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BERNSTEIN INEQUALITY FOR MULTIVARIATE FUNCTIONS WITH SMOOTH FOURIER IMAGES[2]

НЕРІВНІСТЬ БЕРНШТЕЙНА ДЛЯ ФУНКЦІЙ БАГАТЬОХ ЗМІННИХ З ГЛАДКИМИ ЗОБРАЖЕННЯМИ ФУР'Є

Let K be a compact set in $\mathbb{R}^n$ with (O)-property and let $1 \leq p \leq \infty$. Then there exists a constant $C_K < \infty$ independent of f and α such that

$$\|D^\alpha f\|_p \leq C_K \sup_{\xi \in K} |\xi^\alpha| \, \|f\|_{\mathcal{H}_{p,K,3}}$$

for all $\alpha \in \mathbb{Z}_+^n$ and $f \in \mathcal{H}_{p,K,3}$, where $\mathcal{H}_{p,K,3} = \big\{ f \in L^p(\mathbb{R}^n) : \operatorname{supp} \widehat{f} \subset K, D^{(3,3,\ldots,3)} \widehat{f} \in C(\mathbb{R}^n) \big\}$, $\|f\|_{\mathcal{H}_{p,K,3}} = \big\| D^{(3,3,\ldots,3)} \widehat{f} \big\|_\infty$, and $\widehat{f}$ is the Fourier transform of f. Note that K is said to have the (O)-property if there exists a constant $C > 0$ such that

$$\sup_{\mathbf{x} \in K} |\mathbf{x}^{\alpha + e_j}| \geq C \sup_{\mathbf{x} \in K} |\mathbf{x}^\alpha|$$

for all $\alpha \in \mathbb{Z}_+^n$ and $j = 1, 2, \ldots, n$.

Нехай K — компактна множина в $\mathbb{R}^n$, що має (O)-властивість і $1 \leq p \leq \infty$. Тоді існує стала $C_K < \infty$, незалежна від f та α, така, що

$$\|D^\alpha f\|_p \leq C_K \sup_{\xi \in K} |\xi^\alpha| \, \|f\|_{\mathcal{H}_{p,K,3}}$$

для всіх $\alpha \in \mathbb{Z}_+^n$ і $f \in \mathcal{H}_{p,K,3}$, де $\mathcal{H}_{p,K,3} = \big\{ f \in L^p(\mathbb{R}^n) : \operatorname{supp} \widehat{f} \subset K, D^{(3,3,\ldots,3)} \widehat{f} \in C(\mathbb{R}^n) \big\}$, $\|f\|_{\mathcal{H}_{p,K,3}} = \big\| D^{(3,3,\ldots,3)} \widehat{f} \big\|_\infty$ і $\widehat{f}$ є перетворенням Фур'є f. Зауважимо, що K має (O)-властивість, якщо існує стала $C > 0$ така, що

$$\sup_{\mathbf{x} \in K} |\mathbf{x}^{\alpha + e_j}| \geq C \sup_{\mathbf{x} \in K} |\mathbf{x}^\alpha|$$

для всіх $\alpha \in \mathbb{Z}_+^n$ і $j = 1, 2, \ldots, n$.

1. Introduction. Let $1 \leq p \leq \infty$, $\sigma > 0$ and K be a compact set in $\mathbb{R}^n$. We put

$$\mathcal{H}_{p,K} = \big\{ f \in L^p(\mathbb{R}^n) : \ \operatorname{sp}(f) \subset K \big\}, \quad \Delta_\sigma = [-\sigma, \sigma],$$

where $\operatorname{sp}(f) := \operatorname{supp} \widehat{f}$ and $\widehat{f} = \mathcal{F} f$ is the Fourier transform of f. One of the most powerful tools in approximation theory is the following Bernstein inequality, which has various applications:

$$\|Df\|_p \leq \sigma \|f\|_p \quad \forall f \in \mathcal{H}_{p,\Delta_\sigma},$$

where σ is the best constant when p is either infinity or 2. It was studied in [1, 6–15, 17]. As a consequence of the last inequality, we have the following, for $m \in \mathbb{N}$:

$$\|D^m f\|_p \leq \sigma^m \|f\|_p \quad \forall f \in \mathcal{H}_{p,\Delta_\sigma}. \tag{1}$$

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Let us discuss for a moment the best constant for Bernstein inequality. It is interesting to see that σ^m is the best constant for (1) for all $p \in [1,\infty]$. Indeed, assume the contrary that there are $m \geq 1, 0 < c < 1$ and $p \in [1,\infty]$ such that

$$\|D^m f\|_p \leq c\sigma^m \|f\|_p$$

for all $f \in \mathcal{H}_{p,\Delta_\sigma}$. Therefore,

$$\|D^{km} f\|_p \leq c^k \sigma^{km} \|f\|_p, \quad k = 1, 2, \ldots.$$

Hence,

$$\limsup_{k\to\infty} \|D^{km} f\|_p^{1/(km)} \leq c^{1/m}\sigma,$$

which contradicts the following result proved in [2]. Let $1 \leq p \leq \infty$ and $D^m f \in L^p(\mathbb{R})$, $m = 0, 1, 2, \ldots$ Then there always exists the following limit:

$$\lim_{m\to\infty} \|D^m f\|_p^{1/m}$$

and

$$\lim_{m\to\infty} \|D^m f\|_p^{1/m} = \sup\{|\xi| : \xi \in \mathrm{sp}(f)\} \tag{2}$$

because we can choose a function $f \in \mathcal{H}_{p,\Delta_\sigma}$ such that $\sup\big\{|\xi| : \xi \in \mathrm{sp}(f)\big\} = \sigma$. Further, inequality (1) still holds for $0 < p < 1$ (see [14]). So, applying (2) proved in [3] for $0 < p \leq \infty$, we conclude that σ is the best constant for all $p \in (0,\infty]$. Since (1) and (2) still hold for Orlicz's and Lorentz's norms, σ^m is also the best constant for these cases (see [4]).

Applying (1) to each variable, we have the following Bernstein inequality for multivariate functions. Let $1 \leq p \leq \infty$, $\alpha \in \mathbb{Z}_+^n$ and K be a compact set in $\mathbb{R}^n$. Then

$$\|D^\alpha f\|_p \leq \sigma^\alpha \|f\|_p \quad \forall f \in \mathcal{H}_{p,K}, \tag{3}$$

where $\sigma_j = \sup\{|x_j| : \mathbf{x} = (x_1, \ldots, x_n) \in K\}$, $\sigma = (\sigma_1, \ldots, \sigma_n)$, $\sigma^\alpha = \sigma_1^{\alpha_1} \ldots \sigma_n^{\alpha_n}$.

