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REFINEMENTS OF LOCAL FRACTIONAL HILBERT-TYPE INEQUALITIES

УТОЧНЕННЯ ЛОКАЛЬНИХ ДРОБОВИХ НЕРІВНОСТЕЙ ТИПУ ГІЛЬБЕРТА

We study the refinements of several well-known local fractional Hilbert-type inequalities by interpolating the Lebesgue norms of the local fractional Laplace transforms of the functions involved in the inequalities. As an application, the main results are compared with some our results previously known from the literature.

Досліджується уточнення кількох відомих локальних дробових нерівностей типу Гільберта шляхом інтерполяції норм Лебега локальних дробових перетворень Лапласа функцій, що входять в ці нерівності. Як застосування, основні результати роботи порівнюються з деякими нашими результатами, опублікованими раніше.

1. Introduction. Well-known Hilbert inequality (see [4]) in its integral form states that

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}dxdy \le \frac{\pi}{\sin\dfrac{\pi}{p}}\left[\int_0^\infty f^p(x)dx\right]^{\frac{1}{p}}\left[\int_0^\infty g^q(y)dy\right]^{\frac{1}{q}}, \tag{1}$$

where $f,g\colon (0,\infty)\to\mathbb{R}$ are nonnegative integrable functions and p,q is a pair of nonnegative conjugate exponents provided that $\dfrac{1}{p}+\dfrac{1}{q}=1,\ p>1.$ Moreover, the constant $\dfrac{\pi}{\sin(\pi/p)}$ is the best possible in light of the fact that it cannot be replaced by a smaller positive constant in order to keep the inequality valid. Hardy et al. [4] noted that the following equivalent form can be assigned to (1):

$$\int_0^\infty\left[\int_0^\infty \frac{f(x)}{x+y}dx\right]^p dy \le \left(\frac{\pi}{\sin\dfrac{\pi}{p}}\right)^p\int_0^\infty f^p(x)dx, \tag{2}$$

in the sense that (1) implies (2) and vice versa. Over the years, Hilbert-type inequalities (1) and (2) have been extensively studied by numerous authors. A great variety of extensions has included inequalities with more general kernels, weight functions and integration domains, as well as refinements of both initial Hilbert-type inequalities. It should be noticed that these inequalities are still of interest to a large number of authors. An early development of the Hilbert-type inequalities available in the Hardy et al. [4], while the reader is referred to Batbold et al. [2] and Krnić et al. [6] for more recent progress.

Nowadays, an intriguing subject regarding the aforementioned inequalities is their extension on certain fractal spaces based on the local fractional calculus. The local fractional calculus is primarily used in order to describe various real-world phenomena in different fields of study involving non-differentiable problems. In science and engineering, the local fractional calculus is applied to solve ordinary or partial differential equations.

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Moreover, the local fractional calculus is an essential tool in pure mathematics. Recently, by virtue of the local fractional calculus, a numerous extension of classical real inequalities has been extended to hold on certain fractal spaces.

To support the claim, we denote by ${}_aI_b^\alpha f(x)$ and ${}_aI_b^\alpha\big[{}_aI_b^\alpha h(x,y)\big]$ the local fractional integrals

$${}_aI_b^\alpha f(x) = \frac{1}{\Gamma(1+\alpha)} \int\limits_a^b f(x)(dx)^\alpha$$

and

$${}_aI_b^\alpha\big[{}_aI_b^\alpha h(x,y)\big] = \frac{1}{\Gamma^2(1+\alpha)} \int\limits_a^b \int\limits_a^b h(x,y)(dx)^\alpha (dy)^\alpha,$$

where $0 < \alpha \leq 1$ and where Γ stands for a usual Gamma-function defined by $\Gamma(a) = \displaystyle\int_0^\infty t^{a-1}e^{-t}dt$, $a > 0$. Also, for the given calculus $C_\alpha(a,b)$ stands for a set of local fractional continuous functions on the interval (a,b).

