac{x^2}{1+t^2} = \frac{2}{1+t^2}\Lambda_1(t,x).$$

Hence, an integration gives

$$\Lambda_1(t,x(t)) \leq \Lambda_1(t_0,x(t_0))\exp\left(\int\limits_{t_0}^{t}\frac{2}{1+s^2}ds\right) \leq$$

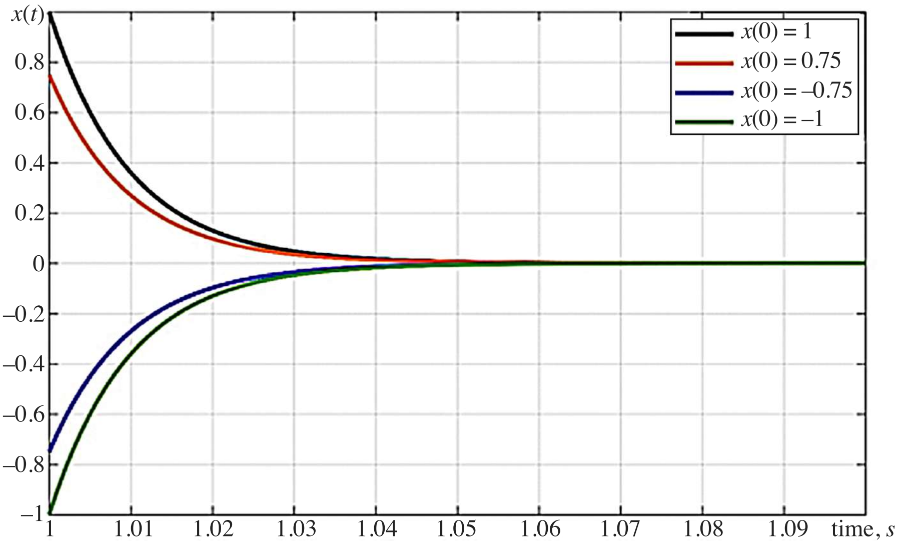

Fig. 2. Behaviors of the trajectories of the solution $x(t)$ of VIDE (10) for $\tau = 1$ and different initial values.

$$\leq \Lambda_1(t_0, x(t_0)) \exp\left(\int\limits_{t_0}^{\infty} \frac{2}{1+s^2} ds \right) =$$

$$= \Lambda_1(t_0, x(t_0)) \exp(\pi - 2 \arctan t_0) = N, \quad N > 0, \quad N \in \mathbb{R},$$

for all $t \geq t_0$ provided that $\Lambda_1(t_0, x(t_0)) \neq 0$. By the last inequality and the above information, we have

$$\frac{1}{2} x^2(t, t_0, x_0) = \Lambda_1(t, x(t)) \leq \Lambda_1(t_0, x(t_0)) \exp(\pi - 2 \arctan t_0) = N.$$

Thus, we get

$$|x(t, t_0, x_0)| \leq 2N.$$

Taking the limit as $t \to \infty$, we obtain

$$\lim_{t\to\infty} |x(t, t_0, x_0)| \leq \lim_{t\to\infty} \sqrt{2N} = \sqrt{2N}.$$

Figure 2 shows the boundedness behaviors of the trajectories of the solutions of VIDE (10).

***Remark* 1.** In the relevant literature, one of the most powerful method to investigate the numerous qualitative properties of trajectories of solutions of ODEs, functional DEs, IEs, IDEs, IDDEs, partial DEs, etc., without solving that equations is the second Lyapunov method. During the investigation(s), whilst using this method, it is needed to construct an appropriate Lyapunov function or functional, which is positive definite such that its time derivative along the solutions of the considered equation is negative semidefinite, negative definite or positive definite. Hence, it can be determined the stability, instability, asymptotically stability, uniformly asymptotically stability, convergence, boundedness, integrability, globally existence and so on of the solutions without any prior information of solutions. Further, in proof(s) of proper results on that concepts, most of times, it is needed to benefit from the Gronwall inequality in derivative of proper Lyapunov function(s) or functional(s) to proceed the proofs. Hence, it can be reached a possible decision on the qualitative properties of solutions for equation(s) considered. However, rarely, without the application of the Gronwall inequality, we can conclude the same results. This case depends on function or functional used through the proof(s), proper assumptions and technique used therein. For example, in the proofs of Theorems 1, 2, Examples 1 and 2 we did not use this inequality. This fact leads some advantages and weaker assumptions.