So, Bernstein inequality is really good when $n = 1$, but we will see that it is no longer good for the multidimensional case. This can be seen by the following example. We put

$$G := \big\{(x,y) \in \mathbb{R}^2 : |xy| \leq 1, |x| \leq 2, |y| \leq 2\big\}.$$

Using (3), one has, for all $f \in \mathcal{H}_{p,G}$,

$$\|D^{(m,m)} f\|_p \leq 2^{2m} \|f\|_p, \quad m = 1, 2, \ldots. \tag{4}$$

On the other hand, it was proved in [3] the following result. Let $0 < p \leq \infty$, $f \in L^p(\mathbb{R}^n)$ and $\mathrm{sp}(f)$ be compact. Then

$$\lim_{|\alpha|\to\infty} \Big(\|D^\alpha f\|_p / \sup_{\xi\in\mathrm{sp}(f)} |\xi^\alpha|\Big)^{1/|\alpha|} = 1. \tag{5}$$

We choose a function $f \in \mathcal{H}_{p,G}$ such that $|xy| = 1$ for some point $(x,y) \in G$ and apply (5) for it to obtain $\|D^{(m,m)} f\|_p^{1/(2m)} \to 1$ when $m \to \infty$, which together with (4) shows that, in general, estimation (3) is rough. So, when $n \geq 2$, to evaluate $\|D^\alpha f\|_p$ we need to find alternative inequalities for (3) and this is our aim.

In this paper, we present a class of compacts in $\mathbb{R}^n$ called compact sets that have (O)-property and obtain the following estimation for multivariate functions. If a compact set K has (O)-property, $f \in L^p(\mathbb{R}^n)$ with a smooth enough Fourier image and $\operatorname{sp}(f) \subset K$. Then there exists a constant $C < \infty$ such that

$$\|D^\alpha f\|_p \leq C \sup_{\xi \in K} |\xi^\alpha|$$

for all $\alpha \in \mathbb{Z}_+^n$.

Note that the hyperbolic cross G mentioned above, every rectangular in $\mathbb{R}^n$ and each compact set $\neq 0$ in $\mathbb{R}$ have (O)-property. Although our proofs in this paper also work with $n = 1$, we will assume that $n \geq 2$ because our results are only meaningful then.

Notations. Let $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, e_j be the unit vector in $\mathbb{R}^n$ such that its j^{th} coordinate equals 1, $D = (D_1, D_2, \ldots, D_n), D_j = \partial/\partial x_j$ for $j = 1, 2, \ldots, n$, $D^\alpha = D_1^{\alpha_1} \ldots D_n^{\alpha_n}$, $\mathbf{x}^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$, $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$ for $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{Z}_+^n, \mathbf{x} = (x_1, x_2, \ldots, x_n)$, $0^0 = 1$, $\dfrac{1}{0} = \infty$ and $\mathcal{S}'(\mathbb{R}^n)$ be the space of tempered distributions. The Fourier transform of a function $f \in L^1(\mathbb{R}^n)$ is given by

$$\widehat{f}(\mathbf{x}) = \int\limits_{\mathbb{R}^n} e^{-i\mathbf{x}\mathbf{y}} f(\mathbf{y}) d\mathbf{y},$$

where $\mathbf{x}\mathbf{y} = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n$.

The Fourier transform of a tempered distribution f is defined via the formula

$$\langle \mathcal{F}f, \varphi \rangle = \langle f, \mathcal{F}\varphi \rangle, \quad \varphi \in \mathcal{S}(\mathbb{R}^n).$$

Denote by $\mu(K)$ the Lebesgue measure of a compact set $K \subset \mathbb{R}^n$. We define

$$\mathcal{H}_{p,K,3} = \left\{ f \in \mathcal{H}_{p,K} : D^{(3,3,\ldots,3)} \widehat{f} \in C(\mathbb{R}^n) \right\}$$

with the norm

$$\|f\|_{\mathcal{H}_{p,K,3}} = \left\| D^{(3,3,\ldots,3)} \widehat{f} \,\right\|_\infty.$$

It should be noticed that if $f \in \mathcal{H}_{p,K,3}$ then generalized derivatives $D^\vartheta \widehat{f}$ belong to $C(\mathbb{R}^n)$ for all $\vartheta \leq (3, 3, \ldots, 3)$. Indeed, first we prove this for $\vartheta = (2, 3, 3, \ldots, 3)$. Let $g \in \mathcal{S}'(\mathbb{R})$. Recall that [5, 16] the tempered distribution Ig is termed a primitive of g if $D(Ig) = g$, that is, $\langle Ig, \varphi' \rangle = -\langle g, \varphi \rangle \quad \forall \varphi \in \mathcal{S}(\mathbb{R})$. Moreover, if A, B are two primitives of g then $A - B$ is a constant. Fix $x_2, x_3, \ldots, x_n$ and define

$$J_\vartheta \widehat{f}(\mathbf{x}) = \int\limits_{-\infty}^{x_1} D^{(3,3,\ldots,3)} \widehat{f}(t, x_2, x_3, \ldots, x_n) dt,$$

where $\vartheta = (2, 3, 3, \ldots, 3)$. Clearly, $J_\vartheta \widehat{f}$ is a primitive of $D^{(3,3,\ldots,3)} \widehat{f}$ and $J_\vartheta \widehat{f}(\mathbf{x}) = 0$ for all $x_1 < -\sup_{\mathbf{z} \in K} |\mathbf{z}|$. On the other hand, $D^\vartheta \widehat{f}$ is also a primitive of $D^{(3,3,\ldots,3)} \widehat{f}$. So, $J_\vartheta \widehat{f}(\mathbf{x}) = D^\vartheta \widehat{f}(\mathbf{x}) + C$, where C is independent of x_1. Then it follows from $J_\vartheta \widehat{f}(\mathbf{x}) - D^\vartheta \widehat{f}(\mathbf{x}) = 0 \ \forall x_1 < -\sup_{\mathbf{z} \in K} |\mathbf{z}|$

that $D^{\vartheta}\widehat{f}(\mathbf{x}) = J_{\vartheta}\widehat{f}(\mathbf{x}) \in C(\mathbb{R}^n)$. Similarly,

$$D^{\gamma}\widehat{f}(\mathbf{x}) = \int\limits_{-\infty}^{x_j} D^{\gamma+e_j}\widehat{f}(x_1, x_2, \ldots, x_{j-1}, t, x_{j+1}, \ldots, x_n)dt \in C(\mathbb{R}^n)$$

for all $j = 1, 2, \ldots, n$ and $\gamma \in \mathbb{Z}_+^n$, $\gamma \leq (3, 3, \ldots, 3) - e_j$.

2. Bernstein inequality for functions with smooth Fourier images.

Definition 2.1. *We say that a compact set $K \subset \mathbb{R}^n$ has (O)-property if there exists a constant $C > 0$ such that*

$$\sup_{\mathbf{x}\in K} |\mathbf{x}^{\alpha+e_j}| \geq C \sup_{\mathbf{x}\in K} |\mathbf{x}^{\alpha}|$$

for all $\alpha \in \mathbb{Z}_+^n$ and $j = 1, 2, \ldots, n$.