In this paper, we refer to a recent paper of Krnić et al. [1], where a general fractal Hilbert-type inequality was obtained for conjugate exponents p and q. Let $\dfrac{1}{p} + \dfrac{1}{q} = 1,\ p > 1,\ 0 < \alpha \leq 1$, and $K \in C_\alpha(a,b)^2,\ \varphi, \psi \in C_\alpha(a,b)$ be nonnegative functions. If the functions F and G are defined by

$$F^p(x) = {}_aI_b^\alpha\big(K(x,y)\psi^{-p}(y)\big), \qquad G^q(y) = {}_aI_b^\alpha\big(K(x,y)\phi^{-q}(x)\big), \tag{3}$$

then for all nonnegative functions $f, g \in C_\alpha(a,b)$ the inequalities

$${}_aI_b^\alpha({}_aI_b^\alpha(K(x,y)f(x)g(y))) \leq [{}_aI_b^\alpha(\varphi Ff)^p(x)]^{\frac{1}{p}}[{}_aI_b^\alpha((\psi Gg)^q(y))]^{\frac{1}{q}}$$

and

$${}_aI_b^\alpha\big((G\psi)^{-p}(y)[{}_aI_b^\alpha(K(x,y)f(x))]^p\big) \leq {}_aI_b^\alpha(\varphi Ff)^p(x) \tag{4}$$

hold and are equivalent.

2. Preliminaries on local fractional calculus. In this section, the basic notations and a brief overview of the local fractional calculus are offered for the reader's convenience. More accurately, key definitions and properties of the local fractional derivative and integral developed by [8] are given (see also [9]).

Let $\mathbb{R}^\alpha$, where $0 < \alpha \leq 1$, be an α-type fractal set of real line numbers. For $a^\alpha,\ b^\alpha \in \mathbb{R}^\alpha$, the addition and multiplication are defined by

$$a^\alpha + b^\alpha := (a+b)^\alpha, \qquad a^\alpha \cdot b^\alpha = a^\alpha b^\alpha := (ab)^\alpha.$$

By making use of these two binary operations, $\mathbb{R}^\alpha$ becomes a field with an additive identity 0^α and a multiplicative identity 1^α.

The starting point in introducing the local fractional calculus on $\mathbb{R}^\alpha$ is the concept of the local fractional continuity. A nondifferentiable function $f\colon \mathbb{R} \to \mathbb{R}^\alpha$ is said to be local fractional continuous at x_0, if, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $|x - x_0| < \delta$ implies that

$$\left|f(x)-f(x_0)\right|<\varepsilon^\alpha.$$

The set of local fractional continuous functions on interval I is denoted by $C_\alpha(I)$.

The local fractional derivative of f of order α at $x=x_0$ is defined by

$$f^{(\alpha)}(x_0)=\left.\frac{d^\alpha f(x)}{dx^\alpha}\right|_{x=x_0}=\lim_{x\to x_0}\frac{\Gamma(1+\alpha)\big(f(x)-f(x_0)\big)}{(x-x_0)^\alpha},$$

where Γ stands for a usual Gamma-function. Now, let

$$f^{(\alpha)}(x)=D_x^\alpha f(x).$$

If there exists $f^{(k+1)\alpha}(x)=\overbrace{D_x^\alpha\ldots D_x^\alpha}^{k+1}f(x)$ for every $x\in I$, then we denote $f\in D_{(k+1)\alpha}(I)$, where $k=0,1,2,\ldots$.

The local fractional integral is defined for a class of local fractional continuous functions. Let $f\in C_\alpha[a,b]$ and let $P=\{t_0,t_1,\ldots,t_N\}$, $N\in\mathbb{N}$, be a partition of interval $[a,b]$ such that $a=t_0<t_1<\ldots<t_{N-1}<t_N=b$. Further, for this partition P, let $\Delta t_j=t_{j+1}-t_j$, $j=0,\ldots,N-1$, and $\Delta t=\max\left\{\Delta t_1,\Delta t_2,\ldots,\Delta t_{N-1}\right\}$. Then the local fractional integral of f on the interval $[a,b]$ of order α (denoted by ${}_aI_b^\alpha f(x)$) is defined by

$${}_aI_b^\alpha f(x)=\frac{1}{\Gamma(1+\alpha)}\int_a^b f(t)(dt)^\alpha=\frac{1}{\Gamma(1+\alpha)}\lim_{\Delta t\to 0}\sum_{j=0}^{N-1}f(t_j)(\Delta t_j)^\alpha.$$

The aforementioned definition implies that ${}_aI_b^\alpha f(x)=0$ if $a=b$, and ${}_aI_b^\alpha f(x)=-{}_bI_a^\alpha f(x)$ if $a<b$.

