DOI: 10.37863/umzh.v74i11.6510

UDC 532.5

Doan Cong Dinh[1] (School Appl. Math. and Informatics, Hanoi Univ. Sci. and Technology, Vietnam)

FUNDAMENTAL SOLUTIONS OF THE STOKES SYSTEM IN QUATERNION ANALYSIS[2]

ФУНДАМЕНТАЛЬНІ РОЗВ'ЯЗКИ СИСТЕМИ СТОКСА У КВАТЕРНІОННОМУ АНАЛІЗІ

The method of quaternionic analysis in fluid mechanics was developed by several generations of mathematicians with numerous important results. We add a small result in this direction. Thus, we introduce a new reformulation of fundamental solutions of the Stokes system within the framework of quaternion analysis and construct integral representations for its solutions.

Метод кватерніонного аналізу в механіці рідин був розроблений кількома поколіннями математиків, і було отримано багато важливих результатів. У даній статті додано один невеликий результат у цьому напрямку. Так, наведено нове переформулювання фундаментальних розв'язків системи Стокса в рамках кватерніонного аналізу та побудовано інтегральні зображення її розв'язків.

1. Introduction. In two-dimensional problems of a continua mechanics, the advantages of complex analysis are applied. In plane problems of the theory of elasticity, the Kolosov – Muskhelishvili formulae provide the most convenient method to represent solutions of the Lamé – Navier system in terms of two holomorphic functions of complex variables. The main advantages of the Kolosov – Muskhelishvili formulae are using holomorphic functions, expansion into series, Cauchy formula and conformal mapping techniques. In general, generalized analytic [18] or pseudoanalytic functions [1] are more widely used in the mathematical physics [10].

In three-dimensional elasticity problem, some generalized Muskhelishvili formulae are constructed by several methods [2, 3, 5, 13, 19]. The displacement field is represented by using two monogenic functions (or antimonogenic functions, Ψ-hyperholomorphic functions) in quaternion analysis.

The treatment of the Stokes system in the plane by complex method is introduced in [12, 20, 21]. In three-dimensional problem, the foundations of Clifford analysis method for boundary-value problem of the Stokes system are in [8, 9, 16, 17]. The main tools to represent solutions in these works are Teodorescu transform, Cauchy-type operator and singular integral operator of Cauchy-type. Recent representations solutions of the Stokes system in Clifford analysis method are in [6, 7].

In this paper we introduce a new reformulation of fundamental solutions of the Stokes system in framework of quaternion analysis, construct integral representation formulae for its solutions. With this new representation, we can see some different structures of the solutions. Some known results: Grigor'ev – Gürlebeck – Legatiuk – Yakovlev solution [6], Papkovich – Neuber solution [4], Naghdi – Hsu solution [14, 15] are included in this new representation.

[1] e-mail: doancongdinhdhbk@gmail.com.
[2] This paper was supported by Hanoi University of Science and Technology (grant number T2020-PC-301).

2. Preliminaries and notations. We recall the notation of the quaternion algebra

$$\mathbb{H} = \left\{ q_0 + q_1 e_1 + q_2 e_2 + q_3 e_3 \,\middle|\, q_0, q_1, q_2, q_3 \in \mathbb{R} \right\},$$

where the quaternion units $e_1,\ e_2,\ e_3$ obey the multiplication rules

$$e_1^2 = e_2^2 = e_3^2 = -1, \qquad e_1 e_2 = -e_2 e_1 = e_3.$$

An element $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ is identified with the reduced quaternion variable $\mathbf{x} = x_1 e_1 + x_2 e_2 + x_3 e_3 \in \mathbb{H}$. An element $q = q_0 + q_1 e_1 + q_2 e_2 + q_3 e_3 \in \mathbb{H}$ has the scalar part $\mathbf{Sc}(q) = q_0$ and the vector-part $\mathbf{Vec}(q) = \mathbf{q} := q_1 e_1 + q_2 e_2 + q_3 e_3$. The Dirac operator in quaternion analysis is given by

$$D = e_1 \frac{\partial}{\partial x_1} + e_2 \frac{\partial}{\partial x_2} + e_3 \frac{\partial}{\partial x_3}.$$

A quaternion-valued function $f \in \mathcal{C}^1(\Omega, \mathbb{H})$ of a reduced quaternion variable $\mathbf{x}$ is in the form

$$f = f_0(\mathbf{x}) + \mathbf{f}(\mathbf{x}) = f_0(\mathbf{x}) + f_1(\mathbf{x}) e_1 + f_2(\mathbf{x}) e_2 + f_3(\mathbf{x}) e_3,$$

where $f_i(\mathbf{x}) \in \mathcal{C}^1(\Omega, \mathbb{R}),\ \Omega \subset \mathbb{R}^3$. The Dirac operator acts on the function f from the left $D^l f$ or from the right $f D^r$ in the forms

$$D^l f = \sum_{k=1}^{3} e_k \frac{\partial f}{\partial x_k} = \sum_{k=1}^{3} \sum_{i=0}^{3} e_k e_i \frac{\partial f_i}{\partial x_k},$$

$$f D^r = \sum_{k=1}^{3} \frac{\partial f}{\partial x_k} e_k = \sum_{k=1}^{3} \sum_{i=0}^{3} e_i e_k \frac{\partial f_i}{\partial x_k}.$$

With two reduced quaternion elements $\mathbf{x}$ and $\mathbf{y}$, we use the notations of scalar product $\mathbf{x} \cdot \mathbf{y}$ and vector product $\mathbf{x} \times \mathbf{y}$,

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + x_3 y_3,$$

$$\mathbf{x} \times \mathbf{y} = (x_2 y_3 - x_3 y_2) e_1 + (x_3 y_1 - x_1 y_3) e_2 + (x_1 y_2 - x_2 y_1) e_3.$$

With $f = f_0(\mathbf{x}) + \mathbf{f}(\mathbf{x})$, we recall the notations

$$\operatorname{grad}(f_0) = \sum_{i=1}^{3} \frac{\partial f_0}{\partial x_i} e_i,$$

$$\operatorname{div}(\mathbf{f}) = \sum_{i=1}^{3} \frac{\partial f_i}{\partial x_i},$$

$$\operatorname{rot}(\mathbf{f}) = \left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3} \right) e_1 + \left(\frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1} \right) e_2 + \left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} \right) e_3.$$

We have the decompositions of $D^l f$ and $f D^r$

$$D^l f = -\operatorname{div}(\mathbf{f}) + \operatorname{grad}(f_0) + \operatorname{rot}(\mathbf{f}),$$

$$fD^r = -\mathrm{div}(\mathbf{f}) + \mathrm{grad}(f_0) - \mathrm{rot}(\mathbf{f}).$$

Solutions of the equation $D^l f = 0$ (or $fD^r = 0$) are called left (or right) monogenic functions. Since $DD = -\Delta$, the left (or right) monogenic functions are harmonic functions. We recall the fundamental solution of the Laplace operator Δ by $H(\mathbf{x},\mathbf{y}) = \dfrac{-1}{4\pi|\mathbf{x}-\mathbf{y}|}$ with $\Delta_{\mathbf{x}} H(\mathbf{x},\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y})$ (the Dirac-delta distribution). Then a fundamental solutions of the Dirac operator D is given by

$$E(\mathbf{x},\mathbf{y}) = -D^l_{\mathbf{x}} H(\mathbf{x},\mathbf{y}) = \frac{\mathbf{y}-\mathbf{x}}{4\pi|\mathbf{x}-\mathbf{y}|^3},$$

since $D^l_{\mathbf{x}} E(x,y) = -D^l_{\mathbf{x}} D^l_{\mathbf{x}} H(\mathbf{x},\mathbf{y}) = \Delta_{\mathbf{x}} H(\mathbf{x},\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y})$. The notation $\Delta_{\mathbf{x}}$ indicates that we apply the Laplace operator with respect to the variable $\mathbf{x}$, the notation $D^l_{\mathbf{x}}$ indicates that we apply the left Dirac operator with respect to the variable $\mathbf{x}$. The notations $D^r_{\mathbf{x}}$, $D^l_{\mathbf{y}}$, $D^r_{\mathbf{y}}$ are used with the analogous meaning. We recall the two basic integral formulae in quaternion analysis which will be frequently used in this paper:

Lemma 2.1 [8, p. 86]. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with $\mathcal{C}^1$-boundary. Let $g, h \in \mathcal{C}^1(\overline{\Omega}, \mathbb{H})$. Then we have the Stokes formula in quaternion analysis*

$$\int\limits_{\partial\Omega} g\mathbf{n}h\, dS(\mathbf{x}) = \int\limits_{\Omega} (gD^r h + gD^l h)\, d\mathbf{x},$$

where $\mathbf{n} = n_1 e_1 + n_2 e_2 + n_3 e_3$ is the outer unit normal vector of the boundary.

