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## $\pi$ -FORMULAE FROM DUAL SERIES OF THE DOUGALL THEOREM

### $\pi$ -ФОРМУЛИ З ДУАЛЬНИХ РЯДІВ ТЕОРЕМИ ДУГАЛЛА

By means of the extended Gould–Hsu inverse series relations, we find that the dual relation of Dougall’s summation theorem for the well-poised  ${}_7F_6$ -series can be used to construct numerous interesting Ramanujan-like infinite-series expressions for  $\pi^{\pm 1}$  and  $\pi^{\pm 2}$ , including an elegant formula for  $\pi^{-2}$  due to Guillera.

За допомогою узагальнених співвідношень Гулда та Хсу для оберненого ряду доведено, що дуальне співвідношення теореми підсумовування Дугалла для добре збалансованого  ${}_7F_6$ -ряду можна використати для побудови багатьох цікавих виразів для нескінченного ряду, подібних до виразів, що були отримані Рамануджаном для  $\pi^{\pm 1}$  і  $\pi^{\pm 2}$ , включаючи елегантну формулу Гільєра для  $\pi^{-2}$ .

**1. Introduction and motivation.** In 1973, Gould and Hsu [27] discovered a useful pair of inverse series relations, which can equivalently be reproduced below. Let  $\{a_i, b_i\}$  be any two complex sequences such that the  $\varphi$ -polynomials defined by

$$\varphi(x; 0) \equiv 1 \quad \text{and} \quad \varphi(x; n) = \prod_{k=0}^{n-1} (a_k + xb_k) \quad \text{for } n \in \mathbb{N}$$

differ from zero for  $x, n \in \mathbb{N}_0$ . Then there hold the inverse series relations

$$f(n) = \sum_{k=0}^n (-1)^k \binom{n}{k} \varphi(k; n) g(k),$$

$$g(n) = \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{a_k + kb_k}{\varphi(n; k+1)} f(k).$$

This inverse pair has wide applications to terminating hypergeometric series identities [9–12, 15, 24]. The duplicate form with applications can be found in [17, 18, 20]. There exist also  $q$ -analogues due to Carlitz [6] which has applications to  $q$ -series identities [13, 14, 16, 19, 25, 26].

The Gould–Hsu inversions have the following extended form (cf. [4, 9, 15]):

$$f(n) = \sum_{k=0}^n (-1)^k \binom{n}{k} \varphi(\lambda + k; n) \varphi(-k; n) \frac{\lambda + 2k}{(\lambda + n)_{k+1}} g(k), \quad (1a)$$

$$g(n) = \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{(a_k + \lambda b_k + kb_k)(a_k - kb_k)}{\varphi(\lambda + n; k+1) \varphi(-n; k+1)} (\lambda + k)_n f(k), \quad (1b)$$

where the shifted factorials are defined by

$$(x)_0 = 1 \quad \text{and} \quad (x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = x(x+1) \dots (x+n-1) \quad \text{for } n \in \mathbb{N}.$$

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There exist numerous hypergeometric series identities (see, for example, [5], Chapter 8, and [7–12, 15, 23, 24]). One of well-known summation theorems originally due to Dougall [22] is about the terminating well-poised  ${}_7F_6$ -series. By examining its dual formulae through (1a), (1b), we find that their limiting relations result unexpectedly in  $\pi$ -related infinite series expressions, including the following elegant formula discovered by Guillera [28–30]:

$$\frac{32}{\pi^2} = \sum_{k=0}^{\infty} \left[ \begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{3}{4} \\ 1, 1, 1, 1, 1 \end{matrix} \right]_k \frac{3 + 34k + 120k^2}{16^k}.$$

By means of the duplicate forms of (1a), (1b), we shall work out, in details, the dual formulae of Dougall’s summation theorem in the next section. Then applications will be presented in Section 3, where several  $\pi$ -related infinite series of Ramanujan-like [32] with the convergence rate “ $\frac{1}{16}$ ” will be illustrated as examples.

Recall that the  $\Gamma$ -function (see, for example, [31], § 8) is defined by the beta integral

$$\Gamma(x) = \int_0^{\infty} u^{x-1} e^{-u} du \quad \text{for } \Re(x) > 0,$$

which admits Euler’s reflection property

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x} \quad \text{with } \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

The asymptotic formula

$$\Gamma(x+n) \approx n^x(n-1)! \quad \text{as } n \rightarrow \infty \tag{2}$$

will be useful in evaluating limits of  $\Gamma$ -function quotients.

For the sake of brevity, the product and quotient of shifted factorials will respectively be abbreviated to

$$[\alpha, \beta, \dots, \gamma]_n = (\alpha)_n(\beta)_n \dots (\gamma)_n,$$

$$\left[ \begin{matrix} \alpha, \beta, \dots, \gamma \\ A, B, \dots, C \end{matrix} \right]_n = \frac{(\alpha)_n(\beta)_n \dots (\gamma)_n}{(A)_n(B)_n \dots (C)_n}.$$

The similar notation will be employed for the  $\Gamma$ -function quotient

$$\Gamma \left[ \begin{matrix} \alpha, \beta, \dots, \gamma \\ A, B, \dots, C \end{matrix} \right] = \frac{\Gamma(\alpha)\Gamma(\beta) \dots \Gamma(\gamma)}{\Gamma(A)\Gamma(B) \dots \Gamma(C)}.$$

**2. Main theorems from duplicate inversions.** The fundamental identity discovered by Dougall [22] (see also [3], § 4.3) for very well-poised terminating  ${}_7F_6$ -series can be stated as

$$\Omega_n(a; b, c, d) := \left[ \begin{matrix} 1+a, 1+a-b-c, 1+a-b-d, 1+a-c-d \\ 1+a-b, 1+a-c, 1+a-d, 1+a-b-c-d \end{matrix} \right]_n =$$

$$= \sum_{k=0}^n \frac{a+2k}{a} \left[ \begin{matrix} a, b, c, d, e, -n \\ 1, 1+a-b, 1+a-c, 1+a-d, 1+a-e, 1+a+n \end{matrix} \right]_k, \quad (3)$$

where the series is 2-balanced because  $1+2a+n = b+c+d+e$ .

For all  $n \in \mathbb{N}_0$ , it is well-known that  $n = \lfloor \frac{n}{2} \rfloor + \lfloor \frac{1+n}{2} \rfloor$ , where  $\lfloor x \rfloor$  denotes the greatest integer not exceeding  $x$ . Then it is not difficult to check that Dougall's formula (3) is equivalent to the following one:

$$\begin{aligned} \Omega_n \left( a; b + \lfloor \frac{n}{2} \rfloor, c, d + \lfloor \frac{1+n}{2} \rfloor \right) &= \left[ \begin{matrix} 1+a-c-d, b+c-a \\ 1+a-d, b-a \end{matrix} \right]_{\lfloor \frac{n}{2} \rfloor} \times \\ &\times \left[ \begin{matrix} 1+a, b+d-a \\ 1+a-c, b+c+d-a \end{matrix} \right]_n \left[ \begin{matrix} 1+a-b-c, c+d-a \\ 1+a-b, d-a \end{matrix} \right]_{\lfloor \frac{1+n}{2} \rfloor} \end{aligned}$$

with its parameters subject to  $\boxed{1+2a = b+c+d+e}$ . Reformulate the above equality as a binomial sum

$$\begin{aligned} &\sum_{k=0}^n (-1)^k \binom{n}{k} [b+k, b-a-k]_{\lfloor \frac{n}{2} \rfloor} [d+k, d-a-k]_{\lfloor \frac{1+n}{2} \rfloor} \times \\ &\times \frac{a+2k}{(a+n)_{k+1}} \left[ \begin{matrix} a, b, c, d, 1+2a-b-c-d \\ 1+a-b, 1+a-c, 1+a-d, b+c+d-a \end{matrix} \right]_k = \\ &= \left[ \begin{matrix} b, 1+a-c-d, b+c-a \\ 1+a-d \end{matrix} \right]_{\lfloor \frac{n}{2} \rfloor} \left[ \begin{matrix} d, 1+a-b-c, c+d-a \\ 1+a-b \end{matrix} \right]_{\lfloor \frac{1+n}{2} \rfloor} \times \\ &\quad \times \left[ \begin{matrix} a, b+d-a \\ 1+a-c, b+c+d-a \end{matrix} \right]_n. \end{aligned}$$