Similarly to the Riemann integral, there is the following analogue of the Newton – Leibnitz formula on the fractal space. Namely, if $f=g^{(\alpha)}\in C_\alpha[a,b]$, then

$${}_aI_b^\alpha f(x)=g(b)-g(a).$$

In particular, if $f(x)=x^{k\alpha}$, $k\in\mathbb{R}$, then

$$\frac{1}{\Gamma(1+\alpha)}\int_a^b x^{k\alpha}(dx)^\alpha=\frac{\Gamma(1+k\alpha)}{\Gamma(1+(k+1)\alpha)}(b^{(k+1)\alpha}-a^{(k+1)\alpha}).$$

For the reader's convenience, from this point the following abbreviations are used:

$$\mathbf{x}=(x_1,x_2,\ldots,x_n),\qquad (d\mathbf{x})^\alpha=\prod_{i=1}^n(dx_i)^\alpha.$$

The starting point in establishing Hilbert-type inequalities is the well-known Hölder inequality. Krnić and Vuković describe the development of multidimensional fractal version of the Hölder inequality in [7]. Let $\sum_{i=1}^m\dfrac{1}{p_i}=1$, $p_i>1$, $i=1,2,\ldots,m$, and let Ω be a fractal surface. If $F_i\in C_\alpha(\Omega^n)$, $i=1,2,\ldots,m$, then the following inequality holds:

$$\frac{1}{\Gamma^n(1+\alpha)}\int\limits_{\Omega^n}\prod_{j=1}^{m}F_j(\mathbf{x})(d\mathbf{x})^{\alpha}\leq\prod_{j=1}^{m}\left(\frac{1}{\Gamma^n(1+\alpha)}\int\limits_{\Omega^n}F_j^{p_j}(\mathbf{x})(d\mathbf{x})^{\alpha}\right)^{\frac{1}{p_j}}.$$

In order to conclude our discussion regarding fractional integrals, we present a variant of the change of variables theorem in the given context. Namely, if $g\in D_\alpha[a,b]$ and $(f\circ g)\in C_\alpha[g(a),g(b)]$, then the relation

$${}_aI_b^{\alpha}(f\circ g)(s)[g'(s)]^{\alpha}={}_{g(a)}I_{g(b)}^{\alpha}f(x)$$

holds.

Throughout the paper is assumed that $\|f\|_p=\left(\dfrac{1}{\Gamma(1+\alpha)}\displaystyle\int_0^{\infty}f^p(x)(dx)^{\alpha}\right)^{\frac{1}{p}}$. Therefore, at this point, the reader should notice that if $\alpha=1$, then the local fractional calculus reduces to the classical real calculus. For further information about the above-mentioned concept of fractional differentiability and integrability, the reader is referred to Yang [8] and references therein.

3. Main results. In order to establish the corresponding refinements, an exstension of the usual Gamma-function. The local fractional Gamma-function $\Gamma_\alpha(\cdot)$ for $0<\alpha\leq 1$ (see also [5]) is defined by

$$\Gamma_\alpha(x)=\frac{1}{\Gamma(1+\alpha)}\int\limits_0^{\infty}E_\alpha(-t^{\alpha})t^{\alpha(x-1)}(dt)^{\alpha},$$

where $E_\alpha(\cdot)$ stands for the Mittag-Leffler function given by

$$E_\alpha(y)=\sum_{k=0}^{\infty}\frac{y^k}{\Gamma(1+\alpha k)}.$$

If $s\in\mathbb{C}$, let $f(x)$ is a function which vanishes for negative values of x. The local fractional Laplace transform $L_\alpha\{f\}(s)$ of order α is defined by

$$L_\alpha\{f\}(s)=\frac{1}{\Gamma(1+\alpha)}\int\limits_0^{\infty}E_\alpha(-s^{\alpha}x^{\alpha})f(x)(dx)^{\alpha}.$$

The following two lemmas are necessary in order to prove main result.