From the definition, we have the following properties of the sets having (O)-property:

If $K_1, K_2, \ldots, K_m$ have (O)-property, then $\cup_{k=1}^m K_k$ has (O)-property (but it is possible that $\cap_{k=1}^m K_k$ does not have (O)-property).

If K has (O)-property, K_1 is an open set satisfying $K_1 \subset \lambda K$ for some $\lambda \in (0, 1)$, then $K \backslash K_1$ has (O)-property.

If H is a compact and $x_j \neq 0$ for any $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in H$ and $j = 1, 2, \ldots, n$, then H has (O)-property.

A compact H_1 has (O)-property if it contains a point $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ such that $|x_j| \leq |a_j|$ for all $\mathbf{x} \in H_1$ and $j = 1, 2, \ldots, n$.

Every rectangular has (O)-property.

The set K, which is defined as follows:

$$K = \big\{\mathbf{x} \in \mathbb{R}^n : |x_\ell x_j| \leq C_{\ell,j}, |x_j| \leq C_j \ \forall j,\ \ell = 1, \ldots, n\big\}$$

has (O)-property, where $C_{\ell,j}, C_j > 0$ for all $j, \ell = 1, \ldots, n$.

The closed ball $B[\mathbf{a}, R]$ and the torus $T[\mathbf{a}, r, R] = \Big\{\mathbf{x} \in \mathbb{R}^n : r \leq \Big(\sum_{j=1}^n (x_j - a_j)^2\Big)^{1/2} \leq \leq R\Big\}$, $0 < r < R$, $\prod_{j=1}^n a_j \neq 0$, have (O)-property, but, for $n > 1$, each ball $B[0, R]$ and the torus $T[0, r, R] = \Big\{\mathbf{x} \in \mathbb{R}^n : r \leq \Big(\sum_{j=1}^n x_j^2\Big)^{1/2} \leq R\Big\}$, $0 < r < R$, do not have (O)-property.

Any polygon in $\mathbb{R}^2$, which has no vertices located on the coordinate axes, has (O)-property.

Now, we state the main theorem of this paper.

Theorem 2.1. *Let K be a compact set in $\mathbb{R}^n$ having (O)-property and $1 \leq p \leq \infty$. Then there exists a constant $C_K < \infty$ independent of f, α such that*

$$\|D^{\alpha} f\|_p \leq C_K \sup_{\xi\in K} |\xi^{\alpha}| \|f\|_{\mathcal{H}_{p,K,3}} \tag{6}$$

for all $\alpha \in \mathbb{Z}_+^n$ and all $f \in \mathcal{H}_{p,K,3}$.

In the sequel we need the following result.

Lemma 2.1 (Nikolskii inequality [10])**.** *Let $0 < q \leq p \leq \infty$, $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) \in \mathbb{R}_+^n$, $f \in L^q(\mathbb{R}^n)$ and $\mathrm{sp}(f) \subset [-\sigma_1, \sigma_1] \times \ldots \times [-\sigma_n, \sigma_n]$. Then $f \in L^p(\mathbb{R}^n)$ and*

$$\|f\|_p \leq C_{p,q} \left(\prod_{j=1}^n \sigma_j\right)^{\frac{1}{q}-\frac{1}{p}} \|f\|_q. \tag{7}$$

***Proof of Theorem* 2.1.** For each $\alpha \in \mathbb{Z}_+^n$, we define $\overline{\alpha} = (\overline{\alpha}_1, \ldots, \overline{\alpha}_n)$, $\underline{\alpha} = (\underline{\alpha}_1, \ldots, \underline{\alpha}_n) \in \mathbb{R}_+^n$ and two sequences of functions of n variables $\{\Phi_j(\mathbf{x})\}_{j=0}^n$, $\{\Psi_j(\mathbf{x})\}_{j=0}^{n-1}$ as follows:

$$\overline{\alpha}_j = \frac{\alpha_j + 1}{\alpha_j + 2}, \quad \underline{\alpha}_j = \alpha_j + 1,$$

$$\Phi_0(\mathbf{x}) = \widehat{f}(\mathbf{x}), \qquad \Phi_j(\mathbf{x}) = \Psi_{j-1}(\mathbf{x}) - \Phi_{j-1}(\mathbf{x}),$$

$$\Psi_{j-1}(\mathbf{x}) = \frac{1}{\overline{\alpha}_j}\Phi_{j-1}(x_1, x_2, \ldots, x_{j-1}, \overline{\alpha}_j x_j, x_{j+1}, \ldots, x_n)$$

for all $j = 1, 2, \ldots, n$. Clearly,

$$D^\beta \Psi_{j-1}(\mathbf{x}) = \big(\overline{\alpha}_j\big)^{\beta_j - 1}(D^\beta \Phi_{j-1})(x_1, x_2, \ldots, x_{j-1}, \overline{\alpha}_j x_j, x_{j+1}, \ldots, x_n) \tag{8}$$

and

$$\mathcal{F}(\Phi_j(\mathbf{x})\mathbf{x}^\alpha) = \mathcal{F}(\Psi_{j-1}(\mathbf{x})\mathbf{x}^\alpha) - \mathcal{F}(\Phi_{j-1}(\mathbf{x})\mathbf{x}^\alpha)$$

for all $\beta \in \mathbb{Z}_+^n$ and $j = 1, 2, \ldots, n$. Hence,

$$\big\|\mathcal{F}(\Phi_j(\mathbf{x})\mathbf{x}^\alpha)\big\|_1 \geq \big\|\mathcal{F}(\Psi_{j-1}(\mathbf{x})\mathbf{x}^\alpha)\big\|_1 - \big\|\mathcal{F}(\Phi_{j-1}(\mathbf{x})\mathbf{x}^\alpha)\big\|_1 \quad \forall j = 1, 2, \ldots, n. \tag{9}$$

We see that

$$\mathcal{F}\big(\Psi_{j-1}(\mathbf{x})\mathbf{x}^\alpha\big)(\mathbf{y}) = \frac{1}{\overline{\alpha}_j}\mathcal{F}\big(\Phi_{j-1}(x_1, \ldots, x_{j-1}, \overline{\alpha}_j x_j, x_{j+1}, \ldots, x_n)\mathbf{x}^\alpha\big)(\mathbf{y}) =$$

$$= \frac{1}{\overline{\alpha}_j^{\alpha_j+1}}\mathcal{F}\big(\Phi_{j-1}(x_1, \ldots, x_{j-1}, \overline{\alpha}_j x_j, x_{j+1}, \ldots, x_n)(x_1, \ldots, x_{j-1}, \overline{\alpha}_j x_j, x_{j+1}, \ldots, x_n)^\alpha\big)(\mathbf{y}) =$$

$$= \frac{1}{\overline{\alpha}_j^{\alpha_j+2}}\mathcal{F}(\Phi_{j-1}(\mathbf{x})\mathbf{x}^\alpha)\left(y_1, y_2, \ldots, y_{j-1}, \frac{y_j}{\overline{\alpha}_j}, y_{j+1}, \ldots, y_n\right).$$