Theorem 2.1 [8, p. 87]. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with $\mathcal{C}^1$-boundary. Let $g \in \mathcal{C}^1(\overline{\Omega}, \mathbb{H})$. Then we have the Cauchy–Pompeiu integral formula*

$$g(\mathbf{y}) = \int\limits_{\partial\Omega} E(\mathbf{x},\mathbf{y})\mathbf{n} g\, dS(\mathbf{x}) - \int\limits_{\Omega} E(\mathbf{x},\mathbf{y}) D^l g\, d\mathbf{x} \quad \forall \mathbf{y} \in \Omega.$$

By straightforward calculations, the following lemma can be proved.

Lemma 2.2. *With $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$, we have*

1) $\sum_{i=1}^3 e_i \mathbf{a} e_i = \mathbf{a}$,
2) $\sum_{i=1}^3 e_i \mathbf{ab} e_i = -\mathbf{ab} - 2\mathbf{ba}$,
3) $\sum_{i=1}^3 e_i \mathbf{abc} e_i = \mathbf{bac} - \mathbf{cab} + \mathbf{cba}$,
4) $\mathbf{Re}(\mathbf{abc} + \mathbf{cba}) = 0$,
5) $\mathbf{Vec}(\mathbf{abc} - \mathbf{cba}) = 0$.

The linearized and stationary equations of the incompressible viscous fluid are modeled by the Stokes system consisting of the equations in the form

$$-\mu\Delta\mathbf{u} + \mathrm{grad}\, p = \mathbf{f},$$

$$\mathrm{div}\, \mathbf{u} = 0.$$

Here $\mathbf{u}$ and p are the velocity and pressure of the fluid flow, respectively, which are the unknowns; $\mathbf{f}$ corresponds to a given forcing term, while μ is the given dynamic viscosity of the fluid. The Stokes system is rewritten in quaternion analysis in the forms

$$\begin{aligned} D^l(\mu D^l \mathbf{u} + p) &= \mathbf{f}, \\ D^l\mathbf{u} + \mathbf{u}D^r &= 0 \end{aligned} \quad \text{or} \quad \begin{aligned} (\mu\mathbf{u}D^r + p)D^r &= \mathbf{f}, \\ D^l\mathbf{u} + \mathbf{u}D^r &= 0. \end{aligned}$$

3. Fundamental solutions of the Stokes system. The three pairs of fundamental solutions $(V_k(\mathbf{x},\mathbf{y}), Q_k(\mathbf{x},\mathbf{y})),\ k=1,2,3,$ of the Stokes system satisfy

$$\mu\Delta_{\mathbf{x}} V_k(\mathbf{x},\mathbf{y}) - \mathrm{grad}_{\mathbf{x}} Q_k(\mathbf{x},\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y})e_k,$$

$$\mathrm{div}_{\mathbf{x}} V_k(\mathbf{x},\mathbf{y}) = 0.$$

They are given by [11]

$$V_k(\mathbf{x},\mathbf{y}) = \frac{-1}{8\mu\pi}\left[\frac{1}{|\mathbf{y}-\mathbf{x}|}e_k + \sum_{i=1}^{3}\frac{(x_k-y_k)(x_i-y_i)e_i}{|\mathbf{y}-\mathbf{x}|^3}\right],$$

$$Q_k(\mathbf{x},\mathbf{y}) = \frac{\partial}{\partial x_k}\left(\frac{1}{4\pi|\mathbf{y}-\mathbf{x}|}\right).$$

We rewrite $V_k(\mathbf{x},\mathbf{y}),\ Q_k(\mathbf{x},\mathbf{y})$ in the forms

$$V_k(\mathbf{x},\mathbf{y}) = \frac{1}{2\mu}\big[e_k H(\mathbf{x},\mathbf{y}) + (x_k-y_k)E(\mathbf{x},\mathbf{y})\big] =$$

$$= \frac{1}{4\mu}\big[3e_k H(\mathbf{x},\mathbf{y}) + (\mathbf{y}-\mathbf{x})e_k E(\mathbf{x},\mathbf{y})\big] =$$

$$= \frac{1}{4\mu}\big[3e_k H(\mathbf{x},\mathbf{y}) + E(\mathbf{x},\mathbf{y})e_k(\mathbf{y}-\mathbf{x})\big],$$

$$Q_k(\mathbf{x},\mathbf{y}) = -\frac{\partial}{\partial x_k}H(\mathbf{x},\mathbf{y}) = \frac{-1}{2}\big[E(\mathbf{x},\mathbf{y})e_k + e_k E(\mathbf{x},\mathbf{y})\big], \quad k=1,2,3.$$

Let $\alpha(\mathbf{x}) = \alpha_0(\mathbf{x}) + \alpha_1(\mathbf{x})e_1 + \alpha_2(\mathbf{x})e_2 + \alpha_3(\mathbf{x})e_3 \in \mathbb{H}$ be a quaternion valued function in variable $\mathbf{x}$. We define

$$\mathcal{L}_{[\alpha]}(\mathbf{x},\mathbf{y}) = \frac{1}{4\mu}[3\alpha H(\mathbf{x},\mathbf{y}) + (\mathbf{y}-\mathbf{x})\alpha E(\mathbf{x},\mathbf{y})] =$$

$$= \frac{\alpha_0}{\mu}H(\mathbf{x},\mathbf{y}) + \sum_{k=1}^{3}\alpha_k V_k(\mathbf{x},\mathbf{y}).$$

Lemma 3.1. *The three pairs of functions* $(V_k(\mathbf{x},\mathbf{y}), Q_k(\mathbf{x},\mathbf{y})),\ k=1,2,3,$ *are the fundamental solutions of the Stokes system*

$$\mu\Delta_{\mathbf{x}} V_k(\mathbf{x},\mathbf{y}) - D^l_{\mathbf{x}} Q_k(\mathbf{x},\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y})e_k,$$

$$D^l_{\mathbf{x}} V_k(\mathbf{x},\mathbf{y}) + V_k(\mathbf{x},\mathbf{y})D^r_{\mathbf{x}} = 0, \quad k=1,2,3.$$