4. Conclusion. We consider a VIDE with constant delay. The uniformly asymptotically stability of solutions, boundedness of solutions as $t \to \infty$ and square integrability of solutions are discussed by using the Lyapunov – Razumikhin technique. Two examples are given to validate our results. Moreover, it is avoided to use the Gronwall inequality in Theorems 1 and 2, Examples 1 and 2. This case leads to delete some reasonless conditions [1 – 7, 9 – 12, 15 – 18, 22 – 31, 33]. The obtained results have weaker conditions, and they allow some new contributions to the literature, improve and generalize the results of Xu [33] and that can be available in the bibliography of this paper. Finally, using the Lyapunov – Razumikhin technique shows a significant difference and advantage of this paper than the related ones in the bibliography of this paper.

References

1. L. C. Becker, *Uniformly continuous L^1 solutions of Volterra equations and global asymptotic stability*, Cubo, **11**, № 3, 1 – 24 (2009).
2. T. A. Burton, *Volterra integral and differential equations, second ed.*, Mathematics in Science and Engineering, vol. 202, Elsevier, Amsterdam (2005).
3. X. Chang, R. Wang, *Stability of perturbed n-dimensional Volterra differential equations*, Nonlinear Anal., **74**, № 5, 1672 – 1675 (2011).
4. P. Eloe, M. Islam, B. Zhang, *Uniform asymptotic stability in linear Volterra integro-differential equations with application to delay systems*, Dynam. Systems and Appl., **9**, № 3, 331 – 344 (2000).
5. M. Funakubo, T. Hara, S. Sakata, *On the uniform asymptotic stability for a linear integro-differential equation of Volterra type*, J. Math. Anal. and Appl., **324**, № 2, 1036 – 1049 (2006).
6. J. R. Graef, C. Tunç, *Continuability and boundedness of multi-delay functional integro-differential equations of the second order*, Rev. R. Acad. Cienc. Exactas Fıs. Nat. Ser. A Math. RACSAM, **109**, № 1, 169 – 173 (2015).
7. J. R. Graef, C. Tunç, S. Şevgin, *Behavior of solutions of nonlinear functional Voltera integro-differential equations with multiple delays*, Dynam. Systems and Appl., **25**, № 1-2, 39 – 46 (2016).
8. J. Hale, *Theory of functional differential equations, second ed.*, Applied Mathematical Sciences, vol. 3, Springer-Verlag, New York, Heidelberg (1977).
9. T. Hara, T. Yoneyama, T. Itoh, *Asymptotic stability criteria for nonlinear Volterra integro-differential equations*, Funkcial. Ekvac., **33**, № 1, 39 – 57 (1990).
10. Y. Hino, S. Murakami, *Stability properties of linear Volterra integro-differential equations in a Banach space*, Funkcial. Ekvac., **48**, № 3, 367 – 392 (2005).
11. C. Tunç, *Stability and boundedness in Volterra integro-differential equations with delay*, Dynam. Systems and Appl., **26**, № 1, 121 – 130 (2017).
12. M. N. Islam, M. M. G. Al-Eid, *Boundedness and stability in nonlinear Volterra integro-differential equations*, Panamer. Math. J., **14**, № 3, 49 – 63 (2004).
13. N. N. Krasovskii, *Stability of motion. Applications of Lyapunov's second method to differential systems and equations with delay* (Translated by J. L. Brenner), Stanford Univ. Press, Stanford, Calif. (1963).
14. S. G. Krein, I. V. Sapronov, *One class of solutions of Volterra equation with regular singularity*, Ukr. Mat. Zh., **49**, № 3, 424 – 432 (1997); *English translation: Ukr. Math. J.*, **49**, № 3, 467 – 476 (1998).
15. V. Lakshmikantham, M. Rama Mohana Rao, *Theory of integro-differential equations*, Stability and Control: Theory, Methods and Applications, vol. 1, Gordon and Breach Sci. Publ., Lausanne (1995).
16. Yu. S. Mishura, *Existence of solutions of abstract Volterra equations in a Banach space and its subsets*, Ukr. Mat. Zh., **52**, № 5, 648 – 657 (2000); *English translation:* Ukr. Math. J., **52**, № 5, 741 – 753 (2001).