This equality matches exactly to (1a) under the assignments  $\lambda \rightarrow a$  and

$$\varphi(x; n) = (b-a+x)_{\lfloor \frac{n}{2} \rfloor} (d-a+x)_{\lfloor \frac{1+n}{2} \rfloor}$$

as well as

$$\begin{aligned} f(n) &= \left[ \begin{matrix} 1+a-c-d, b, b+c-a \\ 1+a-d \end{matrix} \right]_{\lfloor \frac{n}{2} \rfloor} \left[ \begin{matrix} a, b+d-a \\ 1+a-c, b+c+d-a \end{matrix} \right]_n \times \\ &\quad \times \left[ \begin{matrix} 1+a-b-c, d, c+d-a \\ 1+a-b \end{matrix} \right]_{\lfloor \frac{1+n}{2} \rfloor}, \\ g(k) &= \left[ \begin{matrix} a, b, c, d, 1+2a-b-c-d \\ 1+a-b, 1+a-c, 1+a-d, b+c+d-a \end{matrix} \right]_k. \end{aligned}$$

The dual relation corresponding to (1b) can explicitly be stated, according to the parity of  $k$  and  $(a)_k(a+k)_n = (a)_n(a+n)_k$ , as

$$\begin{aligned} & \left[ \begin{matrix} b, c, d, 1+2a-b-c-d \\ 1+a-b, 1+a-c, 1+a-d, b+c+d-a \end{matrix} \right]_n = \\ & = \sum_{k \geq 0} \binom{n}{2k} \frac{(d+3k)(d-a-k)(a+n)_{2k}}{[b+n, b-a-n]_k [d+n, d-a-n]_{k+1}} \times \\ & \times \left[ \begin{matrix} 1+a-c-d, b, b+c-a \\ 1+a-d \end{matrix} \right]_k \left[ \begin{matrix} 1+a-b-c, d, c+d-a \\ 1+a-b \end{matrix} \right]_k \times \\ & \quad \times \left[ \begin{matrix} b+d-a \\ 1+a-c, b+c+d-a \end{matrix} \right]_{2k} - \\ & - \sum_{k \geq 0} \binom{n}{2k+1} \frac{(b+3k+1)(b-a-k-1)(a+n)_{2k+1}}{[b+n, b-a-n]_{k+1} [d+n, d-a-n]_{k+1}} \times \\ & \times \left[ \begin{matrix} 1+a-c-d, b, b+c-a \\ 1+a-d \end{matrix} \right]_k \left[ \begin{matrix} 1+a-b-c, d, c+d-a \\ 1+a-b \end{matrix} \right]_{k+1} \times \\ & \quad \times \left[ \begin{matrix} b+d-a \\ 1+a-c, b+c+d-a \end{matrix} \right]_{2k+1}. \end{aligned}$$

Now multiplying by “ $n^2$ ” across this binomial relation and then letting  $n \rightarrow \infty$ , we may evaluate the limits of the left member by (2) and of the corresponding right member through the Weierstrass  $M$ -test on uniformly convergent series (cf. [33], § 3.106). After some routine simplification, the resulting limiting relation can be expressed explicitly in the following lemma.

**Lemma 1** (infinite series identity).

$$\begin{aligned} & \Gamma \left[ \begin{matrix} 1+a-b, 1+a-c, 1+a-d, b+c+d-a \\ b, c, d, 1+2a-b-c-d \end{matrix} \right] = \\ & = \sum_{k \geq 0} \frac{(d+3k)(a-d)}{(2k)!} \left[ \begin{matrix} b+d-a \\ 1+a-c, b+c+d-a \end{matrix} \right]_{2k} \times \\ & \times \left[ \begin{matrix} 1+a-c-d, b, b+c-a \\ a-d \end{matrix} \right]_k \left[ \begin{matrix} 1+a-b-c, d, c+d-a \\ 1+a-b \end{matrix} \right]_k + \\ & + \sum_{k \geq 0} \frac{(b+3k+1)(a-b)}{(2k+1)!} \left[ \begin{matrix} b+d-a \\ 1+a-c, b+c+d-a \end{matrix} \right]_{2k+1} \times \\ & \times \left[ \begin{matrix} 1+a-c-d, b, b+c-a \\ 1+a-d \end{matrix} \right]_k \left[ \begin{matrix} 1+a-b-c, d, c+d-a \\ a-b \end{matrix} \right]_{k+1}. \end{aligned}$$

According to this lemma, we are going to show two main theorems that will be utilized, in the next section, to deduce infinite series expressions for  $\pi^{\pm 1}$  and  $\pi^{\pm 2}$ .

For the equality in Lemma 1, multiplying both sides by  $(1 + a - c)(b + c + d - a)$  and then unifying the two sums, we derive the following infinite series identity.

**Theorem 1** (infinite series identity).

$$\Gamma \left[ \begin{matrix} 1 + a - b, 2 + a - c, 1 + a - d, 1 - a + b + c + d \\ b, c, d, 1 + 2a - b - c - d \end{matrix} \right] = \sum_{k=0}^{\infty} \mathcal{P}(k) \frac{[b, d, 1 + a - b - c, 1 + a - c - d, b + c - a, c + d - a]_k (b + d - a)_{2k}}{(2k + 1)! [1 + a - b, 1 + a - d]_k [2 + a - c, 1 - a + b + c + d]_{2k}},$$

where  $\mathcal{P}(k)$  is the polynomial given by

$$\mathcal{P}(k) = (1 + a - b - c + k)(d + k)(c + d - a + k)(b + d - a + 2k)(1 + b + 3k) + (1 + 2k)(a - d + k)(1 + a - c + 2k)(b + c + d - a + 2k)(d + 3k).$$

Alternatively, by shifting backward  $k \rightarrow k - 1$  for the second sum and then unifying it to the first one, we get analogously, from Lemma 2 another infinite series identity.

**Theorem 2** (infinite series identity).

$$\Gamma \left[ \begin{matrix} 1 + a - b, 1 + a - c, 1 + a - d, b + c + d - a \\ b, c, d, 1 + 2a - b - c - d \end{matrix} \right] = \sum_{k=0}^{\infty} \mathcal{Q}(k) \frac{[b, d, 1 + a - b - c, 1 + a - c - d, b + c - a, c + d - a]_k (b + d - a)_{2k}}{(2k)! [1 + a - b, 1 + a - d]_k [1 + a - c, b + c + d - a]_{2k}},$$

where  $\mathcal{Q}(k)$  is the rational function given by

$$\mathcal{Q}(k) = (a - d + k)(d + 3k) \times \left\{ 1 + \frac{(2k)(a - b + k)(a - c + 2k)(b + c + d - a - 1 + 2k)(b - 2 + 3k)}{(a - c - d + k)(b - 1 + k)(b + c - a - 1 + k)(b + d - a - 1 + 2k)(d + 3k)} \right\}.$$

**3. Infinite series for  $\pi^{\pm 1}$  and  $\pi^{\pm 2}$ .** By applying Theorems 1 and 2, we can derive numerous infinite series identities. They are recorded below in seven classes whose weight polynomial degrees are not greater than 3. For all the examples, the parameter settings  $[a, b, c, d]$  and eventual references are highlight in their headers. In order to ensure the accuracy, all the summation formulae in this section are verified experimentally by appropriately devised *Mathematica* commands.