Proof. We prove the lemma by straightforward calculations in quaternion analysis:

$$\mu D^l_{\mathbf{x}} V_k(\mathbf{x},\mathbf{y}) + Q_k(\mathbf{x},\mathbf{y}) =$$

$$= \frac{1}{2}D^l_{\mathbf{x}}\big[e_k H(\mathbf{x},\mathbf{y}) + (x_k-y_k)E(\mathbf{x},\mathbf{y})\big] - \frac{1}{2}\big[E(\mathbf{x},\mathbf{y})e_k + e_k E(\mathbf{x},\mathbf{y})\big] =$$

$$= \frac{1}{2}\big[-E(\mathbf{x},\mathbf{y})e_k + e_kE(\mathbf{x},\mathbf{y})\big] - \frac{1}{2}\big[E(\mathbf{x},\mathbf{y})e_k + e_kE(\mathbf{x},\mathbf{y})\big] = -E(\mathbf{x},\mathbf{y})e_k,$$

$$D^l_{\mathbf{x}}(\mu\Delta_{\mathbf{x}}V_k(\mathbf{x},\mathbf{y}) - D^l_{\mathbf{x}}Q_k(\mathbf{x},\mathbf{y})) = D^l_{\mathbf{x}}E(\mathbf{x},\mathbf{y})e_k = \delta(\mathbf{x}-\mathbf{y})e_k, \tag{3.1}$$

$$D^l_{\mathbf{x}}V_k(\mathbf{x},\mathbf{y}) + V_k(\mathbf{x},\mathbf{y})D^r_{\mathbf{x}} = \frac{1}{2\mu}D^l_{\mathbf{x}}\big[e_kH(\mathbf{x},\mathbf{y}) + (x_k-y_k)E(\mathbf{x},\mathbf{y})\big]+$$

$$+\frac{1}{2\mu}\big[e_kH(\mathbf{x},\mathbf{y}) + (x_k-y_k)E(\mathbf{x},\mathbf{y})\big]D^r_{\mathbf{x}} =$$

$$= \frac{1}{2\mu}\big[-E(\mathbf{x},\mathbf{y})e_k + e_kE(\mathbf{x},\mathbf{y}) - e_kE(\mathbf{x},\mathbf{y}) + E(\mathbf{x},\mathbf{y})e_k\big] = 0.$$

4. Integral representation formulae. The classical integral representation formulae for solutions of the Stokes system in matrix form or vector form are in [11, 20]. In this section, we introduce a new representation of the solutions in quaternion analysis.

Lemma 4.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with $\mathcal{C}^1$-boundary, $\mathbf{u},\mathbf{v} \in \mathcal{C}^2(\overline{\Omega},\mathbb{R}^3)$, $p,q \in \mathcal{C}^1(\overline{\Omega},\mathbb{R})$. Suppose that $(\mathbf{u},p)$ and $(\mathbf{v},q)$ are solutions of the systems*

$$\begin{aligned} -\mu\Delta\mathbf{u} + \operatorname{grad} p &= \mathbf{f}, \\ \operatorname{div}\mathbf{u} &= 0, \end{aligned} \qquad \text{and} \qquad \begin{aligned} -\mu\Delta\mathbf{v} + \operatorname{grad} q &= 0, \\ \operatorname{div}\mathbf{v} &= 0. \end{aligned}$$

Then we have the integral formula

$$\int_{\partial\Omega}\Big[(\mu\mathbf{u}D^r + p)\mathbf{n}\mathbf{v} + \mathbf{v}\mathbf{n}(\mu D^l\mathbf{u} + p) - (\mu\mathbf{v}D^r + q)\mathbf{n}\mathbf{u} - \mathbf{u}\mathbf{n}(\mu D^l\mathbf{v} + q)\Big]dS(\mathbf{x}) =$$

$$= \int_{\Omega}(\mathbf{f}\mathbf{v} + \mathbf{v}\mathbf{f})\,d\mathbf{x}.$$

Proof. Applying Lemma 2.1 four times for four pairs of functions $(\mu\mathbf{u}D^r + p, \mathbf{v})$, $(\mathbf{v}, \mu D^l\mathbf{u} + p)$, $(\mu\mathbf{v}D^r + q, \mathbf{u})$, $(\mathbf{u}, \mu D^l\mathbf{v} + q)$, we have

$$\int_{\partial\Omega}\Big[(\mu\mathbf{u}D^r + p)\mathbf{n}\mathbf{v} + \mathbf{v}\mathbf{n}(\mu D^l\mathbf{u} + p) - (\mu\mathbf{v}D^r + q)\mathbf{n}\mathbf{u} - \mathbf{u}\mathbf{n}(\mu D^l\mathbf{v} + q)\Big]dS(\mathbf{x}) =$$

$$= \int_{\Omega}\Big[(\mu\mathbf{u}D^r + p)D^r\mathbf{v} + (\mu\mathbf{u}D^r + p)D^l\mathbf{v} + \mathbf{v}D^l(\mu D^l\mathbf{u} + p) +$$

$$+\mathbf{v}D^r(\mu D^l\mathbf{u} + p) - (\mu\mathbf{v}D^r + q)D^r\mathbf{u} - (\mu\mathbf{v}D^r + q)D^l\mathbf{u} -$$

$$-\mathbf{u}D^l(\mu D^l v\mathbf{v} + q) - \mathbf{u}D^r(\mu D^l\mathbf{v} + q)\Big]d\mathbf{x} =$$

$$= \int_{\Omega}(\mathbf{f}\mathbf{v} + \mathbf{v}\mathbf{f})\,d\mathbf{x}.$$

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with $\mathcal{C}^1$-boundary, $\mathbf{u} \in \mathcal{C}^2(\overline{\Omega}, \mathbb{R}^3)$, $p \in \mathcal{C}^1(\overline{\Omega}, \mathbb{R})$. If $(\mathbf{u}, p)$ satisfy the Stokes system in Ω then, for all $\mathbf{y} \in \Omega$,*

$$p(\mathbf{y}) = \mathbf{Sc}\left(\int\limits_{\partial\Omega} E(\mathbf{x}, \mathbf{y})\mathbf{n}(\mu D^l \mathbf{u} + p)\, dS(\mathbf{x}) - \int\limits_{\Omega} E(\mathbf{x}, \mathbf{y})\mathbf{f}\, d\mathbf{x} \right).$$

Proof. Since $p(\mathbf{y}) = \mathbf{Sc}(\mu D^l \mathbf{u} + p)$ and $D^l(\mu D^l \mathbf{u} + p) = f$, the theorem is proved by applying the Cauchy – Pompeiu integral formula in Theorem 2.1 for the function $\mu D\mathbf{u} + p$ in Ω,

$$D^l \mathbf{u}(\mathbf{y}) + p(\mathbf{y}) = \int\limits_{\partial\Omega} E(\mathbf{x}, \mathbf{y})\mathbf{n}(\mu D^l \mathbf{u} + p)\, dS(\mathbf{x}) - \int\limits_{\Omega} E(\mathbf{x}, \mathbf{y})\mathbf{f}\, d\mathbf{x}.$$

Theorem 4.2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with $\mathcal{C}^1$-boundary, $\mathbf{u} \in \mathcal{C}^2(\overline{\Omega}, \mathbb{R}^3)$, $p \in \mathcal{C}^1(\overline{\Omega}, \mathbb{R})$. If $(\mathbf{u}, p)$ satisfy the Stokes system in Ω then, for all $\mathbf{y} \in \Omega$,*

$$\mathbf{u}(\mathbf{y}) = \int\limits_{\partial\Omega} \Big[E(\mathbf{x}, \mathbf{y})\mathbf{n}\mathbf{u} + \mathcal{L}_{[\mathbf{n}(\mu D^l \mathbf{u} + p)]}(\mathbf{x}, \mathbf{y}) \Big] dS(\mathbf{x}) - \int\limits_{\Omega} \mathcal{L}_{[\mathbf{f}]}(\mathbf{x}, \mathbf{y})\, d\mathbf{x}. \tag{4.1}$$

Proof. First, we construct integral representation formulae for three components of $\mathbf{u}(\mathbf{y})$. Let $\mathbf{y} \in \Omega$ arbitrarily. Choose $\epsilon > 0$ such that

$$B_{\mathbf{y},\epsilon} := \left\{ \mathbf{x} \in \mathbb{R}^3 \,\middle|\, |\mathbf{y} - \mathbf{x}| \le \epsilon \right\} \subset \Omega.$$