Therefore,

$$\Big\|\mathcal{F}(\Psi_{j-1}(\mathbf{x})\mathbf{x}^\alpha))\Big\|_1 = \frac{1}{\overline{\alpha}_j^{\alpha_j+1}}\Big\|\mathcal{F}(\Phi_{j-1}(\mathbf{x})\mathbf{x}^\alpha))\Big\|_1 =$$

$$= \left(1 + \frac{1}{\alpha_j + 1}\right)^{\alpha_j+1}\Big\|\mathcal{F}(\Phi_{j-1}(\mathbf{x})\mathbf{x}^\alpha)\Big\|_1 \quad \forall j = 1, 2, \ldots, n. \tag{10}$$

Observe that

$$\left(1 + \frac{1}{\alpha_j + 1}\right)^{\alpha_j+1} \geq 2 \quad \forall \alpha_j \in \mathbb{Z}_+. \tag{11}$$

Combining (10) and (11), we have

$$\big\|\mathcal{F}(\Psi_{j-1}(\mathbf{x})\mathbf{x}^\alpha)\big\|_1 \geq 2\big\|\mathcal{F}(\Phi_{j-1}(\mathbf{x})\mathbf{x}^\alpha)\big\|_1 \quad \forall \alpha \in \mathbb{Z}_+^n \quad \forall j = 1, 2, \ldots, n. \tag{12}$$

From (9) and (12), we obtain

$$\big\|\mathcal{F}(\Phi_j(\mathbf{x})\mathbf{x}^\alpha)\big\|_1 \geq \big\|\mathcal{F}(\Phi_{j-1}(\mathbf{x})\mathbf{x}^\alpha)\big\|_1 \quad \forall\alpha\in\mathbb{Z}_+^n \quad \forall j=1,2,\ldots,n. \tag{13}$$

Using (13), we deduce that

$$\big\|\mathcal{F}(\Phi_n(\mathbf{x})\mathbf{x}^\alpha)\big\|_1 \geq \big\|\mathcal{F}(\Phi_0(\mathbf{x})\mathbf{x}^\alpha)\big\|_1 = \big\|\mathcal{F}(\widehat{f}(\mathbf{x})\mathbf{x}^\alpha)\big\|_1 \quad \forall\alpha\in\mathbb{Z}_+^n. \tag{14}$$

It is known that, for all $\alpha\in\mathbb{Z}_+^n$,

$$(2\pi)^n D^\alpha f(\mathbf{y}) = i^{|\alpha|}\mathcal{F}\big(\widehat{f}(\mathbf{x})\mathbf{x}^\alpha\big)(-\mathbf{y})$$

and then

$$\big\|\mathcal{F}(\widehat{f}(\mathbf{x})\mathbf{x}^\alpha)\big\|_1 = (2\pi)^n\|D^\alpha f\|_1 \quad \forall\alpha\in\mathbb{Z}_+^n.$$

So, it follows from (14) that

$$\big\|\mathcal{F}(\Phi_n(\mathbf{x})\mathbf{x}^\alpha)\big\|_1 \geq (2\pi)^n\|D^\alpha f\|_1 \quad \forall\alpha\in\mathbb{Z}_+^n. \tag{15}$$

Next, we estimate $\big\|\mathcal{F}(\Phi_n(\mathbf{x})\mathbf{x}^\alpha)\big\|_1$. To do that, we define

$$\mathcal{B} = \big\{\beta\in\mathbb{Z}_+^n : \beta_j\in\{0,1,2\} \text{ for all } j=1,2,\ldots,n\big\},$$

$$M_1 = \sup_{\mathbf{y}\in K} 2^n\|\mathbf{y}\| + 1, \qquad M_{2,f} = \sum_{\alpha\leq(3,3,\ldots,3)} \|D^\alpha\widehat{f}\|_\infty,$$

where $\|\mathbf{y}\| = \sqrt{\sum_{j=1}^n y_j^2}$, $\mathbf{y} = (y_1,\ldots,y_n)$. Since $f\in\mathcal{H}_{p,K,3}$ and K is compact, we have $0<M_1,\ M_{2,f}<\infty$. From

$$\Phi_j(\mathbf{x}) = \frac{1}{\overline{\alpha}_j}\Phi_{j-1}(x_1,x_2,\ldots,x_{j-1},\overline{\alpha}_j x_j,x_{j+1},\ldots,x_n) - \Phi_{j-1}(\mathbf{x})$$

we see that $\mathbf{x}\in\operatorname{supp}\Phi_{j-1}$ or $(x_1,x_2,\ldots,x_{j-1},\overline{\alpha}_j x_j,x_{j+1},\ldots,x_n)\in\operatorname{supp}\Phi_{j-1}$ for any $\mathbf{x}\in$ $\in\operatorname{supp}\Phi_j$, which imply

$$\mu(\operatorname{supp}\Phi_j) \leq (1+1/\overline{\alpha}_j)\mu(\operatorname{supp}\Phi_{j-1}),$$

$$\sup_{\mathbf{x}\in\operatorname{supp}\Phi_j}\|\mathbf{x}\| \leq \max\big\{1,1/\overline{\alpha}_j\big\}\sup_{\mathbf{x}\in\operatorname{supp}\Phi_{j-1}}\|\mathbf{x}\|,$$

$$\sup_{\mathbf{x}\in\operatorname{supp}\Phi_j}|\mathbf{x}^\alpha| \leq \max\big\{1,1/(\overline{\alpha}_j^{\alpha_j})\big\}\sup_{\mathbf{x}\in\operatorname{supp}\Phi_{j-1}}|\mathbf{x}^\alpha|$$

for all $j=1,2,\ldots,n$. Applying these to $j=1,2,\ldots,n$ and using $1/2\leq\overline{\alpha}_j<1$, we conclude that

$$\mu(\operatorname{supp}\Phi_n) \leq 3^n\mu(\operatorname{supp}\Phi_0) = 3^n\mu(\operatorname{sp}(f)) \leq 3^n\mu(K), \tag{16}$$

$$\sup_{\mathbf{x}\in\operatorname{supp}\Phi_j}\|\mathbf{x}\| \leq 2^n\sup_{\mathbf{x}\in\operatorname{supp}\Phi_0}\|\mathbf{x}\| = 2^n\sup_{\mathbf{x}\in\operatorname{sp}(f)}\|\mathbf{x}\| \leq M_1 \tag{17}$$

and

$$\sup_{\mathbf{x}\in\operatorname{supp}\Phi_n}|\mathbf{x}^\alpha| \leq 3^n\sup_{\mathbf{x}\in\operatorname{supp}\Phi_0}|\mathbf{x}^\alpha| = 3^n\sup_{\mathbf{x}\in\operatorname{sp}(f)}|\mathbf{x}^\alpha| \leq 3^n\sup_{\mathbf{x}\in K}|\mathbf{x}^\alpha|. \tag{18}$$

Let $\beta\in\mathcal{B},\ j\in\{1,2,\ldots,n\}$. To estimate $\big|D^\beta\Phi_j(\mathbf{x})\big|$ we divide it into three cases.