**3.1. Series for  $\pi^{-2}$ .**

**Example 1** (Guillera [28–30]:  $\left[ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right]$  in Theorem 1):

$$\frac{32}{\pi^2} = \sum_{k=0}^{\infty} \left[ \begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{3}{4} \\ 1, 1, 1, 1, 1 \end{matrix} \right]_k \frac{120k^2 + 34k + 3}{16^k}.$$

**Example 2** (Chu and Zhang [21]:  $\left[ \frac{3}{2}, \frac{1}{2}, \frac{3}{2}, \frac{1}{2} \right]$  in Theorem 1):

$$\frac{128}{\pi^2} = \sum_{k=0}^{\infty} \left[ \begin{array}{c} \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, -\frac{1}{4} \\ 1, 1, 1, 2, 2 \end{array} \right]_k \frac{120k^2 + 118k + 13}{16^k}.$$

**Example 3**  $\left( \left[ \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2} \right] \right.$  in Theorem 1):

$$\frac{256}{3\pi^2} = \sum_{k=0}^{\infty} \left[ \begin{array}{c} \frac{1}{2}, -\frac{1}{2}, \frac{3}{2}, \frac{1}{4}, \frac{3}{4} \\ 1, 1, 1, 2, 2 \end{array} \right]_k \frac{80k^3 + 148k^2 + 80k + 9}{16^k}.$$

**Example 4**  $\left( \left[ \frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{3}{2} \right] \right.$  in Theorem 1):

$$\frac{512}{\pi^2} = \sum_{k=0}^{\infty} \left[ \begin{array}{c} \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{3}{4}, \frac{5}{4} \\ 1, 1, 1, 2, 2 \end{array} \right]_k \frac{240k^3 + 532k^2 + 336k + 45}{16^k}.$$

**Example 5**  $\left( \left[ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \right] \right.$  in Theorem 1):

$$\frac{32}{\pi^2} = \sum_{k=0}^{\infty} \left[ \begin{array}{c} \frac{3}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{4}, \frac{7}{6} \\ 1, 1, 1, 1, 2, \frac{1}{6} \end{array} \right]_k \frac{3 - 10k - 40k^2}{16^k}.$$

**Example 6**  $\left( \left[ \frac{3}{2}, \frac{1}{2}, \frac{3}{2}, -\frac{1}{2} \right] \right.$  in Theorem 1):

$$\frac{256}{3\pi^2} = \sum_{k=0}^{\infty} \left[ \begin{array}{c} \frac{3}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{4}, -\frac{3}{4}, \frac{7}{6} \\ 1, 1, 1, 2, 3, \frac{1}{6} \end{array} \right]_k \frac{9 - 38k - 40k^2}{16^k}.$$

**Example 7**  $\left( \left[ \frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{3}{2} \right] \right.$  in Theorem 2):

$$\frac{8}{\pi^2} = \sum_{k=1}^{\infty} \left[ \begin{array}{c} \frac{3}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{7}{6} \\ 1, 1, 1, 1, 1, \frac{1}{6} \end{array} \right]_k \frac{k(3 - 18k + 40k^2)}{16^k}.$$

**Example 8**  $\left( \left[ \frac{3}{2}, \frac{1}{2}, \frac{3}{2}, \frac{1}{2} \right]$  in Theorem 2):

$$\frac{24}{\pi^2} = \sum_{k=0}^{\infty} \left[ \begin{array}{c} \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{4}, -\frac{3}{4}, \frac{5}{4}, \frac{5}{6} \\ 1, 1, 1, 1, 2, \frac{1}{4}, -\frac{1}{6} \end{array} \right]_k \frac{3 + 8k + 20k^2}{16^k}.$$

**Example 9**  $\left( \left[ \frac{3}{2}, \frac{3}{2}, -\frac{1}{2}, \frac{1}{2} \right]$  in Theorem 1):

$$\frac{256}{9\pi^2} = \sum_{k=0}^{\infty} \left[ \begin{array}{c} \frac{3}{2}, \frac{5}{2}, -\frac{1}{2}, -\frac{3}{2}, \frac{1}{4}, \frac{3}{4} \\ 1, 1, 1, 2, 2, \frac{1}{2} \end{array} \right]_k \frac{5 + 12k - 68k^2 - 80k^3}{16^k}.$$

### 3.2. Series for $\pi^2$ .

**Example 10** (Chu and Zhang [21]:  $\left[ \frac{3}{2}, 1, 1, 1 \right]$  in Theorem 1):

$$\frac{9\pi^2}{8} = \sum_{k=0}^{\infty} \left[ \begin{array}{c} 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{3}{4} \\ \frac{3}{2}, \frac{5}{4}, \frac{5}{4}, \frac{7}{4}, \frac{7}{4} \end{array} \right]_k \frac{11 + 64k + 111k^2 + 60k^3}{16^k}.$$

**Example 11**  $\left( \left[ \frac{5}{2}, 2, 1, 2 \right]$  in Theorem 1):

$$\frac{225\pi^2}{32} = \sum_{k=0}^{\infty} \left[ \begin{array}{c} 2, 2, \frac{1}{2}, \frac{1}{2}, \frac{3}{4}, \frac{5}{4} \\ 1, \frac{3}{2}, \frac{7}{4}, \frac{7}{4}, \frac{9}{4}, \frac{9}{4} \end{array} \right]_k \frac{68 + 206k + 197k^2 + 60k^3}{16^k}.$$

**Example 12**  $\left( \left[ \frac{5}{2}, 1, 2, 2 \right]$  in Theorem 1):

$$\frac{135\pi^2}{64} = \sum_{k=0}^{\infty} \left[ \begin{array}{c} 2, \frac{1}{2}, -\frac{1}{2}, \frac{5}{3}, \frac{1}{4}, \frac{3}{4} \\ \frac{5}{2}, \frac{2}{3}, \frac{5}{4}, \frac{7}{4}, \frac{7}{4}, \frac{9}{4} \end{array} \right]_k \frac{21 + 93k + 110k^2 + 40k^3}{16^k}.$$

**Example 13**  $\left( \left[ \frac{5}{2}, 1, 2, 1 \right]$  in Theorem 2):

$$\frac{3\pi^2}{32} = 1 + \sum_{k=1}^{\infty} \left[ \begin{array}{c} 1, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{4}, -\frac{3}{4} \\ \frac{5}{2}, \frac{3}{4}, \frac{3}{4}, \frac{5}{4}, \frac{5}{4} \end{array} \right]_k \frac{3 + 3k - 22k^2 - 40k^3}{16^k}.$$

**Example 14**  $\left( \left[ \frac{7}{2}, 1, 2, 1 \right] \right.$  in Theorem 2):

$$\frac{15\pi^2}{256} = \frac{1}{3} + \sum_{k=1}^{\infty} \left[ \begin{matrix} 1, -\frac{1}{2}, -\frac{3}{2}, -\frac{3}{4}, -\frac{5}{4} \\ \frac{7}{2}, \frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \frac{7}{4} \end{matrix} \right]_k \frac{1 - 3k + 2k^2 + 8k^3}{16^k}.$$

**Example 15**  $\left( \left[ \frac{7}{2}, 2, 2, 2 \right] \right.$  in Theorem 2):

$$\frac{27\pi^2}{128} = \sum_{k=0}^{\infty} \left[ \begin{matrix} 2, -\frac{1}{2}, -\frac{1}{2}, \frac{4}{3}, \frac{1}{4}, -\frac{1}{4} \\ \frac{5}{2}, \frac{1}{3}, \frac{5}{4}, \frac{5}{4}, \frac{7}{4}, \frac{7}{4} \end{matrix} \right]_k \frac{2 - 21k - 66k^2 - 40k^3}{16^k}.$$

**Example 16**  $\left( \left[ \frac{7}{2}, 1, 2, 2 \right] \right.$  in Theorem 2):

$$\frac{405\pi^2}{256} = 18 + \sum_{k=1}^{\infty} \left[ \begin{matrix} 2, -\frac{1}{2}, \frac{3}{2}, -\frac{3}{2}, -\frac{1}{4}, -\frac{3}{4} \\ \frac{1}{2}, \frac{7}{2}, \frac{3}{4}, \frac{5}{4}, \frac{5}{4}, \frac{7}{4} \end{matrix} \right]_k \frac{48 - 59k - 194k^2 - 120k^3}{16^k}.$$