Denote $\Omega_\epsilon := \Omega \setminus B_{\mathbf{y},\epsilon}$. Applying Lemma 4.1 for two pairs of the functions $(\mathbf{u}(\mathbf{x}), p(\mathbf{x}))$ and $(V_k(\mathbf{x}, \mathbf{y}), Q_k(\mathbf{x}, \mathbf{y}))$ in the domain Ω_ϵ, we have

$$\mathcal{A}_{k,\partial\Omega_\epsilon}(\mathbf{y}) - \mathcal{B}_{k,\partial\Omega_\epsilon}(\mathbf{y}) = \mathcal{T}_{k,\Omega_\epsilon}(\mathbf{y}), \tag{4.2}$$

where

$$\mathcal{A}_{k,\partial\Omega}(\mathbf{y}) = \int\limits_{\partial\Omega} \Big[(\mu \mathbf{u} D^r + p)\mathbf{n} V_k(\mathbf{x}, \mathbf{y}) + V_k(\mathbf{x}, \mathbf{y})\mathbf{n}(\mu D^l \mathbf{u} + p) \Big] dS(\mathbf{x}),$$

$$\mathcal{B}_{k,\partial\Omega}(\mathbf{y}) = \int\limits_{\partial\Omega} \Big[(\mu V_k(\mathbf{x}, \mathbf{y}) D^r_{\mathbf{x}} + Q_k(\mathbf{x}, \mathbf{y}))\mathbf{n}\mathbf{u} + \mathbf{u}\mathbf{n}(\mu D^l_{\mathbf{x}} V_k(\mathbf{x}, \mathbf{y}) + Q_k(\mathbf{x}, \mathbf{y})) \Big] dS(\mathbf{x}),$$

$$\mathcal{T}_{k,\Omega}(\mathbf{y}) = \int\limits_{\Omega} (V_k(\mathbf{x}, \mathbf{y})\mathbf{f} + \mathbf{f} V_k(\mathbf{x}, \mathbf{y}))d\mathbf{x}.$$

Since $\partial\Omega_\epsilon = \partial\Omega \cup \partial B_{\mathbf{y},\epsilon}$, the equality (4.2) becomes

$$\mathcal{A}_{k,\partial\Omega_\epsilon}(\mathbf{y}) - \mathcal{B}_{k,\partial\Omega}(\mathbf{y}) + \mathcal{B}_{k,\partial B_{\mathbf{y},\epsilon}}(\mathbf{y}) = \mathcal{T}_{k,\Omega_\epsilon}(\mathbf{y}).$$

Let $\epsilon \to 0^+$ in the above equality. Then we obtain

$$\lim_{\epsilon \to 0^+} \mathcal{A}_{k,\partial\Omega_\epsilon}(\mathbf{y}) = \mathcal{A}_{k,\partial\Omega}(\mathbf{y}), \qquad \lim_{\epsilon \to 0^+} \mathcal{T}_{k,\Omega_\epsilon}(\mathbf{y}) = \mathcal{T}_{k,\Omega}(\mathbf{y}).$$

We shall find the limit $\lim_{\epsilon\to 0^+}\mathcal{B}_{k,\partial B_{\mathbf{y},\epsilon}}(\mathbf{y})$. By (3.1) we have

$$\mu D^l_{\mathbf{x}} V_k(\mathbf{x},\mathbf{y}) + Q_k(\mathbf{x},\mathbf{y}) = -E(\mathbf{x},\mathbf{y})e_k,$$

$$\mu V_k(\mathbf{x},\mathbf{y}) D^r_{\mathbf{x}} + Q_k(\mathbf{x},\mathbf{y}) = -e_k E(\mathbf{x},\mathbf{y}),$$

$$\lim_{\epsilon\to 0^+}\mathcal{B}_{k,\partial B_{\mathbf{y},\epsilon}}(\mathbf{y}) = -\lim_{\epsilon\to 0^+}\int\limits_{|\mathbf{y}-\mathbf{x}|=\epsilon}\big[e_k E(\mathbf{x},\mathbf{y})\mathbf{n}\mathbf{u}(\mathbf{x}) + \mathbf{u}(\mathbf{x})\mathbf{n}E(\mathbf{x},\mathbf{y})e_k\big]dS(\mathbf{x}) =$$

$$= -\lim_{\epsilon\to 0^+}\int\limits_{|\mathbf{y}-\mathbf{x}|=\epsilon}\big[e_k E(\mathbf{x},\mathbf{y})\mathbf{n}\mathbf{u}(\mathbf{y}) + \mathbf{u}(\mathbf{y})\mathbf{n}E(\mathbf{x},\mathbf{y})e_k\big]dS(\mathbf{x}) =$$

$$= -\big[e_k\mathbf{u}(\mathbf{y}) + \mathbf{u}(\mathbf{y})e_k\big] = 2u_k(\mathbf{y}).$$

We obtain the representation for $u_k(\mathbf{y})$:

$$u_k(\mathbf{y}) = \frac{1}{2}\big[\mathcal{B}_{k,\partial\Omega}(\mathbf{y}) + \mathcal{T}_{k,\Omega}(\mathbf{y}) - \mathcal{A}_{k,\partial\Omega}(\mathbf{y})\big], \quad k=1,2,3.$$

Combining the three components of the function u, we obtain

$$\mathbf{u}(\mathbf{y}) = \frac{1}{2}\left[\sum_{k=1}^{3} e_k\mathcal{B}_{k,\partial\Omega}(\mathbf{y}) + \sum_{k=1}^{3} e_k\mathcal{T}_{k,\Omega}(\mathbf{y}) - \sum_{k=1}^{3} e_k\mathcal{A}_{k,\partial\Omega}(\mathbf{y})\right]. \tag{4.3}$$

We shall simplify the three terms in (4.3). Simplifying the first term of (4.3):

$$\sum_{k=1}^{3} e_k\mathcal{B}_{k,\partial\Omega}(\mathbf{y}) = \int\limits_{\partial\Omega}\sum_{k=1}^{3} e_k\Big[(\mu V_k(\mathbf{x},\mathbf{y})D^r_{\mathbf{x}} + Q_k(\mathbf{x},\mathbf{y}))\mathbf{n}\mathbf{u} +$$

$$+\mathbf{u}\mathbf{n}(\mu D^l_{\mathbf{x}} V_k(\mathbf{x},\mathbf{y}) + Q_k(\mathbf{x},\mathbf{y}))\Big]dS(\mathbf{x}) =$$

$$= -\int\limits_{\partial\Omega}\sum_{k=1}^{3} e_k\big[e_k E(\mathbf{x},\mathbf{y})\mathbf{n}\mathbf{u} + \mathbf{u}\mathbf{n}E(\mathbf{x},\mathbf{y})e_k\big]dS(\mathbf{x}) =$$

$$= \int\limits_{\partial\Omega}\left[3E(\mathbf{x},\mathbf{y})\mathbf{n}\mathbf{u} - \sum_{k=1}^{3} e_k\mathbf{u}\mathbf{n}E(\mathbf{x},\mathbf{y})e_k\right]dS(\mathbf{x}) =$$

$$= \int\limits_{\partial\Omega}\big[2E(\mathbf{x},\mathbf{y})\mathbf{n}\mathbf{u} - \mathbf{n}\mathbf{u}E(\mathbf{x},\mathbf{y}) + E(\mathbf{x},\mathbf{y})\mathbf{u}\mathbf{n}\big]dS(\mathbf{x}) =$$

$$= 2\int\limits_{\partial\Omega}\mathbf{Vec}\big[E(\mathbf{x},\mathbf{y})\mathbf{n}\mathbf{u}\big]dS(\mathbf{x}). \tag{4.4}$$

Simplifying the second term of (4.3):

$$\sum_{k=1}^{3} e_k\mathcal{T}_{k,\Omega}(\mathbf{y}) = \int\limits_{\Omega}\left(\sum_{k=1}^{3} e_k V_k(\mathbf{x},\mathbf{y})\mathbf{f} + \sum_{k=1}^{3} e_k\mathbf{f}V_k(\mathbf{x},\mathbf{y})\right)d\mathbf{x},$$

$$\sum_{k=1}^{3} e_k V_k(\mathbf{x},\mathbf{y})\mathbf{f} = -\frac{2}{\mu} H(\mathbf{x},\mathbf{y})\mathbf{f}, \tag{4.5}$$

$$\sum_{k=1}^{3} e_k \mathbf{f} V_k(\mathbf{x},\mathbf{y}) = \frac{1}{2\mu}\sum_{k=1}^{3} e_k \mathbf{f}\big[e_k H(\mathbf{x},\mathbf{y}) + (x_k - y_k)E(\mathbf{x},\mathbf{y})\big] =$$

$$= \frac{1}{2\mu} H(\mathbf{x},\mathbf{y})\mathbf{f} + \frac{1}{2\mu}(\mathbf{x}-\mathbf{y})\mathbf{f}E(\mathbf{x},\mathbf{y}). \tag{4.6}$$