***Case* 1:** $\beta_j = 0$. From (8) and the definition of $\Phi_j(\mathbf{x})$, we have

$$\left|D^\beta \Phi_j(\mathbf{x})\right| = \left|\frac{1}{\overline{\alpha}_j}(D^\beta \Phi_{j-1})(x_1, x_2, \ldots, x_{j-1}, \overline{\alpha}_j x_j, x_{j+1}, \ldots, x_n) - D^\beta \Phi_{j-1}(\mathbf{x})\right| \leq$$

$$\leq \left|\left(\frac{1}{\overline{\alpha}_j} - 1\right) D^\beta \Phi_{j-1}(x_1, x_2, \ldots, x_{j-1}, \overline{\alpha}_j x_j, x_{j+1}, \ldots, x_n)\right| +$$

$$+ \left|D^\beta \Phi_{j-1}(x_1, x_2, \ldots, x_{j-1}, \overline{\alpha}_j x_j, x_{j+1}, \ldots, x_n) - D^\beta \Phi_{j-1}(\mathbf{x})\right|$$

and then

$$\left|D^\beta \Phi_j(\mathbf{x})\right| \leq \left|\left(\frac{1}{\overline{\alpha}_j} - 1\right) D^\beta \Phi_{j-1}(x_1, x_2, \ldots, x_{j-1}, \overline{\alpha}_j x_j, x_{j+1}, \ldots, x_n)\right| +$$

$$+ \left|\int\limits_{x_j}^{\overline{\alpha}_j x_j} D^{\beta + e_j} \Phi_{j-1}(x_1, x_2, \ldots, x_{j-1}, t, x_{j+1}, \ldots, x_n) dt\right| \leq$$

$$\leq \left|\frac{1}{\overline{\alpha}_j} - 1\right| \left\|D^\beta \Phi_{j-1}\right\|_\infty + \left|(\overline{\alpha}_j - 1) x_j\right| \left\|D^{\beta + e_j} \Phi_{j-1}\right\|_\infty.$$

Combining this, $|\overline{\alpha}_j - 1| = (\alpha_j + 2)^{-1} \leq (\alpha_j + 1)^{-1}$, $\left|\frac{1}{\overline{\alpha}_j} - 1\right| = (\alpha_j + 1)^{-1}$ and $|x_j| \leq M_1$ $\forall \mathbf{x} \in \operatorname{supp} \Phi_j$, we get

$$\left|D^\beta \Phi_j(\mathbf{x})\right| \leq \frac{M_1 \left(\|D^{\beta + e_j} \Phi_{j-1}\|_\infty + \|D^\beta \Phi_{j-1}\|_\infty\right)}{\alpha_j + 1} \quad \forall \mathbf{x} \in \operatorname{supp} \Phi_j. \tag{19}$$

***Case* 2:** $\beta_j = 1$. Using (8) and the definition of $\Phi_j(\mathbf{x})$, we obtain

$$\left|D^\beta \Phi_j(\mathbf{x})\right| = \left|D^\beta \Phi_{j-1}(x_1, x_2, \ldots, x_{j-1}, \overline{\alpha}_j x_j, x_{j+1}, \ldots, x_n) - D^\beta \Phi_{j-1}(\mathbf{x})\right|$$

and then

$$\left|D^\beta \Phi_j(\mathbf{x})\right| \leq \left|\int\limits_{x_j}^{\overline{\alpha}_j x_j} D^{\beta + e_j} \Phi_{j-1}(x_1, x_2, \ldots, x_{j-1}, t, x_{j+1}, \ldots, x_n) dt\right| \leq$$

$$\leq \left|(\overline{\alpha}_j - 1) x_j\right| \left\|D^{\beta + e_j} \Phi_{j-1}\right\|_\infty,$$

which implies

$$\left|D^\beta \Phi_j(\mathbf{x})\right| \leq \frac{1}{\alpha_j + 2} |x_j| \left\|D^{\beta + e_j} \Phi_{j-1}\right\|_\infty \quad \forall \mathbf{x} \in \mathbb{R}^n.$$

Therefore, by (17) and $|x_j| \leq M_1$ $\forall \mathbf{x} \in \operatorname{supp} \Phi_j$, we get

$$\left|D^\beta \Phi_j(\mathbf{x})\right| \leq \frac{M_1}{\alpha_j + 1} \left\|D^{\beta + e_j} \Phi_{j-1}\right\|_\infty \quad \forall \mathbf{x} \in \operatorname{supp} \Phi_j. \tag{20}$$

***Case* 3:** $\beta_j = 2$. We have

$$\big|D^\beta\Phi_j(\mathbf{x})\big| = \overline{\alpha}_j(D^\beta\Phi_{j-1})(x_1,x_2,\ldots,x_{j-1},\overline{\alpha}_jx_j,x_{j+1},\ldots,x_n) - D^\beta\Phi_{j-1}(\mathbf{x}) \le$$

$$\le \Big|(\overline{\alpha}_j-1)\big((D^\beta\Phi_{j-1})(x_1,x_2,\ldots,x_{j-1},\overline{\alpha}_jx_j,x_{j+1},\ldots,x_n) - D^\beta\Phi_{j-1}(\mathbf{x})\big)\Big| +$$

$$+\Big|(D^\beta\Phi_{j-1})(x_1,x_2,\ldots,x_{j-1},\overline{\alpha}_jx_j,x_{j+1},\ldots,x_n) - D^\beta\Phi_{j-1}(\mathbf{x})\Big|$$

and then

$$\big|D^\beta\Phi_j(\mathbf{x})\big| \le \big|(\overline{\alpha}_j-1)D^\beta\Phi_{j-1}(\mathbf{x})\big| + \left|\int\limits_{x_j}^{\overline{\alpha}_jx_j} D^{\beta+e_j}\Phi_{j-1}(t,x_2,\ldots,x_n)dt\right| \le$$

$$\le \big|(\overline{\alpha}_j-1)\big|\big\|D^\beta\Phi_{j-1}\big\|_\infty + |(\overline{\alpha}_j-1)x_j|\big\|D^{\beta+e_j}\Phi_{j-1}\big\|_\infty.$$