Lemma 1. *Let $a>-1$. If $f_i\in C_\alpha(\mathbb{R}_+)$, $i=1,\ldots,n$, are nonnegative functions with the local fractional Laplace transforms $L_\alpha\{f_i\}(t)$, $i=1,2,\ldots,n$, respectively, then*

$$\frac{1}{\Gamma^n(1+\alpha)}\int\limits_{\mathbb{R}_+^n}\frac{\prod_{i=1}^{n}f_i(x_i)}{\left(\sum_{j=1}^{n}x_j^{\alpha}\right)^{a+1}}(dx_1)^{\alpha}(dx_2)^{\alpha}\ldots(dx_n)^{\alpha}=$$

$$=\frac{1}{\Gamma_\alpha(a+1)}\frac{1}{\Gamma(1+\alpha)}\int\limits_0^{\infty}t^{\alpha a}\prod_{i=1}^{n}L_\alpha\{f_i\}(t)(dt)^{\alpha}. \tag{5}$$

Proof. The proof is a simple application of the Fubini theorem (see [8]) as follows:

$$\frac{1}{\Gamma(1+\alpha)}\int\limits_0^\infty t^{\alpha a}\prod_{i=1}^n L_\alpha\{f_i\}(t)(dt)^\alpha =$$

$$= \frac{1}{\Gamma(1+\alpha)}\int\limits_0^\infty t^{\alpha a}\left(\frac{1}{\Gamma(1+\alpha)}\prod_{i=1}^n\int\limits_0^\infty E_\alpha(-x_i^\alpha t^\alpha) f_i(x_i)(dx_i)^\alpha\right)(dt)^\alpha =$$

$$= \frac{1}{\Gamma^n(1+\alpha)}\int\limits_{\mathbb{R}_+^n}\prod_{i=1}^n f_i(x_i)\left(\frac{1}{\Gamma(1+\alpha)}\int\limits_0^\infty E_\alpha\big(-t^\alpha(x_1^\alpha+\ldots+x_n^\alpha)\big)t^{\alpha a}(dt)^\alpha\right)\times$$

$$\times(dx_1)^\alpha(dx_2)^\alpha\ldots(dx_n)^\alpha. \tag{6}$$

By using the substitution $u = t(x_1+\ldots+x_n)$, we obtain

$$\frac{1}{\Gamma(1+\alpha)}\int\limits_0^\infty E_\alpha\big(-t^\alpha(x_1^\alpha+\ldots+x_n^\alpha)\big)t^{\alpha a}(dt)^\alpha =$$

$$= \frac{1}{\left(\sum_{i=1}^n x_i^\alpha\right)^{a+1}}\frac{1}{\Gamma(1+\alpha)}\int\limits_0^\infty E_\alpha(-u^\alpha)u^{\alpha((a+1)-1)}(du)^\alpha =$$

$$= \frac{1}{\left(\sum_{i=1}^n x_i^\alpha\right)^{a+1}}\Gamma_\alpha(a+1). \tag{7}$$

Therefore, from (6) and (7) we get (5).

Lemma 1 is proved.

Lemma 2. *Let* $\dfrac{1}{p}+\dfrac{1}{q}=1$ *with* $p>1$ *and* $B,\ C\in\mathbb{R}$. *If* $f\in C_\alpha(\mathbb{R}_+)$ *is nonnegative function with the local fractional Laplace transform* $L_\alpha\{f\}(t)$, *then*

$$\frac{1}{\Gamma(1+\alpha)}\int\limits_0^\infty y^{\alpha(p-1-p(B+C))}L_\alpha\{f\}^p(y)(dy)^\alpha \le$$

$$\le \Gamma_\alpha(1-pC)\Gamma_\alpha^{p-1}(1-qB)\frac{1}{\Gamma(1+\alpha)}\int\limits_0^\infty x^{p(B+C)-1}f^p(x)(dx)^\alpha. \tag{8}$$

Proof. For the kernel $K(x,y)=E_\alpha(-x^\alpha y^\alpha)$, with functions $\varphi(x)=x^{\alpha B}$ and $\psi(y)=y^{\alpha C}$, following (3), we get

$$F^p(x) = \frac{1}{\Gamma(1+\alpha)}\int\limits_0^\infty \psi^{-p}(y)K(x,y)(dy)^\alpha = \frac{1}{\Gamma(1+\alpha)}\int\limits_0^\infty y^{-\alpha pC}E_\alpha(-x^\alpha y^\alpha)(dy)^\alpha.$$