Combining (4.5), (4.6), we have

$$\sum_{k=1}^{3} e_k \mathcal{T}_{k,\Omega}(\mathbf{y}) = \frac{-1}{2\mu}\int_{\Omega}\big[3H(\mathbf{x},\mathbf{y})\mathbf{f} + (\mathbf{y}-\mathbf{x})\mathbf{f}E(\mathbf{x},\mathbf{y})\big]d\mathbf{x} =$$

$$= -2\int_{\Omega}\mathcal{L}_{[\mathbf{f}]}(\mathbf{x},\mathbf{y})\,d\mathbf{x}. \tag{4.7}$$

Simplifying the third term of (4.3):

$$\sum_{k=1}^{3} e_k \mathcal{A}_{k,\partial\Omega}(\mathbf{y}) = \sum_{k=1}^{3}\int_{\partial\Omega} e_k\big[(\mu\mathbf{u}D^r + p)\mathbf{n}V_k(\mathbf{x},\mathbf{y}) +$$

$$+ V_k(\mathbf{x},\mathbf{y})\mathbf{n}(\mu D^l\mathbf{u} + p)\big]dS(\mathbf{x}),$$

$$\sum_{k=1}^{3} e_k(\mu\mathbf{u}D^r + p)\mathbf{n}V_k(\mathbf{x},\mathbf{y}) = \frac{1}{2\mu}\sum_{k=1}^{3} e_k(\mu\mathbf{u}D^r + p)\mathbf{n}\big[e_k H(\mathbf{x},\mathbf{y}) + (x_k - y_k)E(\mathbf{x},\mathbf{y})\big] =$$

$$= \frac{1}{2\mu}\sum_{k=1}^{3} e_k(\mu\mathbf{u}D^r + p)\mathbf{n}e_k H(\mathbf{x},\mathbf{y}) + \frac{1}{2\mu}(\mathbf{x}-\mathbf{y})(\mu\mathbf{u}D^r + p)\mathbf{n}E(\mathbf{x},\mathbf{y}) =$$

$$= \frac{1}{2}D^l\mathbf{u}\mathbf{n}H(\mathbf{x},\mathbf{y}) + \mathbf{n}D^l uH(\mathbf{x},\mathbf{y}) + \frac{1}{2\mu}p\mathbf{n}H(\mathbf{x},\mathbf{y}) +$$

$$+ \frac{1}{2\mu}(\mathbf{x}-\mathbf{y})(-\mu D^l\mathbf{u} + p)\mathbf{n}E(\mathbf{x},\mathbf{y}), \tag{4.8}$$

$$\sum_{k=1}^{3} e_k V_k(\mathbf{x},\mathbf{y})\mathbf{n}(\mu D^l\mathbf{u} + p) = \frac{1}{2\mu}\sum_{k=1}^{3} e_k[e_k H(\mathbf{x},\mathbf{y}) +$$

$$+ (x_k - y_k)E(\mathbf{x},\mathbf{y})]\mathbf{n}(\mu D^l\mathbf{u} + p) =$$

$$= \frac{-3}{2\mu}H(\mathbf{x},\mathbf{y})\mathbf{n}(\mu D^l\mathbf{u} + p) + \frac{1}{2\mu}(\mathbf{x}-\mathbf{y})E(\mathbf{x},\mathbf{y})\mathbf{n}(\mu D^l\mathbf{u} + p) =$$

$$= -\frac{2}{\mu}H(\mathbf{x},\mathbf{y})\mathbf{n}(\mu D^l\mathbf{u} + p). \tag{4.9}$$

Combining (4.8), (4.9), we get

$$\sum_{k=1}^{3} e_k \mathcal{A}_{k,\partial\Omega} = \frac{-1}{4\mu} \int\limits_{\partial\Omega} \Big\{ 3 \Big[\mu \mathbf{n} D^l \mathbf{u} - \mu D^l \mathbf{u}\mathbf{n} + 2p\mathbf{n} \Big] H(\mathbf{x},\mathbf{y}) +$$

$$+ (\mathbf{y}-\mathbf{x}) \Big[\mu \mathbf{n} D^l \mathbf{u} - \mu D^l \mathbf{u}\mathbf{n} + 2p\mathbf{n} \Big] E(\mathbf{x},\mathbf{y}) \Big\}\, dS(\mathbf{x}) =$$

$$= -2 \int\limits_{\partial\Omega} \mathcal{L}_{[\mu\mathbf{n}\times D^l\mathbf{u}+p\mathbf{n}]}(\mathbf{x},\mathbf{y})\, dS(\mathbf{x}) =$$

$$= -2 \int\limits_{\partial\Omega} \mathcal{L}_{[\mathbf{n}(\mu D^l \mathbf{u}+p)]}(\mathbf{x},\mathbf{y})\, dS(\mathbf{x}) - 2\mu \int\limits_{\partial\Omega} \mathcal{L}_{[\mathbf{n}\cdot D^l\mathbf{u}]}(\mathbf{x},\mathbf{y})\, dS(\mathbf{x}) =$$

$$= -2 \int\limits_{\partial\Omega} \mathcal{L}_{[\mathbf{n}(\mu D^l \mathbf{u}+p)]}(\mathbf{x},\mathbf{y})\, dS(\mathbf{x}) - 2 \int\limits_{\partial\Omega} H(\mathbf{x},\mathbf{y})\mathbf{n}\cdot D^l\mathbf{u}\, dS(\mathbf{x}). \tag{4.10}$$

Finally, from (4.4), (4.7), (4.10), we obtain an integral representation formula for $\mathbf{u}$:

$$\mathbf{u}(\mathbf{y}) = \int\limits_{\partial\Omega} \big[\mathbf{Vec}(E(\mathbf{x},\mathbf{y})\mathbf{n}\mathbf{u}) + H(\mathbf{x},\mathbf{y})\mathbf{n}\cdot D^l\mathbf{u} \big] dS(\mathbf{x}) +$$

$$+ \int\limits_{\partial\Omega} \mathcal{L}_{[\mathbf{n}(\mu D^l\mathbf{u}+p)]}(\mathbf{x},\mathbf{y})\, dS(\mathbf{x}) - \int\limits_{\Omega} \mathcal{L}_{[\mathbf{f}]}(\mathbf{x},\mathbf{y})\, d\mathbf{x}. \tag{4.11}$$

We shall prove that

$$\int\limits_{\partial\Omega} \big[\mathbf{Vec}(E(\mathbf{x},\mathbf{y})\mathbf{n}\mathbf{u}) + H(\mathbf{x},\mathbf{y})\mathbf{n}\cdot D^l\mathbf{u} \big]\, dS(x) = \int\limits_{\partial\Omega} E(\mathbf{x},\mathbf{y})\mathbf{n}\mathbf{u}\, dS(\mathbf{x}). \tag{4.12}$$

Let $\mathbf{y} \in \Omega$ be choosen arbitrarily. Choose $\epsilon > 0$ such that

$$B_{\mathbf{y},\epsilon} := \big\{ x \in \mathbb{R}^3 \,\big|\, |\mathbf{y}-\mathbf{x}| \le \epsilon \big\} \subset \Omega.$$