Hence,

$$\big|D^\beta\Phi_j(\mathbf{x})\big| \le \frac{M_1\Big(\big\|D^\beta\Phi_{j-1}\big\|_\infty + \|D^{\beta+e_j}\Phi_{j-1}\|_\infty\Big)}{\alpha_j+1} \tag{21}$$

for all $\mathbf{x} \in \operatorname{supp}\Phi_j$. Combining (19)–(21), we conclude that, for all $\beta \in \mathcal{B}$, $j \in \{1,2,\ldots,n\}$,

$$\|D^\beta\Phi_j\|_\infty \le \frac{M_1\Big(\big\|D^\beta\Phi_{j-1}\big\|_\infty + \|D^{\beta+e_j}\Phi_{j-1}\|_\infty\Big)}{\alpha_j+1}. \tag{22}$$

Applying (22) to $j = 1,2,\ldots,n$, we obtain the following estimate for $\beta \in \mathcal{B}$:

$$\|D^\beta\Phi_n\|_\infty \le \frac{M_1^n\sum_{\gamma\le(1,1,\ldots,1)}\|D^{\beta+\gamma}\Phi_0\|_\infty}{\prod_{j=1}^n(\alpha_j+1)}$$

and then

$$\big\|D^\beta\Phi_n\big\|_\infty \le \frac{M_1^nM_{2,f}}{\prod_{j=1}^n(\alpha_j+1)}. \tag{23}$$

Note that

$$\sup_{\mathbf{y}\in\mathbb{R}^n}\big|\mathbf{y}^\beta\mathcal{F}\big(\Phi_n(\mathbf{x})\mathbf{x}^\alpha\big)(\mathbf{y})\big| = \sup_{\mathbf{y}\in\mathbb{R}^n}\big|\mathcal{F}\big(D^\beta\big(\Phi_n(\mathbf{x})\mathbf{x}^\alpha\big)\big)(\mathbf{y})\big| \le \big\|D^\beta\big(\Phi_n(\mathbf{x})\mathbf{x}^\alpha\big)\big\|_1. \tag{24}$$

From (16), (24) and the fact that $\big\|D^\beta\big(\Phi_n(\mathbf{x})\mathbf{x}^\alpha\big)\big\|_1 \le \mu(\operatorname{supp}\Phi_n)\big\|D^\beta\big(\Phi_n(\mathbf{x})\mathbf{x}^\alpha\big)\big\|_\infty$, we get

$$\sup_{\mathbf{y}\in\mathbb{R}^n}\big|\mathbf{y}^\beta\mathcal{F}\big(\Phi_n(\mathbf{x})\mathbf{x}^\alpha\big)(\mathbf{y})\big| \le 3^n\mu(\operatorname{sp}(f))\big\|D^\beta\big(\Phi_n(\mathbf{x})\mathbf{x}^\alpha\big)\big\|_\infty \tag{25}$$

for all $\beta \in \mathcal{B}$. Using Leibniz's rule, we have

$$D^{\beta}\big(\Phi_n(\mathbf{x})\mathbf{x}^{\alpha}\big) = \sum_{\gamma\le\beta}\binom{\beta}{\gamma} D^{\beta-\gamma}\Phi_n(\mathbf{x}) D^{\gamma}\mathbf{x}^{\alpha}.$$

Hence, it follows from (23) and (24) that, for $\beta\in\mathcal{B}$,

$$\big|D^{\beta}\big(\Phi_n(\mathbf{x})\mathbf{x}^{\alpha}\big)\big| \le \sum_{\gamma\le\beta}\binom{\beta}{\gamma}\big|D^{\beta-\gamma}\Phi_n(\mathbf{x})\big|\big|D^{\gamma}\mathbf{x}^{\alpha}\big| \le \sum_{\gamma\le\beta}\binom{\beta}{\gamma}\frac{M_1^n M_{2,f}}{\prod_{j=1}^n(\alpha_j+1)}|D^{\gamma}\mathbf{x}^{\alpha}|.$$

So, since $|D^{\gamma}\mathbf{x}^{\alpha}| = \left|\prod_{j=1}^n \left(x_j^{\alpha_j-\gamma_j}\alpha_j(\alpha_j-1)\ldots(\alpha_j-\gamma_j+1)\right)\right| \le \left|\prod_{j=1}^n \left(x_j^{\alpha_j-\gamma_j}\alpha_j^{\gamma_j}\right)\right|$, we see that

$$\big|D^{\beta}\big(\Phi_n(\mathbf{x})\mathbf{x}^{\alpha}\big)\big| \le \sum_{\gamma\in\mathcal{B},\gamma\le\alpha}\binom{\beta}{\gamma}\frac{M_1^n M_{2,f}\alpha^{\gamma}}{\prod_{j=1}^n(\alpha_j+1)}|\mathbf{x}^{\alpha-\gamma}|$$

for all $\mathbf{x}\in\mathbb{R}^n$. So,

$$\big\|D^{\beta}\big(\Phi_n(\mathbf{x})\mathbf{x}^{\alpha}\big)\big\|_{\infty} \le \frac{4^n M_1^n M_{2,f}\alpha^{\beta}}{\prod_{j=1}^n(\alpha_j+1)} \sup_{\gamma\in\mathcal{B},\gamma\le\alpha}\ \sup_{\mathbf{x}\in\operatorname{supp}\Phi_n}|\mathbf{x}^{\alpha-\gamma}| \tag{26}$$

for all $\beta\in\mathcal{B}$. From (18), (25) and (26), we obtain

$$\sup_{\mathbf{y}\in\mathbb{R}^n}\big|\mathbf{y}^{\beta}\mathcal{F}\big(\Phi_n(\mathbf{x})\mathbf{x}^{\alpha}\big)(\mathbf{y})\big| \le \frac{M_{3,f}\alpha^{\beta}}{\prod_{j=1}^n(\alpha_j+1)} \sup_{\gamma\in\mathcal{B},\gamma\le\alpha}\ \sup_{\mathbf{x}\in\operatorname{sp}(f)}|\mathbf{x}^{\alpha-\gamma}| \quad \forall\beta\in\mathcal{B}, \tag{27}$$

where $M_{3,f} = 36^n M_1^n M_{2,f}\mu(K)$. Note that

$$\left(\prod_{j=1}^n\big(\alpha_j+1+|y_j|\big)\right)^2 |\mathcal{F}(\Phi_n(\mathbf{x})\mathbf{x}^{\alpha})(\mathbf{y})| \le 2^n\left(\prod_{j=1}^n\big((\alpha_j+1)^2+y_j^2\big)\right)\big|\mathcal{F}(\Phi_n(\mathbf{x})\mathbf{x}^{\alpha})(\mathbf{y})\big| \le$$

$$\le 2^n\sum_{\beta\in\mathcal{B}}\underline{\alpha}^{(2,2,\ldots,2)-\beta}\big|\mathbf{y}^{\beta}\mathcal{F}\big(\Phi_n(\mathbf{x})\mathbf{x}^{\alpha}\big)(\mathbf{y})\big|$$

for all $\mathbf{y}\in\mathbb{R}^n$. Therefore, using (27), we get

$$\sup_{\mathbf{y}\in\mathbb{R}^n}\left(\prod_{j=1}^n\big(\alpha_j+1+|y_j|\big)\right)^2\big|\mathcal{F}(\Phi_n(\mathbf{x})\mathbf{x}^{\alpha})(\mathbf{y})\big| \le$$

$$\le \sum_{\beta\in\mathcal{B}}\left(\frac{M_{3,f}\alpha^{\beta}\underline{\alpha}^{(2,2,\ldots,2)-\beta}}{\prod_{j=1}^n(\alpha_j+1)} \sup_{\gamma\in\mathcal{B},\gamma\le\alpha}\ \sup_{\mathbf{x}\in\operatorname{sp}(f)}|\mathbf{x}^{\alpha-\gamma}|\right).$$