17. Y. Raffoul, *Boundedness in nonlinear functional differential equations with applications to Volterra integro-differential equations*, J. Integral Equat. Appl., **16**, № 4, 375 – 388 (2004).
18. Y. Raffoul, *Exponential stability and instability in finite delay nonlinear Volterra integro-differential equations*, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal., **20**, № 1, 95 – 106 (2013).
19. R. Reissig, G. Sansone, R. Conti, *Nonlinear differential equations of higher order*, Noordhoff Int. Publ., Leyden (1974).
20. A. M. Samoilenko, N. A. Perestyuk, *Impulsive differential equations. With a preface by Yu. A. Mitropol'skii and a supplement by S. I. Trofimchuk*, World Sci. Ser. Nonlinear Sci. Ser. A Monographs and Treatises, vol. 14, World Sci. Publ. Co., Inc., River Edge, NJ (1995).
21. A. M. Samoilenko, O. A. Boichuk, S. A. Krivosheya, *Boundary value problems for systems of linear integro-differential equations with a degenerate kernel*, Ukr. Mat. Zh., **48**, № 11, 1576 – 1579 (1996); *English translation:* Ukr. Math. J., **48**, № 11, 1785 – 1789 (1997).
22. C. Tunç, *Properties of solutions to Volterra integro-differential equations with delay*, Appl. Math. Inf. Sci., **10**, № 5, 1775 – 1780 (2016).
23. C. Tunç, *Qualitative properties in nonlinear Volterra integro-differential equations with delay*, J. Taibah Univ. Sci., **11**, № 2, 309 – 314 (2017).
24. C. Tunç, *Asymptotic stability and boundedness criteria for nonlinear retarded Volterra integro-differential equations*, J. King Saud Univ.-Sci., **30**, № 4, 3531 – 3536 (2018).
25. C. Tunç, O. Tunç, *On behaviors of functional Volterra integro-differential equations with multiple time-lags*, J. Taibah Univ. Sci., **12**, № 2, 173 – 179 (2018).
26. C. Tunç, O. Tunç, *New results on the stability, integrability and boundedness in Volterra integro-differential equations*, Bull. Comput. Appl. Math., **6**, № 1, 41 – 58 (2018).
27. C. Tunç, O. Tunç, *New qualitative criteria for solutions of Volterra integro-differential equations*, Arab J. Basic and Appl. Sci., **25**, № 3, 158 – 165 (2018).
28. C. Tunç, O. Tunç, *A note on the qualitative analysis of Volterra integro-differential equations*, J. Taibah Univ. Sci., **13**, № 1, 490 – 496 (2019).
29. J. Vanualailai, S. Nakagiri, *Stability of a system of Volterra integro-differential equations*, J. Math. Anal. and Appl., **281**, № 2, 602 – 619 (2003).
30. Ke Wang, *Uniform asymptotic stability in functional-differential equations with infinite delay*, Ann. Different. Equat., **9**, № 3, 325 – 335 (1993).
31. Q. Wang, *The stability of a class of functional differential equations with infinite delays*, Ann. Different. Equat., **16**, № 1, 89 – 97 (2000).
32. A. M. Wazwaz, *Linear and nonlinear integral equations. Methods and applications*, Higher Education Press, Beijing; Springer, Heidelberg (2011).
33. Anshi Xu, *Uniform asymptotic stability in functional-differential equations with infinite delay*, Chinese Sci. Bull., **43**, № 12, 1000 – 1003 (1998).
34. B. Zhou, A. V. Egorov, *Razumikhin and Krasovskii stability theorems for time-varying time-delay systems*, Automatica J. IFAC, **71**, 281 – 291 (2016).
35. M. Bohner, O. Tunç, C. Tunç, *Qualitative analysis of Caputo fractional integro-differential equations with constant delays*, Comput. Appl. Math., **40**, № 6, Paper 214 (2021).
36. C. Tunç, A. K. Golmankhaneh, *On stability of a class of second alpha-order fractal differential equations*, AIMS Mathematics, **5**, № 3, 2126 – 2142 (2020).
37. O. Tunç, *Stability, instability, boundedness and integrability of solutions of a class of integro-delay differential equations*, J. Nonlinear and Convex Anal., **23**, № 4, 801 – 819 (2022).
38. C. Tunç, O. Tunç, *New results on the qualitative analysis of integro-differential equations with constant time-delay*, J. Nonlinear and Convex Anal., **23**, № 3, 435 – 448 (2022).
39. C. Tunç, O. Tunç, *On the stability, integrability and boundedness analyses of systems of integro-differential equations with time-delay retardation*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM, **115**, № 3, Paper 115 (2021).
40. O. Tunç, C. Tunç, *Solution estimates to Caputo proportional fractional derivative delay integro-differential equations*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM, **117**, № 1, Paper 12 (2023).

Received 24.04.20