Denote $\Omega_\epsilon := \Omega \setminus B_{\mathbf{y},\epsilon}$. Applying Lemma 2.1, we have

$$\int\limits_{\partial\Omega_\epsilon} \big[E(\mathbf{x},\mathbf{y})\mathbf{n}\mathbf{u} + H(\mathbf{x},\mathbf{y})\mathbf{n}D^l\mathbf{u} \big] dS(\mathbf{x}) =$$

$$= \int\limits_{\Omega_\epsilon} \Big[E(\mathbf{x},\mathbf{y})D^l\mathbf{u} + \Big(D^l_{\mathbf{x}} H(\mathbf{x},\mathbf{y}) D^l\mathbf{u} - H(\mathbf{x},\mathbf{y})\Delta\mathbf{u} \Big) \Big] d\mathbf{x} =$$

$$= -\int\limits_{\Omega_\epsilon} H(\mathbf{x},\mathbf{y})\Delta\mathbf{u}\, d\mathbf{x} \quad \big(\text{by definition } E(\mathbf{x},\mathbf{y}) = -D^l_{\mathbf{x}} H(\mathbf{x},\mathbf{y})\big).$$

Let $\epsilon \to 0^+$ in the above equality. Then we obtain

$$\int\limits_{\partial\Omega} \big[E(\mathbf{x},\mathbf{y})\mathbf{n}\mathbf{u} + H(\mathbf{x},\mathbf{y})\mathbf{n}D^l\mathbf{u} \big] dS(\mathbf{x}) - \mathbf{u}(\mathbf{y}) = -\int\limits_{\Omega} H(\mathbf{x},\mathbf{y})\Delta\mathbf{u}\, d\mathbf{x}.$$

It implies that $\int_{\partial\Omega}\Big[\mathbf{Sc}(E(\mathbf{x},\mathbf{y})\mathbf{n}u) - H(\mathbf{x},\mathbf{y})\mathbf{n}\cdot D^l\mathbf{u}\Big]dS(\mathbf{x}) = 0$. Then

$$\int\limits_{\partial\Omega}\big[\mathbf{Vec}(E(\mathbf{x},\mathbf{y})\mathbf{nu}) + H(\mathbf{x},\mathbf{y})\mathbf{n}\cdot D\mathbf{u}\big]\,dS(\mathbf{x}) =$$

$$= \int\limits_{\partial\Omega}\big[\mathbf{Vec}(E(\mathbf{x},\mathbf{y})\mathbf{nu}) + \mathbf{Sc}(E(\mathbf{x},\mathbf{y})\mathbf{nu})\big]dS(\mathbf{x}) = \int\limits_{\partial\Omega} E(\mathbf{x},\mathbf{y})\mathbf{nu}\,dS(\mathbf{x}).$$

From (4.11), (4.12), we get the formula (4.1).

Theorem 4.2 is proved.

5. Solutions of homogeneous Stokes system. We consider the homogeneous system

$$\begin{aligned} &-\Delta\mathbf{u} + \operatorname{grad} p = 0,\\ &\operatorname{div}\ \mathbf{u} = 0. \end{aligned} \tag{5.1}$$

The representation (4.1) becomes

$$\mathbf{u}(\mathbf{y}) = \int\limits_{\partial\Omega}\Big[E(\mathbf{x},\mathbf{y})\mathbf{nu} + \mathcal{L}_{[\mathbf{n}(\mu D^l\mathbf{u}+p)]}(\mathbf{x},\mathbf{y})\Big]dS(\mathbf{x}). \tag{5.2}$$

In the following, we shall show that in a bounded domain with $\mathcal{C}^1$-boundary, the representation (5.2) covers some results in [4, 6, 14, 15].

The Grigor'ev – Gürlebeck – Legatiuk – Yakovlev solution [6].

Theorem 5.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with $\mathcal{C}^1$-boundary, $\mathbf{u} \in \mathcal{C}^2(\overline{\Omega},\mathbb{R}^3)$, $p \in \mathcal{C}^1(\overline{\Omega},\mathbb{R})$. The pair $(\mathbf{u},p)$ is a solutions of the system* (5.1) *in Ω if and only if there exist three quaternion valued functions h, g, G in Ω, satisfying the conditions*

$$D^l h = 0, \quad D^l g = 0, \quad D^l G = g,$$

$$\mathbf{Sc}(G) = 0, \quad \mathbf{Sc}(g) = \frac{1}{4\mu}p, \quad \mathbf{Sc}(h(\mathbf{x}) + g(\mathbf{x})\mathbf{x}) = 0,$$

such that $\mathbf{u}$, p admit the representations

$$\begin{aligned} &\mathbf{u}(\mathbf{x}) = h(\mathbf{x}) + 3G(\mathbf{x}) + g(\mathbf{x})\mathbf{x},\\ &p(\mathbf{x}) = 4\mu\mathbf{Sc}(g(\mathbf{x})). \end{aligned} \tag{5.3}$$

Proof. Suppose that $(\mathbf{u},p)$ is a pair of solutions of the system (5.1) in Ω. Rewrite the representation (5.2) in the form

$$u(y) = \int\limits_{\partial\Omega} E\mathbf{nu}\,dS(\mathbf{x}) - \frac{1}{4\mu}\int\limits_{\partial\Omega} E(\mathbf{x},\mathbf{y})\mathbf{n}(\mu D^l\mathbf{u} + p)\mathbf{x}\,dS(\mathbf{x}) +$$

$$+\frac{3}{4\mu}\int\limits_{\partial\Omega} H(\mathbf{x},\mathbf{y})\mathbf{n}(\mu D^l\mathbf{u} + p)\,dS(\mathbf{x}) +$$

$$+\frac{1}{4\mu}\int\limits_{\partial\Omega} E(\mathbf{x},\mathbf{y})\mathbf{n}(\mu D^l\mathbf{u}+p)\mathbf{y}\,dS(\mathbf{x}).$$

Define

$$h(\mathbf{y})=\frac{1}{4\mu}\int\limits_{\partial\Omega} E(\mathbf{x},\mathbf{y})\mathbf{n}\big[\mu\mathbf{u}-(\mu D^l\mathbf{u}+p)\mathbf{x}\big]dS(\mathbf{x}), \tag{5.4}$$

$$g(\mathbf{y})=\frac{1}{4\mu}\int\limits_{\partial\Omega} E(\mathbf{x},\mathbf{y})\mathbf{n}(\mu D^l\mathbf{u}+p)\,dS(\mathbf{x})=\frac{1}{4\mu}(\mu D^l\mathbf{u}+p), \tag{5.5}$$

$$G(\mathbf{y})=\frac{1}{4\mu}\int\limits_{\partial\Omega}\big[H(\mathbf{x},\mathbf{y})\mathbf{n}(\mu D^l\mathbf{u}+p)+\mu E(\mathbf{x},\mathbf{y})\mathbf{n}\mathbf{u}\big]dS(\mathbf{x}). \tag{5.6}$$

Then $D^lg=0,\ D^lG=g$, and, from (4.12), it follows that $\mathbf{Sc}(G)=0$. We obtain the representation (5.3).

Conversely, suppose that there are three quaternionic valued functions $h,\ g,\ G$ in Ω such that $D^lh=0,\ D^lg=0,\ D^lG=g,\ \mathbf{Sc}(g)=\dfrac{1}{4\mu}p,\ \mathbf{Sc}(h(\mathbf{x})+g(\mathbf{x})\mathbf{x})=0$. Let $\mathbf{u}$, p defined by (5.3). We shall prove that $(\mathbf{u},p)$ satisfy the system (5.1). We have

$$u_0(\mathbf{x})=h_0(\mathbf{x})-\mathbf{g}(\mathbf{x})\cdot\mathbf{x}=0,$$

$$\mu D^l\mathbf{u}(\mathbf{x})+p(\mathbf{x})=\mu D^l[h(\mathbf{x})+3G(\mathbf{x})+g(\mathbf{x})\mathbf{x}]+4\mu g_0(\mathbf{x})=$$

$$=3\mu g(\mathbf{x})+\mu\sum_{i=1}^{3}e_ig(\mathbf{x})e_i+4\mu g_0(\mathbf{x})=$$

$$=3\mu g(\mathbf{x})-3\mu g_0(\mathbf{x})+\mu\mathbf{g}(\mathbf{x})+4\mu g_0(\mathbf{x})=4\mu g(\mathbf{x}),$$