**3.3. Series for  $\pi^2/\Gamma^3$ .**

**Example 17**  $\left( \left[ \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, -\frac{2}{3} \right] \right.$  in Theorem 1):

$$\frac{98\pi^2}{3\Gamma\left(\frac{2}{3}\right)^3} = \sum_{k=0}^{\infty} \left[ \begin{matrix} \frac{1}{3}, -\frac{2}{3}, \frac{5}{6}, -\frac{5}{6}, \frac{11}{6}, \frac{10}{9}, \frac{1}{12}, -\frac{5}{12} \\ 1, \frac{3}{2}, \frac{1}{4}, \frac{3}{4}, \frac{13}{6}, \frac{1}{9}, \frac{13}{12}, \frac{19}{12} \end{matrix} \right]_k \frac{118 + 45k - 1098k^2 - 1080k^3}{16^k}.$$

**Example 18**  $\left( \left[ \frac{3}{2}, \frac{1}{3}, \frac{1}{3}, \frac{4}{3} \right] \right.$  in Theorem 1):

$$\frac{637\pi^2}{16\Gamma\left(\frac{2}{3}\right)^3} = \sum_{k=0}^{\infty} \left[ \begin{matrix} \frac{1}{3}, \frac{4}{3}, \frac{5}{6}, -\frac{5}{6}, \frac{11}{6}, \frac{13}{9}, \frac{1}{12}, \frac{7}{12} \\ 1, \frac{3}{2}, \frac{3}{4}, \frac{5}{4}, \frac{13}{6}, \frac{4}{9}, \frac{19}{12}, \frac{25}{12} \end{matrix} \right]_k \frac{1080k^3 + 2286k^2 + 1395k + 161}{16^k}.$$

**Example 19**  $\left( \left[ \frac{1}{2}, \frac{2}{3}, -\frac{1}{3}, -\frac{1}{3} \right] \right.$  in Theorem 1):

$$\frac{275\pi^2}{\Gamma\left(\frac{1}{3}\right)^3} = \sum_{k=0}^{\infty} \left[ \begin{matrix} \frac{2}{3}, -\frac{1}{3}, \frac{7}{6}, -\frac{7}{6}, \frac{13}{6}, \frac{11}{9}, -\frac{1}{12}, \frac{5}{12} \\ 1, \frac{3}{2}, \frac{1}{4}, \frac{3}{4}, \frac{11}{6}, \frac{2}{9}, \frac{17}{12}, \frac{23}{12} \end{matrix} \right]_k \frac{125 - 351k - 1602k^2 - 1080k^3}{16^k}.$$



**Example 20**  $\left( \left[ \begin{matrix} 3 & 2 & 5 & -1 \\ 2 & 3 & 3 & -3 \end{matrix} \right] \right.$  in Theorem 1):

$$\frac{825\pi^2}{8\Gamma\left(\frac{1}{3}\right)^3} = \sum_{k=0}^{\infty} \left[ \begin{matrix} 2 & -1 & 1 & -1 & 7 & 11 & -1 & -7 \\ 3 & 3 & 6 & 6 & 9 & 9 & 12 & 12 \\ 1, & \frac{3}{2}, & \frac{3}{4}, & \frac{5}{4}, & \frac{17}{6}, & \frac{2}{9}, & \frac{11}{12}, & \frac{17}{12} \end{matrix} \right]_k \frac{53 - 315k - 1278k^2 - 1080k^3}{16^k}.$$

**Example 21**  $\left( \left[ \begin{matrix} 3 & 2 & -1 & 5 \\ 2 & 3 & -3 & 3 \end{matrix} \right] \right.$  in Theorem 1):

$$\frac{2805\pi^2}{4\Gamma\left(\frac{1}{3}\right)^3} = \sum_{k=0}^{\infty} \left[ \begin{matrix} 2 & 5 & 7 & -7 & 13 & 14 & 5 & 11 \\ 3 & 3 & 6 & 6 & 9 & 9 & 12 & 12 \\ 1, & \frac{3}{2}, & \frac{3}{4}, & \frac{5}{4}, & \frac{11}{6}, & \frac{5}{9}, & \frac{23}{12}, & \frac{29}{12} \end{matrix} \right]_k \frac{1080k^3 + 2790k^2 + 2151k + 478}{16^k}.$$

**Example 22**  $\left( \left[ \begin{matrix} -1 & -2 & -2 & -2 \\ 2 & 3 & 3 & 3 \end{matrix} \right] \right.$  in Theorem 2):

$$\frac{3872\pi^2}{243\Gamma\left(\frac{2}{3}\right)^3} = \sum_{k=0}^{\infty} \left[ \begin{matrix} -2 & -5 & 5 & 11 & -11 & 4 & -5 & -11 \\ 3 & 3 & 6 & 6 & 9 & 9 & 12 & 12 \\ 1, & \frac{1}{2}, & -\frac{1}{4}, & -\frac{3}{4}, & \frac{7}{6}, & -\frac{5}{9}, & \frac{7}{12}, & \frac{13}{12} \end{matrix} \right]_k \frac{1080k^3 - 954k^2 - 585k + 242}{16^k}.$$

**Example 23**  $\left( \left[ \begin{matrix} 1 & -2 & 4 & -2 \\ 2 & 3 & 3 & -3 \end{matrix} \right] \right.$  in Theorem 2):

$$\frac{2380\pi^2}{27\Gamma\left(\frac{2}{3}\right)^3} = \sum_{k=0}^{\infty} \left[ \begin{matrix} -2 & -5 & -1 & -5 & 5 & 4 & -11 & -17 \\ 3 & 3 & 6 & 6 & 9 & 9 & 12 & 12 \\ 1, & \frac{1}{2}, & \frac{1}{4}, & -\frac{1}{4}, & \frac{13}{6}, & -\frac{5}{9}, & \frac{1}{12}, & \frac{7}{12} \end{matrix} \right]_k \frac{1080k^3 - 1278k^2 + 99k + 170}{16^k}.$$

**Example 24**  $\left( \left[ \begin{matrix} 1 & 4 & -2 & -2 \\ 2 & 3 & -3 & -3 \end{matrix} \right] \right.$  in Theorem 2):

$$\frac{770\pi^2}{27\Gamma\left(\frac{2}{3}\right)^3} = \sum_{k=0}^{\infty} \left[ \begin{matrix} 1 & -2 & 5 & 11 & -11 & 7 & -5 & 1 \\ 3 & 3 & 6 & 6 & 9 & 9 & 12 & 12 \\ 1, & \frac{1}{2}, & \frac{1}{4}, & -\frac{1}{4}, & \frac{7}{6}, & -\frac{2}{9}, & \frac{13}{12}, & \frac{19}{12} \end{matrix} \right]_k \frac{55 + 441k - 234k^2 - 1080k^3}{16^k}.$$

**Example 25**  $\left( \left[ \begin{matrix} 3 & 4 & -2 & 4 \\ 2 & 3 & -3 & 3 \end{matrix} \right] \right.$  in Theorem 2):

$$\frac{-1001\pi^2}{18\Gamma\left(\frac{2}{3}\right)^3} = \sum_{k=0}^{\infty} \left[ \begin{matrix} 1 & 4 & 5 & 11 & -11 & 10 & 1 & 7 \\ 3 & 3 & 6 & 6 & 9 & 9 & 12 & 12 \\ 1, & \frac{1}{2}, & \frac{1}{4}, & \frac{3}{4}, & \frac{7}{6}, & \frac{1}{9}, & \frac{19}{12}, & \frac{25}{12} \end{matrix} \right]_k \frac{1080k^3 + 1422k^2 + 351k + 44}{16^k}.$$