By using the substitution $t = xy$, we obtain

$$F^p(x) = x^{\alpha(pC-1)}\Gamma_\alpha(1-pC). \tag{9}$$

Similarly, we have

$$G^q(y) = \frac{1}{\Gamma(1+\alpha)} \int_0^\infty \varphi^{-q}(x)K(x,y)(dx)^\alpha y^{\alpha(qB-1)}\Gamma_\alpha(1-qB). \tag{10}$$

Finally, combining (4), (9) and (10), we get (8).

Lemma 2 is proved.

At this point, we are ready to present and prove main result.

Theorem 1. *Let* $s > 0$ *and* $B_i,\ C_i \in \mathbb{R},\ i = 1,2,\ldots,n$. *Let* $\sum_{i=1}^n \frac{1}{p_i} = 1$ *with* $p_i > 1$, $\frac{1}{p_i} + \frac{1}{q_i} = 1,\ i = 1,2,\ldots,n$, *and* $\sum_{i=1}^n \gamma_i = 0$. *If* $f_i \in C_\alpha(\mathbb{R}_+),\ i = 1,2,\ldots,n$, *are nonnegative functions and* $L_\alpha\{f_i\}(t)$ *is the local frational Laplace transforms of* f_i, *then*

$$\frac{1}{\Gamma^n(1+\alpha)} \int_{\mathbb{R}_+^n} \frac{\prod_{i=1}^n f_i(x_i)}{\left(\sum_{j=1}^n x_j^\alpha\right)^s}(dx_1)^\alpha(dx_2)^\alpha\ldots(dx_n)^\alpha \le$$

$$\le \frac{1}{\Gamma_\alpha(s)} \prod_{i=1}^n \|t^{\alpha(\frac{1}{q_i}+\frac{s-n}{p_i}-\gamma_i)} L_\alpha\{f_i\}(t)\|_{p_i} \le$$

$$\le \frac{1}{\Gamma_\alpha(s)} \prod_{i=1}^n \left(\Gamma_\alpha^{\frac{1}{q_i}}(1-q_iB_i)\Gamma_\alpha^{\frac{1}{p_i}}(1-p_iC_i)\right)\times$$

$$\times \prod_{i=1}^n \left(\frac{1}{\Gamma(1+\alpha)} \int_0^\infty x_i^{\alpha(n-s+p_i\gamma_i-1)} f_i^{p_i}(x_i)(dx_i)^\alpha\right)^{\frac{1}{p_i}}. \tag{11}$$

Proof. In view of Lemma 1, setting the exponents $\beta_i,\ i = 1,\ldots,n$, such that $\sum_{i=1}^n \beta_i = s-1$, we have

$$\frac{1}{\Gamma^n(1+\alpha)} \int_{\mathbb{R}_+^n} \frac{\prod_{i=1}^n f_i(x_i)}{\left(\sum_{j=1}^n x_j\right)^s}(dx_1)^\alpha(dx_2)^\alpha\ldots(dx_n)^\alpha =$$

$$= \frac{1}{\Gamma_\alpha(s)}\frac{1}{\Gamma(1+\alpha)} \int_0^\infty t^{\alpha(s-1)} \prod_{i=1}^n L_\alpha\{f_i\}(t)(dt)^\alpha =$$

$$= \frac{1}{\Gamma_\alpha(s)}\frac{1}{\Gamma(1+\alpha)} \int_0^\infty \prod_{i=1}^n \left(t^{\alpha\beta_i} L_\alpha\{f_i\}\right)(dt)^\alpha.$$

Additionally, by applying Hölder's inequality (see [3]), we obtain

$$\frac{1}{\Gamma^n(1+\alpha)} \int\limits_{\mathbb{R}^n_+} \frac{\prod_{i=1}^n f_i(x_i)}{\left(\sum_{j=1}^n x_j^\alpha\right)^s} (dx_1)^\alpha (dx_2)^\alpha \dots (dx_n)^\alpha \leq$$

$$\leq \frac{1}{\Gamma_\alpha(s)} \prod_{i=1}^n \left(\frac{1}{\Gamma(1+\alpha)} \int\limits_0^\infty t^{\alpha p_i \beta_i} L_\alpha\{f_i\}^{p_i}(t)(dt)^\alpha \right)^{\frac{1}{p_i}}. \tag{12}$$