Then it follows from $\alpha^{\beta}\underline{\alpha}^{(2,2,\ldots,2)-\beta} \le \underline{\alpha}^{(2,2,\ldots,2)} = \left(\prod_{j=1}^n(\alpha_j+1)\right)^2$ that

$$\sup_{\mathbf{y}\in\mathbb{R}^n}\left(\prod_{j=1}^n\big(\alpha_j+1+|y_j|\big)\right)^2\big|\mathcal{F}(\Phi_n(\mathbf{x})\mathbf{x}^{\alpha})(\mathbf{y})\big| \le$$

$$\leq 3^n M_{3,f} \left(\prod_{j=1}^{n} (\alpha_j + 1) \right) \sup_{\gamma \in \mathcal{B}, \gamma \leq \alpha} \sup_{\mathbf{x} \in \mathrm{sp}(f)} |\mathbf{x}^{\alpha - \gamma}|. \tag{28}$$

It is easy to see that

$$\left\| \mathcal{F}(\Phi_n(\mathbf{x})\mathbf{x}^\alpha) \right\|_1 \leq \left(\int\limits_{\mathbb{R}^n} \frac{1}{\left(\prod_{j=1}^{n} \left(\alpha_j + 1 + |y_j| \right) \right)^2} d\mathbf{y} \right) \times$$

$$\times \left(\sup_{\mathbf{y} \in \mathbb{R}^n} \left(\prod_{j=1}^{n} \left(\alpha_j + 1 + |y_j| \right) \right)^2 \left| \mathcal{F}(\Phi_n(\mathbf{x})\mathbf{x}^\alpha)(\mathbf{y}) \right| \right).$$

Then it follows from (28) and

$$\int\limits_{\mathbb{R}^n} \frac{1}{\left(\prod_{j=1}^{n} \left(\alpha_j + 1 + |y_j| \right) \right)^2} d\mathbf{y} = \frac{2^n}{\prod_{j=1}^{n} (\alpha_j + 1)}$$

that

$$\left\| \mathcal{F}(\Phi_n(\mathbf{x})\mathbf{x}^\alpha) \right\|_1 \leq 6^n M_{3,f} \sup_{\gamma \in \mathcal{B}, \gamma \leq \alpha} \sup_{\mathbf{x} \in \mathrm{sp}(f)} |\mathbf{x}^{\alpha - \gamma}|. \tag{29}$$

From (15) and (29), we deduce that

$$\|D^\alpha f\|_1 \leq C_{1,K} M_{2,f} \sup_{\gamma \leq (2,2,\ldots,2), \gamma \leq \alpha} \sup_{\xi \in \mathrm{sp}(f)} |\xi^{\alpha - \gamma}| \tag{30}$$

for all $\alpha \in \mathbb{Z}_+^n$, where $C_{1,K} = (432\pi)^n M_1^n \mu(K)$ is independent of f. Moreover, since $\mathrm{sp}(f) \subset K$,

$$M_{2,f} \leq C_{2,K} \|f\|_{\mathcal{H}_{p,K,3}},$$

which together with (30) implies

$$\|D^\alpha f\|_1 \leq C_{1,K} C_{2,K} \sup_{\gamma \leq (2,2,\ldots,2), \gamma \leq \alpha} \sup_{\xi \in \mathrm{sp}(f)} |\xi^{\alpha - \gamma}| \|f\|_{\mathcal{H}_{p,K,3}}. \tag{31}$$

Using Nikolskii inequality and $\mathrm{sp}(f) \subset K$, we obtain that $f \in L^p(\mathbb{R}^n)$ and

$$\|D^\alpha f\|_p \leq C_{p,K} \|D^\alpha f\|_1$$

for all $\alpha \in \mathbb{Z}_+^n$. Combining this with (31), one has

$$\|D^\alpha f\|_p \leq C_{1,K} C_{2,K} C_{p,K} \sup_{\gamma \leq (2,2,\ldots,2), \gamma \leq \alpha} \sup_{\xi \in K} |\xi^{\alpha - \gamma}| \quad \forall \alpha \in \mathbb{Z}_+^n. \tag{32}$$

Because K has (O)-property, there exists a constant $C_{3,K}$ such that

$$\sup_{\gamma \leq (2,2,\ldots,2), \gamma \leq \alpha} \sup_{\xi \in K} |\xi^{\alpha - \gamma}| \leq C_{3,K} \sup_{\mathbf{x} \in K} |\mathbf{x}^\alpha| \tag{33}$$

for all $\alpha \in \mathbb{Z}_+^n$. From (32) and (33), we confirm (6).

Theorem 2.1 is proved.

By Theorem 2.1, we have the following corollary.

***Corollary* 2.1.** *Let K be a compact set in $\mathbb{R}^n$ having (O)-property, $1 \leq p \leq \infty$ and $f \in \mathcal{H}_{p,K,3}$. Then there exists a constant $C_{K,f} < \infty$ such that*

$$\|D^\alpha f\|_p \leq C_{K,f} \sup_{\xi \in K} |\xi^\alpha|$$

for all $\alpha \in \mathbb{Z}_+^n$. In particular,

$$\limsup_{|\alpha| \to \infty} \left(\|D^\alpha f\|_p / \sup_{\xi \in K} |\xi^\alpha| \right)^{1/|\alpha|} \leq 1.$$

***Remark* 2.1.** Theorem 2.1 has the potential to be applied in computational science because one can calculate (evaluate) $\sup_{\xi \in K} |\xi^\alpha|$ and $\|f\|_{\mathcal{H}_{p,K,3}}$ for $f \in \mathcal{H}_{p,K,3}$ while it is virtually impossible to calculate $\|D^\alpha f\|_p$ directly.