**Example 26**  $\left( \left[ \begin{matrix} 3 & 4 & 4 & -2 \\ 2 & 3 & 3 & -3 \end{matrix} \right] \right.$  in Theorem 2):

$$\frac{385\pi^2}{36\Gamma\left(\frac{2}{3}\right)^3} = \sum_{k=0}^{\infty} \left[ \begin{matrix} 1 & -2 & -1 & -5 & 5 & 7 & -5 & -11 \\ 3 & -3 & -6 & -6 & 6 & 9 & -12 & -12 \end{matrix} \right]_k \frac{55 + 189k + 90k^2 - 1080k^3}{16^k}.$$

**Example 27**  $\left( \left[ \begin{matrix} 1 & -1 & 2 & -1 \\ 2 & -3 & 3 & -3 \end{matrix} \right] \right.$  in Theorem 2):

$$\frac{910\pi^2}{9\Gamma\left(\frac{1}{3}\right)^3} = \sum_{k=0}^{\infty} \left[ \begin{matrix} -1 & -4 & 1 & 7 & -7 & 5 & -7 & -13 \\ 3 & -3 & 6 & 6 & -6 & 9 & -12 & -12 \end{matrix} \right]_k \frac{1080k^3 - 774k^2 - 225k + 91}{16^k}.$$

**Example 28**  $\left( \left[ \begin{matrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 3 & 3 \end{matrix} \right] \right.$  in Theorem 2):

$$\frac{1225\pi^2}{6\Gamma\left(\frac{1}{3}\right)^3} = \sum_{k=0}^{\infty} \left[ \begin{matrix} 2 & -1 & 1 & 7 & -7 & 8 & -1 & -7 \\ 3 & -3 & 6 & 6 & -6 & 9 & -12 & -12 \end{matrix} \right]_k \frac{98 + 153k - 414k^2 - 1080k^3}{16^k}.$$

### 3.4. Series for $\Gamma^3/\pi^2$ .

**Example 29**  $\left( \left[ \begin{matrix} -1 & -5 & 1 & 1 \\ 2 & -6 & 6 & 6 \end{matrix} \right] \right.$  in Theorem 1):

$$\frac{180\Gamma\left(\frac{2}{3}\right)^3}{\pi^2} = \sum_{k=0}^{\infty} \left[ \begin{matrix} 1 & -1 & 5 & -5 & 7 & -1 & 5 & 19 \\ 6 & -6 & 6 & -6 & 6 & -12 & 12 & 18 \end{matrix} \right]_k \frac{35 + 228k - 540k^2 - 2160k^3}{16^k}.$$

**Example 30**  $\left( \left[ \begin{matrix} -1 & 1 & 1 & 7 \\ 2 & 6 & 6 & 6 \end{matrix} \right] \right.$  in Theorem 1):

$$\frac{8748\Gamma\left(\frac{2}{3}\right)^3}{7\pi^2} = \sum_{k=0}^{\infty} \left[ \begin{matrix} 1 & 5 & -5 & 7 & 11 & 11 & 17 & 25 \\ 6 & 6 & -6 & 6 & 6 & -12 & 12 & 18 \end{matrix} \right]_k \frac{593 + 1344k - 1404k^2 - 2160k^3}{16^k}.$$

**Example 31**  $\left( \left[ \begin{matrix} 1 & 1 & -5 & 7 \\ 2 & 6 & -6 & 6 \end{matrix} \right] \right.$  in Theorem 1):

$$\frac{960\Gamma\left(\frac{2}{3}\right)^3}{7\pi^2} = \sum_{k=0}^{\infty} \left[ \begin{matrix} 1 & -1 & 7 & -7 & 13 & 5 & 11 & 25 \\ 6 & -6 & 6 & -6 & 6 & -12 & 12 & 18 \end{matrix} \right]_k \frac{65 + 372k - 756k^2 - 2160k^3}{16^k}.$$

**Example 32**  $\left( \left[ \frac{1}{2}, \frac{1}{6}, \frac{7}{6}, -\frac{5}{6} \right]$  in Theorem 1):

$$\frac{960\Gamma\left(\frac{2}{3}\right)^3}{\pi^2} = \sum_{k=0}^{\infty} \left[ \begin{array}{cccccccc} \frac{1}{6}, -\frac{1}{6}, \frac{5}{6}, -\frac{5}{6}, \frac{7}{6}, -\frac{1}{12}, -\frac{7}{12}, \frac{19}{18} \\ 1, 1, \frac{1}{2}, \frac{3}{2}, \frac{2}{3}, \frac{4}{3}, \frac{7}{3}, \frac{1}{18} \end{array} \right]_k \frac{245 - 204k - 2052k^2 - 2160k^3}{16^k}.$$

**Example 33**  $\left( \left[ \frac{1}{2}, \frac{1}{6}, \frac{7}{6}, \frac{7}{6} \right]$  in Theorem 1):

$$\frac{7776\Gamma\left(\frac{2}{3}\right)^3}{7\pi^2} = \sum_{k=0}^{\infty} \left[ \begin{array}{cccccccc} \frac{1}{6}, \frac{5}{6}, \frac{5}{6}, -\frac{5}{6}, \frac{7}{6}, \frac{11}{6}, \frac{5}{12}, \frac{11}{12}, \frac{25}{18} \\ 1, 2, \frac{3}{2}, \frac{3}{2}, \frac{1}{3}, \frac{2}{3}, \frac{4}{3}, -\frac{1}{6}, \frac{7}{18} \end{array} \right]_k \frac{360k^2 + 546k + 191}{16^k}.$$

**Example 34**  $\left( \left[ \frac{3}{2}, \frac{1}{6}, \frac{1}{6}, \frac{7}{6} \right]$  in Theorem 1):

$$\frac{1024\Gamma\left(\frac{2}{3}\right)^3}{21\pi^2} = \sum_{k=0}^{\infty} \left[ \begin{array}{cccccccc} \frac{1}{6}, -\frac{1}{6}, \frac{7}{6}, -\frac{7}{6}, \frac{13}{6}, -\frac{1}{12}, \frac{5}{12}, \frac{25}{18} \\ 1, 1, \frac{1}{2}, \frac{3}{2}, \frac{4}{3}, \frac{5}{3}, \frac{7}{3}, \frac{7}{18} \end{array} \right]_k \frac{2160k^3 + 2268k^2 + 60k + 13}{16^k}.$$

**Example 35**  $\left( \left[ -\frac{1}{2}, -\frac{1}{6}, \frac{5}{6}, \frac{5}{6} \right]$  in Theorem 1):

$$\frac{2916\Gamma\left(\frac{1}{3}\right)^3}{5\pi^2} = \sum_{k=0}^{\infty} \left[ \begin{array}{cccccccc} -\frac{1}{6}, \frac{5}{6}, \frac{7}{6}, -\frac{7}{6}, \frac{13}{6}, \frac{7}{12}, \frac{13}{12}, \frac{23}{18} \\ 1, 2, \frac{3}{2}, \frac{3}{2}, -\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{5}{18} \end{array} \right]_k \frac{697 + 1056k - 1836k^2 - 2160k^3}{16^k}.$$

**Example 36**  $\left( \left[ \frac{1}{2}, -\frac{1}{6}, \frac{11}{6}, \frac{5}{6} \right]$  in Theorem 1):

$$\frac{2592\Gamma\left(\frac{1}{3}\right)^3}{25\pi^2} = \sum_{k=0}^{\infty} \left[ \begin{array}{cccccccc} -\frac{1}{6}, \frac{5}{6}, \frac{7}{6}, -\frac{7}{6}, \frac{13}{6}, \frac{1}{12}, \frac{7}{12}, \frac{23}{18} \\ 1, 2, \frac{3}{2}, \frac{3}{2}, \frac{1}{3}, \frac{2}{3}, \frac{5}{3}, \frac{5}{18} \end{array} \right]_k \frac{223 - 888k - 3348k^2 - 2160k^3}{16^k}.$$