By putting

$$\beta_i = \frac{1}{q_i} - (B_i + C_i), \quad i = 1, \dots, n, \tag{13}$$

in Lemma 2, we get

$$\left(\frac{1}{\Gamma(1+\alpha)} \int\limits_0^\infty t^{\alpha p_i \beta_i} L_\alpha\{f_i\}^{p_i}(t)(dt)^\alpha \right)^{\frac{1}{p_i}} =$$

$$= \left(\frac{1}{\Gamma(1+\alpha)} \int\limits_0^\infty t^{\alpha(p_i - 1 - p_i(B_i + C_i))} L_\alpha\{f_i\}^{p_i}(t)(dt)^\alpha \right)^{\frac{1}{p_i}} \leq$$

$$\leq \Gamma_\alpha^{\frac{1}{q_i}}(1 - q_i B_i) \Gamma_\alpha^{\frac{1}{p_i}}(1 - p_i C_i) \left(\frac{1}{\Gamma(1+\alpha)} \int\limits_0^\infty x_i^{\alpha(p_i(B_i + C_i) - 1)} f_i^{p_i}(x_i)(dx_i)^\alpha \right)^{\frac{1}{p_i}}. \tag{14}$$

If we put $B_i + C_i = \dfrac{n-s}{p_i} + \gamma_i$, then the relation (13) implies $\sum_{i=1}^n \gamma_i = 0$. Finally, the inequality (11) follows from relations (12) and (14).

Theorem 1 is proved.

***Remark* 1.** By putting $B_i = \dfrac{n-s}{p_i q_i}$, $C_i = \dfrac{n-s}{p_i^2}$ for $i = 1, 2, \dots, n$ in Theorem 1, we obtain

$$\frac{1}{\Gamma^n(1+\alpha)} \int\limits_{\mathbb{R}^n_+} \frac{\prod_{i=1}^n f_i(x_i)}{\left(\sum_{j=1}^n x_j^\alpha\right)^s} (dx_1)^\alpha (dx_2)^\alpha \dots (dx_n)^\alpha \leq$$

$$\leq \frac{1}{\Gamma_\alpha(s)} \prod_{i=1}^n \left\| t^{\alpha\left(\frac{1}{q_i} + \frac{s-n}{p_i}\right)} L_\alpha\{f_i\}(t) \right\|_{p_i} \leq \frac{1}{\Gamma_\alpha(s)} \prod_{i=1}^n \Gamma_\alpha\left(\frac{s-n+p_i}{p_i} \right) \times$$

$$\times \prod_{i=1}^n \left(\frac{1}{\Gamma(1+\alpha)} \int\limits_0^\infty x_i^{\alpha(n-s-1)} f_i^{p_i}(x_i)(dx_i)^\alpha \right)^{\frac{1}{p_i}}. \tag{15}$$

The inequality (15) is a refinement of result from Krnić and Vuković [7].

We restate Theorem 1 for the case $n = 2$. This result is interesting in its own right, since it is a refinement of the inequalities from Batbold et al. [1]. By putting $p_1 = q_2 = p$, $p_2 = q_1 = q$,

$B_i = A_i,\ C_i = \dfrac{2-s}{q_i} - A_{i+1},\ \gamma_i = A_i - A_{i+1}$ for $i = 1, 2$ (the indices are taken modulo 2) in Theorem 1, we get the following result.