Theorem 2.2. *Let K be a compact set in $\mathbb{R}^n$ having (O)-property, $1 \leq p \leq \infty$ and $f \in \mathcal{H}_{p,K,3}$. Then*

$$\lim_{|\alpha| \to \infty} \left(\|D^\alpha f\|_p / \sup_{\xi \in K} |\xi^\alpha| \right) = 0.$$

Proof. For any $\lambda \in \left[\frac{1}{2}, 1\right)$ we define $\psi_\lambda(\mathbf{x}) = f(\lambda \mathbf{x})$ and put $H = \bigcup_{\frac{1}{2} \leq \kappa \leq 1} (\kappa K)$, where $\kappa K = \{\kappa \mathbf{x} \colon \mathbf{x} \in K\}$. Clearly, $\mathrm{sp}(\psi_\lambda) = \lambda\, \mathrm{sp}(f)$. So,

$$\mathrm{sp}\,(\psi_\lambda) \subset \lambda K \subset H. \tag{34}$$

Since $D^\alpha \psi_\lambda(\mathbf{x}) = \lambda^{|\alpha|} D^\alpha f(\lambda \mathbf{x})$,

$$\|D^\alpha \psi_\lambda\|_p = \lambda^{|\alpha| - n/p} \|D^\alpha f\|_p \tag{35}$$

for all $\alpha \in \mathbb{Z}_+^n$. From (34) we obtain $\mathrm{sp}\,(f - \psi_\lambda) \subset H$. Thus, by Theorem 2.1,

$$\|D^\alpha (f - \psi_\lambda)\|_p \leq C_K \sup_{\xi \in H} |\xi^\alpha| \|f - \psi_\lambda\|_{\mathcal{H}_{p,K,3}} \leq C_K \sup_{\xi \in K} |\xi^\alpha| \|f - \psi_\lambda\|_{\mathcal{H}_{p,K,3}}. \tag{36}$$

Then

$$\|D^\alpha f\|_p / \sup_{\xi \in K} |\xi^\alpha| \leq \left(\|D^\alpha (f - \psi_\lambda)\|_p + \|D^\alpha \psi_\lambda\|_p \right) / \sup_{\xi \in K} |\xi^\alpha| \leq$$

$$\leq \left(C_K \sup_{\xi \in K} |\xi^\alpha| \|f - \psi_\lambda\|_{\mathcal{H}_{p,K,3}} + \lambda^{|\alpha| - n/p} \|D^\alpha f\|_p \right) / \sup_{\xi \in K} |\xi^\alpha| \tag{37}$$

for all $\alpha \in \mathbb{Z}_+^n$, where the second inequality comes from (35). Hence,

$$\|D^\alpha f\|_p / \sup_{\xi \in K} |\xi^\alpha| \leq C_K \|f - \psi_\lambda\|_{\mathcal{H}_{p,K,3}} / (1 - \lambda^{|\alpha| - n/p})$$

for all $\alpha \in \mathbb{Z}_+^n$, $|\alpha| > n$. Consequently,

$$\limsup_{|\alpha| \to \infty} \left(\|D^\alpha f\|_p / \sup_{\xi \in K} |\xi^\alpha| \right) \leq C_K \|f - \psi_\lambda\|_{\mathcal{H}_{p,K,3}}. \tag{38}$$

Note that

$$\|f-\psi_\lambda\|_{\mathcal{H}_{p,K,3}} = \sup_{\mathbf{x}\in\mathbb{R}^n} \left|D^{(3,3,\ldots,3)}(\widehat{f-\psi_\lambda})(\mathbf{x})\right| =$$

$$= \sup_{\mathbf{x}\in H} \left|(D^{(3,3,\ldots,3)}\widehat{f})(\mathbf{x}) - \lambda^{3n}(D^{(3,3,\ldots,3)}\widehat{f})(\lambda\mathbf{x})\right| \leq$$

$$\leq \sup_{\mathbf{x}\in H} \Big(|D^{(3,3,\ldots,3)}\widehat{f}(\mathbf{x}) - (D^{(3,3,\ldots,3)}\widehat{f})(\lambda\mathbf{x})| + (1-\lambda^{3n})|(D^{(3,3,\ldots,3)}f)(\lambda\mathbf{x})|\Big).$$

Combining this and (38), we obtain

$$\limsup_{|\alpha|\to\infty}(\|D^\alpha f\|_p/\sup_{\xi\in K}|\xi^\alpha|) \leq$$

$$\leq C_K \sup_{\mathbf{x}\in H} \Big(|D^{(3,3,\ldots,3)}\widehat{f}(\mathbf{x}) - (D^{(3,3,\ldots,3)}\widehat{f})(\lambda\mathbf{x})| + (1-\lambda^{3n})|(D^{(3,3,\ldots,3)}\widehat{f})(\lambda\mathbf{x})|\Big),$$

which implies

$$\limsup_{|\alpha|\to\infty}\left(\|D^\alpha f\|_p/\sup_{\xi\in K}|\xi^\alpha|\right) \leq$$

$$\leq C_K \sup_{\mathbf{x}\in H} |D^{(3,3,\ldots,3)}\widehat{f}(\mathbf{x}) - (D^{(3,3,\ldots,3)}\widehat{f})(\lambda\mathbf{x})| + C_K(1-\lambda^{3n})\|D^{(3,3,\ldots,3)}\widehat{f}\|_\infty. \tag{39}$$

Since $D^{(3,3,\ldots,3)}\widehat{f} \in C(\mathbb{R}^n)$ and $\operatorname{supp} D^{(3,3,\ldots,3)}\widehat{f}$ is a compact set, we deduce that the function $D^{(3,3,\ldots,3)}\widehat{f}$ is uniformly continuous on $\mathbb{R}^n$. Combing this with (39), by letting $\lambda\to 1^-$, we conclude that

$$\lim_{|\alpha|\to\infty}(\|D^\alpha f\|_p/\sup_{\xi\in K}|\xi^\alpha|) = 0.$$

Theorem 2.2 is proved.

***Remark* 2.2.** It follows from Theorem 2.1 that $\sup_{f\in\mathcal{H}_{p,K,3}} A_{\alpha,f} \leq C_K$ for all $\alpha\in\mathbb{Z}_+^n$, and then the sequence $(A_{\alpha,f})_{\alpha\in\mathbb{Z}_+^n}$ is bounded, where

$$A_{\alpha,f} = \|D^\alpha f\|_p/\left(\sup_{\xi\in K}|\xi^\alpha|\|f\|_{\mathcal{H}_{p,K,3}}\right).$$

Using Theorem 2.2, we obtain the stronger result: $A_{\alpha,f}\to 0$ as $|\alpha|\to\infty$.

***Corollary* 2.2.** *Let $\sigma\in\mathbb{R}_+^n$, $1\leq p\leq\infty$ and $f\in\mathcal{H}_{p,\Delta_\sigma,3}$, where $\Delta_\sigma = [-\sigma_1,\sigma_1]\times\ldots$ $\ldots\times[-\sigma_n,\sigma_n]$. Then*

$$\lim_{|\alpha|\to\infty}\|D^\alpha f\|_p/\sigma^\alpha = 0.$$

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