$$D^l(\mu D^l\mathbf{u}+p)=4\mu D^lg(\mathbf{x})=0,$$

$$u=\mathbf{h}+3G+g_0(x)x+(x_3\mathbf{g}_2-x_2\mathbf{g}_3)e_1+(x_1\mathbf{g}_3-x_3\mathbf{g}_1)e_2+(x_2\mathbf{g}_1-x_1\mathbf{g}_2)e_3,$$

$$\operatorname{div}(\mathbf{u})=\operatorname{div}(\mathbf{h})+3\operatorname{div}(G)+3g_0+x_1\frac{\partial\mathbf{g}_0}{\partial x_1}+x_2\frac{\partial\mathbf{g}_0}{\partial x_2}+x_3\frac{\partial\mathbf{g}_0}{\partial x_3}+$$

$$+x_3\frac{\partial\mathbf{g}_2}{\partial x_1}-x_2\frac{\partial\mathbf{g}_3}{\partial x_1}+x_1\frac{\partial\mathbf{g}_3}{\partial x_2}-x_3\frac{\partial\mathbf{g}_1}{\partial x_2}+x_2\frac{\partial\mathbf{g}_1}{\partial x_3}-x_1\frac{\partial\mathbf{g}_2}{\partial x_3}. \tag{5.7}$$

Since $D^lh=0,\ D^lG=g$, substituting $\operatorname{div}(\mathbf{h})=0,\ \operatorname{div}(G)=-g_0$ into (5.7), we get

$$\operatorname{div}(\mathbf{u})=x_1\left(\frac{\partial\mathbf{g}_0}{\partial x_1}+\frac{\partial\mathbf{g}_3}{\partial x_2}-\frac{\partial\mathbf{g}_2}{\partial x_3}\right)+x_2\left(\frac{\partial\mathbf{g}_0}{\partial x_2}-\frac{\partial\mathbf{g}_3}{\partial x_1}+\frac{\partial\mathbf{g}_1}{\partial x_3}\right)+$$

$$+x_3\left(\frac{\partial\mathbf{g}_0}{\partial x_3}+\frac{\partial\mathbf{g}_2}{\partial x_1}-\frac{\partial\mathbf{g}_1}{\partial x_2}\right)=0.$$

Theorem 5.1 is proved.

The Papkovich – Neuber solution [4].

Theorem 5.2. *Let* $\Omega \subset \mathbb{R}^3$ *be a bounded, is simply connected domain with* $\mathcal{C}^1$*-boundary,* $\mathbf{u} \in \mathcal{C}^2(\overline{\Omega}, \mathbb{R}^3)$, $p \in \mathcal{C}^1(\overline{\Omega}, \mathbb{R})$. *The pair* $(\mathbf{u}, p)$ *is a solutions of the system* (5.1) *in* Ω *if and only if there exist a harmonic vectors* Φ *and a harmonic scalar function* Ψ *in* Ω *such that*

$$\begin{aligned} \mathbf{u} &= \Phi - \frac{1}{2}\mathrm{grad}(\mathbf{x} \cdot \Phi + \Psi), \\ p &= -\mu \mathrm{div}(\Phi). \end{aligned} \tag{5.8}$$

Proof. Suppose that $(\mathbf{u}, p)$ is a pair of solutions of the system (5.1) in Ω. The representation formula of p reads

$$p(\mathbf{y}) = \mathbf{Sc} \int\limits_{\partial\Omega} E(\mathbf{x}, \mathbf{y})\mathbf{n}(\mu D^l \mathbf{u} + p)\, dS(\mathbf{x}).$$

Define

$$\Phi(\mathbf{y}) = 4G = \frac{1}{\mu} \int\limits_{\partial\Omega} \big[H(\mathbf{x}, \mathbf{y})(\mu \mathbf{n} D^l \mathbf{u} + p\mathbf{n}) + E(\mathbf{x}, \mathbf{y})\mathbf{n}\mathbf{u}\big] dS(\mathbf{x}).$$

Then $\Delta\Phi = 0$. Further,

$$-\mu\, \mathrm{div}\, [\Phi(\mathbf{y})] = \mu \mathbf{Sc}\big[D^l_{\mathbf{y}} \Phi(\mathbf{y})\big] = \mathbf{Sc} \int\limits_{\partial\Omega} E(\mathbf{x}, \mathbf{y})(\mu \mathbf{n} D^l \mathbf{u} + p\mathbf{n})\, dS(\mathbf{x}) =$$

$$= \mathbf{Sc} \int\limits_{\partial\Omega} E(\mathbf{x}, \mathbf{y})\mathbf{n}(\mu D^l \mathbf{u} + p)\, dS(\mathbf{x}) = p(\mathbf{y}),$$

$$\mathbf{u} - \left(\Phi - \frac{1}{2}\,\mathrm{grad}(\mathbf{x} \cdot \Phi)\right) = h(\mathbf{x}) + 3G + g(\mathbf{x})\mathbf{x} - 4G + 2\,\mathrm{grad}(\mathbf{x} \cdot G),$$

$$D\left[\mathbf{u} - \left(\Phi - \frac{1}{2}\,\mathrm{grad}(\mathbf{x} \cdot \Phi)\right)\right] = -DG + D(g(\mathbf{x})\mathbf{x}) - 2\Delta(\mathbf{x} \cdot G) =$$

$$= -g + \sum_{i=1}^{3} e_i g(\mathbf{x}) e_i - 4\mathrm{div}(G) = -g - 3g_0 + \mathbf{g} + 4g_0 = 0.$$

There exists a real-valued harmonic function Ψ such that

$$\mathbf{u} - \left(\Phi - \frac{1}{2}\,\mathrm{grad}(\mathbf{x} \cdot \Phi)\right) = \mathrm{grad}(-\Psi).$$

It leads to the Papkovich – Neuber representation

$$\mathbf{u} = \Phi - \frac{1}{2}\,\mathrm{grad}(\mathbf{x} \cdot \Phi + \Psi).$$

Conversely, suppose that Φ is a harmonic vectors and Ψ is a harmonic scalar function in Ω. The pair $(\mathbf{u}, p)$ is defined by (5.8). Then

$$\mu D^l \mathbf{u} + p = \mu D^l \Phi - \frac{\mu}{2}(D^l)^2(\mathbf{x} \cdot \Phi + \Psi) + p =$$

$$= \mu D^l \Phi + \frac{\mu}{2} \Delta(\mathbf{x} \cdot \Phi + \Psi) + p =$$

$$= \mu D^l \Phi + \mu \operatorname{div} \Phi + p = \mu D^l \Phi,$$

$$D^l(\mu D^l \mathbf{u} + p) = -\Delta \Phi = 0.$$

That means $(\mathbf{u}, p)$ satisfy the system (5.1).

Theorem 5.2 is proved.

The Naghdi – Hsu solution [14, 15].