**Example 37**  $\left( \left[ \frac{1}{2}, \frac{5}{6}, -\frac{1}{6}, -\frac{1}{6} \right]$  in Theorem 1):

$$\frac{32\Gamma\left(\frac{1}{3}\right)^3}{5\pi^2} = \sum_{k=0}^{\infty} \left[ \begin{array}{cccccccc} \frac{1}{6}, -\frac{1}{6}, \frac{5}{6}, -\frac{5}{6}, \frac{11}{6}, \frac{1}{12}, \frac{7}{12}, \frac{23}{18} \\ 1, 1, \frac{1}{2}, \frac{3}{2}, \frac{2}{3}, \frac{4}{3}, \frac{5}{3}, \frac{5}{18} \end{array} \right]_k \frac{2160k^3 + 1188k^2 - 84k + 11}{16^k}.$$

**Example 38**  $\left( \left[ \frac{1}{2}, \frac{5}{6}, -\frac{1}{6}, \frac{11}{6} \right]$  in Theorem 1):

$$\frac{2592\Gamma\left(\frac{1}{3}\right)^3}{55\pi^2} = \sum_{k=0}^{\infty} \left[ \begin{array}{c} \frac{1}{6}, -\frac{1}{6}, \frac{5}{6}, \frac{7}{6}, \frac{11}{6}, \frac{13}{12}, \frac{19}{12}, \frac{29}{18} \\ 1, 2, \frac{3}{2}, \frac{3}{2}, \frac{2}{3}, \frac{4}{3}, -\frac{1}{3}, \frac{11}{18} \end{array} \right]_k \frac{151 + 264k - 2052k^2 - 2160k^3}{16^k}.$$

**Example 39**  $\left( \left[ \frac{3}{2}, -\frac{1}{6}, \frac{5}{6}, \frac{5}{6} \right]$  in Theorem 1):

$$\frac{256\Gamma\left(\frac{1}{3}\right)^3}{9\pi^2} = \sum_{k=0}^{\infty} \left[ \begin{array}{c} \frac{1}{6}, -\frac{1}{6}, \frac{5}{6}, -\frac{5}{6}, \frac{11}{6}, \frac{1}{12}, -\frac{5}{12}, \frac{23}{18} \\ 1, 1, \frac{1}{2}, \frac{3}{2}, \frac{4}{3}, \frac{5}{3}, \frac{8}{3}, \frac{5}{18} \end{array} \right]_k \frac{55 - 348k - 2700k^2 - 2160k^3}{16^k}.$$

**Example 40**  $\left( \left[ \frac{3}{2}, \frac{5}{6}, \frac{5}{6}, \frac{11}{6} \right]$  in Theorem 1):

$$\frac{6912\Gamma\left(\frac{1}{3}\right)^3}{55\pi^2} = \sum_{k=0}^{\infty} \left[ \begin{array}{c} \frac{1}{6}, -\frac{1}{6}, \frac{5}{6}, \frac{7}{6}, \frac{11}{6}, \frac{7}{12}, \frac{13}{12}, \frac{29}{18} \\ 1, 2, \frac{3}{2}, \frac{3}{2}, \frac{2}{3}, \frac{4}{3}, \frac{5}{3}, \frac{11}{18} \end{array} \right]_k \frac{2160k^3 + 3564k^2 + 1680k + 251}{16^k}.$$

**Example 41**  $\left( \left[ -\frac{1}{2}, \frac{1}{6}, \frac{7}{6}, \frac{1}{6} \right]$  in Theorem 2):

$$\frac{2673\Gamma\left(\frac{2}{3}\right)^3}{16\pi^2} = \sum_{k=0}^{\infty} \left[ \begin{array}{c} \frac{1}{6}, \frac{5}{6}, -\frac{5}{6}, \frac{11}{6}, -\frac{11}{6}, -\frac{1}{12}, \frac{5}{12}, \frac{13}{18} \\ 1, 1, \frac{1}{2}, \frac{3}{2}, \frac{1}{3}, -\frac{1}{3}, -\frac{2}{3}, -\frac{5}{18} \end{array} \right]_k \frac{11 + 1380k + 1188k^2 - 2160k^3}{16^k}.$$

**Example 42**  $\left( \left[ -\frac{1}{2}, \frac{7}{6}, -\frac{5}{6}, \frac{7}{6} \right]$  in Theorem 2):

$$\frac{13365\Gamma\left(\frac{2}{3}\right)^3}{16\pi^2} = \sum_{k=0}^{\infty} \left[ \begin{array}{c} \frac{1}{6}, -\frac{1}{6}, \frac{5}{6}, -\frac{5}{6}, \frac{7}{6}, \frac{11}{12}, \frac{17}{12}, \frac{19}{18} \\ 1, 1, \frac{1}{2}, \frac{3}{2}, \frac{2}{3}, -\frac{2}{3}, -\frac{5}{3}, \frac{1}{18} \end{array} \right]_k \frac{2160k^3 - 2484k^2 - 1092k + 385}{16^k}.$$

**Example 43**  $\left( \left[ \frac{1}{2}, \frac{7}{6}, \frac{1}{6}, \frac{7}{6} \right]$  in Theorem 2):

$$\frac{675\Gamma\left(\frac{2}{3}\right)^3}{2\pi^2} = \sum_{k=0}^{\infty} \left[ \begin{array}{c} \frac{1}{6}, -\frac{1}{6}, \frac{5}{6}, -\frac{5}{6}, \frac{7}{6}, \frac{5}{12}, \frac{11}{12}, \frac{19}{18} \\ 1, 1, \frac{1}{2}, \frac{3}{2}, \frac{1}{3}, \frac{2}{3}, -\frac{2}{3}, \frac{1}{18} \end{array} \right]_k \frac{2160k^3 - 972k^2 - 660k + 175}{16^k}.$$

**Example 44**  $\left( \left[ \begin{array}{cccc} 3 & 7 & 7 & 7 \\ 2 & 6 & 6 & 6 \end{array} \right] \text{ in Theorem 2} \right)$ :

$$\frac{180\Gamma\left(\frac{2}{3}\right)^3}{\pi^2} = \sum_{k=0}^{\infty} \left[ \begin{array}{cccccccc} \frac{1}{6}, -\frac{1}{6}, \frac{5}{6}, -\frac{5}{6}, \frac{7}{6}, -\frac{1}{12}, \frac{5}{12}, \frac{19}{18} \\ 1, 1, \frac{1}{2}, \frac{3}{2}, \frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{1}{18} \end{array} \right]_k \frac{35 + 228k - 540k^2 - 2160k^3}{16^k}.$$

**Example 45**  $\left( \left[ \begin{array}{cccc} -1 & -1 & 11 & -1 \\ 2 & 6 & 6 & 6 \end{array} \right] \text{ in Theorem 2} \right)$ :

$$\frac{1053\Gamma\left(\frac{1}{3}\right)^3}{32\pi^2} = \sum_{k=0}^{\infty} \left[ \begin{array}{cccccccc} -\frac{1}{6}, \frac{7}{6}, -\frac{7}{6}, \frac{13}{6}, -\frac{13}{6}, -\frac{5}{12}, \frac{1}{12}, \frac{11}{18} \\ 1, 1, \frac{1}{2}, \frac{3}{2}, -\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}, -\frac{7}{18} \end{array} \right]_k \frac{2160k^3 - 756k^2 - 1668k + 65}{16^k}.$$

**Example 46**  $\left( \left[ \begin{array}{cccc} -1 & 5 & -1 & 5 \\ 2 & 6 & 6 & 6 \end{array} \right] \text{ in Theorem 2} \right)$ :

$$\frac{3969\Gamma\left(\frac{1}{3}\right)^3}{32\pi^2} = \sum_{k=0}^{\infty} \left[ \begin{array}{cccccccc} \frac{1}{6}, -\frac{1}{6}, \frac{5}{6}, \frac{7}{6}, -\frac{7}{6}, \frac{7}{12}, \frac{13}{12}, \frac{17}{18} \\ 1, 1, \frac{1}{2}, \frac{3}{2}, \frac{1}{3}, -\frac{1}{3}, -\frac{4}{3}, -\frac{1}{18} \end{array} \right]_k \frac{2160k^3 - 2052k^2 - 1092k + 245}{16^k}.$$