Theorem 2. *Let* $p > 1,\ \dfrac{1}{p} + \dfrac{1}{q} = 1$ *and* $s > 0$. *If* $f, g \in C_\alpha(\mathbb{R}_+)$ *are nonnegative functions and* $L_\alpha\{f\}(t),\ L_\alpha\{g\}(t)$ *are the local fractional Laplace transforms of* f *and* g, *respectively*, *then*

$$\frac{1}{\Gamma^2(1+\alpha)} \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x^\alpha + y^\alpha)^s} (dx)^\alpha (dy)^\alpha \le$$

$$\le \frac{1}{\Gamma_\alpha(s)} \left\| t^{\alpha(\frac{1}{q} + \frac{s-2}{p} + A_2 - A_1)} L_\alpha\{f\}(t) \right\|_p \left\| t^{\alpha(\frac{1}{p} + \frac{s-2}{q} + A_1 - A_2)} L_\alpha\{g\}(t) \right\|_q \le$$

$$\le L \left[\frac{1}{\Gamma(1+\alpha)} \int_0^\infty x^{\alpha(1-s+p(A_1-A_2))} f^p(x) (dx)^\alpha \right]^{\frac{1}{p}} \times$$

$$\times \left[\frac{1}{\Gamma(1+\alpha)} \int_0^\infty y^{\alpha(1-s+q(A_2-A_1))} g^q(y) (dy)^\alpha \right]^{\frac{1}{q}},$$

where $L = \big[B_\alpha(1 - pA_2, s - 1 + pA_2)\big]^{\frac{1}{p}} \big[B_\alpha(1 - qA_1, s - 1 + qA_1)\big]^{\frac{1}{q}}$.

By putting $A_1 = \dfrac{2-s}{2q}$ and $A_2 = \dfrac{2-s}{2p}$ in Theorem 2, we have a refinement of inequality from [1].

***Corollary* 1.** *Let* $p > 1,\ \dfrac{1}{p} + \dfrac{1}{q} = 1$ *and* $s > 0$. *If* $f, g \in C_\alpha(\mathbb{R}_+)$ *are nonnegative functions and* $L_\alpha\{f\}(t),\ L_\alpha\{g\}(t)$ *are the local fractional Laplace transforms of* f *and* g, *respectively*, *then*

$$\frac{1}{\Gamma^2(1+\alpha)} \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x^\alpha + y^\alpha)^s} (dx)^\alpha (dy)^\alpha \le$$

$$\le \frac{1}{\Gamma_\alpha(s)} \left\| t^{\alpha(\frac{1}{q} + \frac{s}{2} - 1)} L_\alpha\{f\}(t) \right\|_p \left\| t^{\alpha(\frac{1}{p} + \frac{s}{2} - 1)} L_\alpha\{g\}(t) \right\|_q \le$$

$$\le B_\alpha\Big(\frac{s}{2}, \frac{s}{2}\Big) \left[\frac{1}{\Gamma(1+\alpha)} \int_0^\infty x^{\alpha(p-1) - \frac{\alpha s p}{2}} f^p(x) (dx)^\alpha \right]^{\frac{1}{p}} \times$$

$$\times \left[\frac{1}{\Gamma(1+\alpha)} \int_0^\infty y^{\alpha(q-1) - \frac{\alpha s q}{2}} g^q(y) (dy)^\alpha \right]^{\frac{1}{q}}.$$

References

1. Ts. Batbold, M. Krnić, P. Vuković, *A unified approach to fractal Hilbert-type inequalities*, J. Inequal. and Appl., **2019**, Article 117 (2019).

2. Ts. Batbold, M. Krnić, J. Pečarić, P. Vuković, *Further development of Hilbert-type inequalities*, Element, Zagreb (2017).
3. G.-S. Chen, *Generalizations of Hölder's and some related integral inequalities on fractal space*, J. Funct. Spaces and Appl., **2013** (2013).
4. G. H. Hardy, J. E. Littlewood, G. Pólya, *Inequalities*, second ed., Cambridge Univ. Press, Cambridge (1967).
5. G. Jumarie, *Fractional Euler's integral of first and second kinds. Application to fractional Hermite's polynomials and to probability density of fractional order*, J. Appl. Math. Inform., **28**, № 1-2, 257 – 273 (2010).
6. M. Krnić, J. Pečarić, I. Perić, P. Vuković, *Recent advances in Hilbert-type inequalities*, Element, Zagreb (2012).
7. M. Krnić, P. Vuković, *Multidimensional Hilbert-type inequalities obtained via local fractional calculus*, Acta Appl. Math. (to appear).
8. X. J. Yang, *Advanced local fractional calculus and its applications*, World Sci. Publ., New York (2012).
9. X. J. Yang, *Local fractional functional analysis and its applications*, Asian Acad. Publ., Hong Kong (2011).

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