Theorem 5.3. *Let* $\Omega \subset \mathbb{R}^3$ *be a bounded domain with* $\mathcal{C}^1$*-boundary,* $\mathbf{u} \in \mathcal{C}^2(\overline{\Omega}, \mathbb{R}^3)$*,* $p \in \in \mathcal{C}^1(\overline{\Omega}, \mathbb{R})$*. The pair* $(\mathbf{u}, p)$ *is a solutions of the system* (5.1) *in* Ω *if and only if there exists a harmonic vector* Φ *in* Ω *such that*

$$\begin{aligned} &\mathbf{u} = \Phi - \operatorname{grad} \int\limits_{\Omega} H(\mathbf{x}, \mathbf{y}) \operatorname{div}(\Phi)\, d\mathbf{x}, \\ &p = -\mu \operatorname{div}(\Phi). \end{aligned} \tag{5.9}$$

Proof. Suppose that $(\mathbf{u}, p)$ is a pair of solutions of the system (5.1) in Ω. Define

$$\Phi(\mathbf{x}) = 4G = \frac{1}{\mu} \int\limits_{\partial\Omega} \big[H(\mathbf{x}, \mathbf{y})(\mathbf{n}D\mathbf{u} + p\mathbf{n}) + E(\mathbf{x}, \mathbf{y})\mathbf{n}\mathbf{u}\big]\, dS(\mathbf{x}).$$

Then $\Delta\Phi = 0$ and $p = -\mu \operatorname{div}(\Phi)$. We shall prove that $\mathbf{u} = \Phi - \operatorname{grad} \int_{\Omega} H(\mathbf{x}, \mathbf{y}) \operatorname{div}(\Phi)\, d\mathbf{x}$. We have

$$\mathbf{u}(\mathbf{y}) - \Phi(\mathbf{y}) = h(\mathbf{y}) + 3G(\mathbf{y}) + g(\mathbf{y})\mathbf{y} - 4G = h(\mathbf{y}) - G(\mathbf{y}) + g(\mathbf{y})\mathbf{y} =$$

$$= \frac{1}{4\mu} \left[(\mu D^l \mathbf{u}(\mathbf{y}) + p(\mathbf{y}))\mathbf{y} - \int\limits_{\partial\Omega} E(\mathbf{x}, \mathbf{y})\mathbf{n}(\mu D^l \mathbf{u} + p)\mathbf{x}\, dS(\mathbf{x}) - \right.$$

$$\left. - \int\limits_{\partial\Omega} H(\mathbf{x}, \mathbf{y})\mathbf{n}(\mu D^l \mathbf{u} + p)\, dS(\mathbf{x}) \right] =$$

$$= -\frac{1}{4\mu} \int\limits_{\Omega} E(\mathbf{x}, \mathbf{y}) D^l[(\mu D^l \mathbf{u} + p)\mathbf{x}]\, d\mathbf{x} + \frac{1}{4\mu} \int\limits_{\Omega} E(\mathbf{x}, \mathbf{y})(\mu D^l \mathbf{u} + p)\, d\mathbf{x} =$$

$$= -\frac{1}{4\mu} \int\limits_{\Omega} E(\mathbf{x}, \mathbf{y})[-3p + \mu D^l \mathbf{u}]\, d\mathbf{x} + \frac{1}{4\mu} \int\limits_{\Omega} E(\mathbf{x}, \mathbf{y})(\mu D^l \mathbf{u} + p)\, d\mathbf{x} =$$

$$= \frac{1}{\mu} \int\limits_{\Omega} E(\mathbf{x}, \mathbf{y}) p\, d\mathbf{x} = \frac{1}{\mu} D^l_{\mathbf{y}} \int\limits_{\Omega} H(\mathbf{x}, \mathbf{y}) p\, d\mathbf{x} = -\operatorname{grad}_{\mathbf{y}} \int\limits_{\Omega} H(\mathbf{x}, \mathbf{y}) \operatorname{div}(\Phi)\, d\mathbf{x}.$$

Conversely, suppose that Φ is a harmonic vectors in Ω. The pair $(\mathbf{u}, p)$ is defined by (5.9). Then

$$\mu D^l \mathbf{u} + p = \mu D^l \Phi - \mu (D^l)^2 \int\limits_{\Omega} H(\mathbf{x}, \mathbf{y}) \operatorname{div}(\Phi)\, d\mathbf{x} + p =$$

$$= \mu D^l \Phi + \mu\Delta \int\limits_{\Omega} H(\mathbf{x}, \mathbf{y}) \mathrm{div}(\Phi)\, d\mathbf{x} + p =$$

$$= \mu D^l \Phi + \mu \mathrm{div}(\Phi) + p = \mu D^l \Phi,$$

$$D^l(\mu D^l \mathbf{u} + p) = -\Delta\Phi = 0.$$

That means $(\mathbf{u}, p)$ satisfy the system (5.1).

Theorem 5.3 is proved.

6. Conclusions. In this paper, we reformulate the fundamental solutions of the Stokes system in quaternion forms. Then using these fundamental solutions, we construct new integral representations for solutions of the Stokes system in framework of quaternion analysis. In classical real analysis, matrix notations are used in the integral representation formulae. In quaternion analysis, integral representation formulae are more compact and we can see some different structures of the solutions.

References

1. L. Bers, *Theory of pseudo-analytic functions*, New York Univ., New York (1953).
2. S. Bock, K. Gürlebeck, D. Legatiuk, H. M. Nguyen, *Ψ-hyperholomorphic functions and a Kolosov – Muskhelishvili formula*, Math. Methods Appl. Sci., **38**, № 18, 5114 – 5123 (2015).
3. S. Bock, *On monogenic series expansions with applications to linear elasticity*, Adv. Appl. Clifford Algebras, **24**, 931 – 943 (2014).
4. T. T. Cong, J. Blake, *General solution of the Stokes flow equations*, J. Math. Anal. and Appl., **90**, 72 – 84 (1982).
5. Y. Grigor'ev, *Three-dimensional analogue of Kolosov – Muskhelishvili formulae*, Modern Trends in Hypercomplex Analysis, Trends Math., Birkhäuser, Cham (2016).
6. Yu. Grigor'ev, K. Gürlebeck, D. Legatiuk, A. Yakovlev, *On quaternionic functions for the solution of an ill-posed Cauchy problem for a viscous fluid*, AIP Conf. Proc., **2116**, № 1, Article 160005 (2019).
7. Yu. Grigor'ev, *Quaternionic functions and their applications in a viscous fluid flow*, Complex Anal. and Oper. Theory, **12**, 491 – 508 (2018).
8. K. Gürlebeck, W. Sprössig, *Quaternionic and Clifford calculus for physicists and engineers*, Math. Methods Practice, Wiley, Chichester (1997).
9. K. Gürlebeck, W. Sprössig, *Quaternionic analysis and elliptic boundary value problems*, Birkhäuser, Basel and Akademie-Verlag, Berlin (1989).
10. V. V. Kravchenko, *Applied quaternionic analysis*, Birkhäuser, Basel (2009).
11. O. A. Ladyzhenskaya, *The mathematical theory of viscous incompressible flow, 2nd english ed.*, Gordon and Breach, New York etc. (1969) (translated from Russian).
12. W. E. Langlois, M. O. Deville, *Slow viscous flow, second ed.*, Springer Int. Publ., Switzerland (2014).
13. L.-W. Liu, H.-K. Hong, *Clifford algebra valued boundary integral equations for three-dimensional elasticity*, Appl. Math. Model., **54**, 246 – 267 (2018).
14. P. M. Naghdi, C. S. Hsu, *On the representation of displacements in linear elasticity in terms of three stress functions*, J. Math. and Mech., **10**, 233 – 245 (1961).
15. B. S. Padmaathi, G. P. R. Sekhar, T. Amaranath, *A note on complete general solutions of Stokes equations*, Quart. J. Mech. and Appl. Math., **51**, № 3, 383 – 388 (1998).
16. W. Sprössig, K. Gürlebeck, *On the treatment of fluid problems by methods of Clifford analysis*, Math. and Comput. Simulation, **44**, № 4, 401 – 413 (1997).
17. W. Sprössig, *Quaternionic analysis in fluid mechanics*, Birkhäuser, Boston, MA (2000).
18. I. N. Vekua, *Generalized analytic functions*, Pergamon Press, London (1962).
19. D. Weisz-Patrault, S. Bock, K. Gürlebeck, *Three-dimensional elasticity based on quaternion-valued potentials*, Int. J. Solids and Structures, **51**, № 19, 3422 – 3430 (2014).
20. M. Zabarankin, *Cauchy integral formula for generalized analytic functions in hydrodynamics*, Proc. Roy. Soc. A: Math., Phys. and Eng. Sci., **468**, 3745 – 3764 (2012).
21. M. Zabarankin, *The framework of k-harmonically analytic functions for three-dimensional Stokes flow problems, Part I*, SIAM J. Appl. Math., **89**, № 3, 845 – 880 (2008).

Received 01.04.21