**Example 47**  $\left( \left[ \begin{array}{cccc} 1 & 5 & 5 & 5 \\ 2 & 6 & 6 & 6 \end{array} \right] \text{ in Theorem 2} \right)$ :

$$\frac{63\Gamma\left(\frac{1}{3}\right)^3}{10\pi^2} = \sum_{k=0}^{\infty} \left[ \begin{array}{cccccccc} \frac{1}{6}, -\frac{1}{6}, \frac{5}{6}, \frac{7}{6}, -\frac{7}{6}, \frac{1}{12}, \frac{7}{12}, \frac{17}{18} \\ 1, 1, \frac{1}{2}, \frac{3}{2}, -\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, -\frac{1}{18} \end{array} \right]_k \frac{432k^3 - 108k^2 - 132k + 7}{16^k}.$$

**Example 48**  $\left( \left[ \begin{array}{cccc} 3 & 11 & -1 & 11 \\ 2 & 6 & 6 & 6 \end{array} \right] \text{ in Theorem 2} \right)$ :

$$\frac{84\Gamma\left(\frac{1}{3}\right)^3}{5\pi^2} = \sum_{k=0}^{\infty} \left[ \begin{array}{cccccccc} \frac{1}{6}, -\frac{1}{6}, \frac{5}{6}, -\frac{5}{6}, \frac{11}{6}, \frac{7}{12}, \frac{13}{12}, \frac{23}{18} \\ 1, 1, \frac{1}{2}, \frac{3}{2}, -\frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{5}{18} \end{array} \right]_k \frac{2160k^3 - 324k^2 - 516k + 77}{16^k}.$$

**Example 49**  $\left( \left[ \begin{array}{cccc} 3 & 11 & 11 & -1 \\ 2 & 6 & 6 & 6 \end{array} \right] \text{ in Theorem 2} \right)$ :

$$\frac{84\Gamma\left(\frac{1}{3}\right)^3}{\pi^2} = \sum_{k=0}^{\infty} \left[ \begin{array}{cccccccc} \frac{1}{6}, -\frac{1}{6}, \frac{5}{6}, \frac{7}{6}, -\frac{7}{6}, \frac{1}{12}, -\frac{5}{12}, \frac{17}{18} \\ 1, 1, \frac{1}{2}, \frac{3}{2}, \frac{1}{3}, \frac{2}{3}, \frac{5}{3}, -\frac{1}{18} \end{array} \right]_k \frac{175 + 228k - 972k^2 - 2160k^3}{16^k}.$$

3.5. Series for  $\pi^{-1}$ .

**Example 50**  $\left( \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{3} \right) \right)$  in Theorem 1):

$$\frac{15\sqrt{3}}{\pi} = \sum_{k=0}^{\infty} \left[ \begin{matrix} \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \\ 1, 1, 1, \frac{11}{12}, \frac{17}{12} \end{matrix} \right]_k \frac{135k^2 + 75k + 8}{16^k}.$$

**Example 51**  $\left( \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{2}{3} \right) \right)$  in Theorem 1):

$$\frac{21\sqrt{3}}{\pi} = \sum_{k=0}^{\infty} \left[ \begin{matrix} \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \\ 1, 1, 1, \frac{13}{12}, \frac{19}{12} \end{matrix} \right]_k \frac{810k^3 + 684k^2 + 141k + 10}{16^k}.$$

**Example 52**  $\left( \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{4} \right) \right)$  in Theorem 1):

$$\frac{48}{\pi} = \sum_{k=0}^{\infty} \left[ \begin{matrix} \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{5}{8} \\ 1, 1, 1, \frac{7}{8}, \frac{11}{8} \end{matrix} \right]_k \frac{480k^2 + 212k + 15}{16^k}.$$

**Example 53**  $\left( \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{4} \right) \right)$  in Theorem 1):

$$\frac{80}{3\pi} = \sum_{k=0}^{\infty} \left[ \begin{matrix} \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{3}{8}, \frac{7}{8} \\ 1, 1, 1, \frac{9}{8}, \frac{13}{8} \end{matrix} \right]_k \frac{640k^3 + 560k^2 + 112k + 7}{16^k}.$$

**Example 54**  $\left( \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{6}, \frac{1}{2} \right) \right)$  in Theorem 1):

$$\frac{256}{3\pi\sqrt{3}} = \sum_{k=0}^{\infty} \left[ \begin{matrix} \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{6}, \frac{5}{6} \\ 1, 1, 1, \frac{4}{3}, \frac{5}{3} \end{matrix} \right]_k \frac{720k^3 + 804k^2 + 236k + 15}{16^k}.$$

**Example 55**  $\left( \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{6} \right) \right)$  in Theorem 1):

$$\frac{192}{\pi\sqrt{3}} = \sum_{k=0}^{\infty} \left[ \begin{matrix} \frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{12}, \frac{7}{12} \\ 1, 1, 1, \frac{4}{3}, \frac{4}{3} \end{matrix} \right]_k \frac{6480k^3 + 4284k^2 + 840k + 35}{16^k}.$$

**Example 56** (Chu and Zhang [21]:  $\left[ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{5}{6} \right]$  in Theorem 1):

$$\frac{384}{\pi\sqrt{3}} = \sum_{k=0}^{\infty} \left[ \begin{array}{cccccc} \frac{1}{2}, \frac{5}{6}, \frac{5}{6}, \frac{5}{12}, \frac{11}{12} \\ 1, 1, 1, \frac{2}{3}, \frac{5}{3} \end{array} \right]_k \frac{1080k^2 + 798k + 55}{16^k}.$$

**Example 57** ( $\left[ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{5} \right]$  in Theorem 1):

$$\frac{105\sqrt{5-2\sqrt{5}}}{\pi} = \sum_{k=0}^{\infty} \left[ \begin{array}{cccccc} \frac{1}{2}, \frac{1}{5}, \frac{1}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{10} \\ 1, 1, 1, \frac{13}{10}, \frac{17}{20}, \frac{27}{20} \end{array} \right]_k \frac{3750k^3 + 2525k^2 + 505k + 24}{16^k}.$$

**Example 58** ( $\left[ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{2}{5} \right]$  in Theorem 1):

$$\frac{45\sqrt{5+2\sqrt{5}}}{\pi} = \sum_{k=0}^{\infty} \left[ \begin{array}{cccccc} \frac{1}{2}, \frac{1}{5}, \frac{2}{5}, \frac{2}{5}, \frac{3}{5}, \frac{7}{10} \\ 1, 1, 1, \frac{11}{10}, \frac{19}{20}, \frac{29}{20} \end{array} \right]_k \frac{3750k^3 + 2800k^2 + 595k + 42}{16^k}.$$

**Example 59** ( $\left[ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{5} \right]$  in Theorem 1):

$$\frac{55\sqrt{5+2\sqrt{5}}}{3\pi} = \sum_{k=0}^{\infty} \left[ \begin{array}{cccccc} \frac{1}{2}, \frac{2}{5}, \frac{3}{5}, \frac{3}{5}, \frac{4}{5}, \frac{3}{10} \\ 1, 1, 1, \frac{9}{10}, \frac{21}{20}, \frac{31}{20} \end{array} \right]_k \frac{1250k^3 + 1025k^2 + 215k + 16}{16^k}.$$

**Example 60** ( $\left[ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{4}{5} \right]$  in Theorem 1):

$$\frac{195\sqrt{5-2\sqrt{5}}}{\pi} = \sum_{k=0}^{\infty} \left[ \begin{array}{cccccc} \frac{1}{2}, \frac{1}{5}, \frac{2}{5}, \frac{4}{5}, \frac{4}{5}, \frac{9}{10} \\ 1, 1, 1, \frac{7}{10}, \frac{23}{20}, \frac{33}{20} \end{array} \right]_k \frac{3750k^3 + 3350k^2 + 655k + 36}{16^k}.$$

**Example 61** ( $\left[ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{8} \right]$  in Theorem 1):

$$\frac{480}{\pi(\sqrt{2}+1)} = \sum_{k=0}^{\infty} \left[ \begin{array}{cccccc} \frac{1}{2}, \frac{1}{8}, \frac{1}{8}, \frac{7}{8}, \frac{1}{16}, \frac{9}{16} \\ 1, 1, 1, \frac{11}{8}, \frac{13}{16}, \frac{21}{16} \end{array} \right]_k \frac{15360k^3 + 9920k^2 + 1888k + 63}{16^k}.$$

**Example 62**  $\left( \left[ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{8} \right] \right.$  in Theorem 1  $\left. \right)$ :

$$\frac{224}{3\pi(\sqrt{2}-1)} = \sum_{k=0}^{\infty} \left[ \begin{array}{c} \frac{1}{2}, \frac{3}{8}, \frac{3}{8}, \frac{5}{8}, \frac{3}{16}, \frac{11}{16} \\ 1, 1, 1, \frac{9}{8}, \frac{15}{16}, \frac{23}{16} \end{array} \right]_k \frac{5120k^3 + 3776k^2 + 800k + 55}{16^k}.$$

**Example 63**  $\left( \left[ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{5}{8} \right] \right.$  in Theorem 1  $\left. \right)$ :

$$\frac{288}{\pi(\sqrt{2}-1)} = \sum_{k=0}^{\infty} \left[ \begin{array}{c} \frac{1}{2}, \frac{3}{8}, \frac{5}{8}, \frac{5}{8}, \frac{5}{16}, \frac{13}{16} \\ 1, 1, 1, \frac{7}{8}, \frac{17}{16}, \frac{25}{16} \end{array} \right]_k \frac{15360k^3 + 12736k^2 + 2656k + 195}{16^k}.$$

**Example 64**  $\left( \left[ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{7}{8} \right] \right.$  in Theorem 1  $\left. \right)$ :

$$\frac{1056}{\pi(\sqrt{2}+1)} = \sum_{k=0}^{\infty} \left[ \begin{array}{c} \frac{1}{2}, \frac{1}{8}, \frac{7}{8}, \frac{7}{8}, \frac{7}{16}, \frac{15}{16} \\ 1, 1, 1, \frac{5}{8}, \frac{19}{16}, \frac{27}{16} \end{array} \right]_k \frac{15360k^3 + 14144k^2 + 2656k + 105}{16^k}.$$

**Example 65**  $\left( \left[ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{6} \right] \right.$  in Theorem 2  $\left. \right)$ :

$$\frac{10\sqrt{3}}{\pi} = \sum_{k=0}^{\infty} \left[ \begin{array}{c} -\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{12}, -\frac{5}{12} \\ 1, 1, 1, \frac{1}{3}, \frac{1}{3} \end{array} \right]_k \frac{2160k^3 - 372k^2 + 68k + 5}{16^k}.$$

**Example 66**  $\left( \left[ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{3} \right] \right.$  in Theorem 2  $\left. \right)$ :

$$\frac{6\sqrt{3}}{\pi} = \sum_{k=0}^{\infty} \left[ \begin{array}{c} -\frac{1}{2}, \frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, -\frac{2}{3} \\ 1, 1, 1, \frac{1}{12}, \frac{7}{12} \end{array} \right]_k \frac{135k^3 - 48k^2 - 7k + 2}{16^k}.$$

**Example 67**  $\left( \left[ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{4} \right] \right.$  in Theorem 2  $\left. \right)$ :

$$\frac{20}{\pi} = \sum_{k=0}^{\infty} \left[ \begin{array}{c} -\frac{1}{2}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{8}, -\frac{5}{8} \\ 1, 1, 1, \frac{1}{8}, \frac{5}{8} \end{array} \right]_k \frac{960k^3 - 232k^2 - 38k + 5}{16^k}.$$



**Example 68**  $\left( \left[ \frac{1}{2}, \frac{1}{2}, \frac{1}{6}, \frac{1}{2} \right]$  in Theorem 2):

$$\frac{10\sqrt{3}}{9\pi} = \sum_{k=0}^{\infty} \left[ \begin{array}{cccc} \frac{1}{2}, \frac{1}{4}, -\frac{1}{4}, \frac{5}{6}, -\frac{5}{6} \\ 1, 1, 1, \frac{1}{3}, \frac{2}{3} \end{array} \right]_k \frac{k(120k^2 - 26k + 5)}{16^k}.$$

**Example 69**  $\left( \left[ \frac{1}{2}, \frac{1}{2}, \frac{7}{6}, -\frac{1}{2} \right]$  in Theorem 2):

$$\frac{6\sqrt{3}}{\pi} = \sum_{k=0}^{\infty} \left[ \begin{array}{cccc} -\frac{1}{2}, -\frac{1}{4}, -\frac{3}{4}, \frac{1}{6}, -\frac{1}{6} \\ 1, 1, 1, \frac{1}{3}, \frac{2}{3} \end{array} \right]_k \frac{720k^3 - 300k^2 - 4k + 3}{16^k}.$$

**Example 70**  $\left( \left[ \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{7}{6} \right]$  in Theorem 2):

$$\frac{27\sqrt{3}}{\pi} = \sum_{k=0}^{\infty} \left[ \begin{array}{cccc} -\frac{3}{2}, \frac{1}{6}, \frac{7}{6}, \frac{1}{12}, \frac{7}{12} \\ 1, 1, 1, \frac{1}{3}, -\frac{2}{3} \end{array} \right]_k \frac{21 + 292k - 420k^2 - 2160k^3}{16^k}.$$

**Example 71**  $\left( \left[ \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{4} \right]$  in Theorem 2):

$$\frac{96}{\pi} = \sum_{k=0}^{\infty} \left[ \begin{array}{cccc} -\frac{3}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, -\frac{3}{8} \\ 1, 1, 1, \frac{3}{8}, -\frac{1}{8} \end{array} \right]_k \frac{9 + 102k - 424k^2 - 960k^3}{16^k}.$$

**Example 72**  $\left( \left[ \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{4} \right]$  in Theorem 2):

$$\frac{160}{\pi} = \sum_{k=0}^{\infty} \left[ \begin{array}{cccc} -\frac{3}{2}, \frac{5}{4}, -\frac{5}{4}, -\frac{1}{8}, -\frac{5}{8} \\ 1, 1, 1, \frac{1}{8}, -\frac{3}{8} \end{array} \right]_k \frac{960k^3 - 232k^2 - 710k + 75}{16^k}.$$

**Example 73**  $\left( \left[ \frac{1}{2}, \frac{1}{2}, -\frac{1}{6}, \frac{1}{2} \right]$  in Theorem 2):

$$\frac{28}{3\sqrt{3}\pi} = \sum_{k=0}^{\infty} \left[ \begin{array}{cccc} \frac{1}{2}, \frac{1}{4}, -\frac{1}{4}, \frac{7}{6}, -\frac{7}{6}, \frac{13}{12} \\ 1, 1, 1, \frac{2}{3}, \frac{4}{3}, \frac{1}{12} \end{array} \right]_k \frac{k(60k^2 - 8k - 7)}{16^k}.$$

**Example 74**  $\left( \left[ \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{3} \right]$  in Theorem 2):

$$\frac{162\sqrt{3}}{5\pi} = \sum_{k=0}^{\infty} \left[ \begin{array}{c} -\frac{3}{2}, -\frac{1}{3}, -\frac{2}{3}, -\frac{4}{3}, \frac{4}{3} \\ 1, 1, 1, \frac{1}{12}, -\frac{5}{12} \end{array} \right]_k \frac{135k^3 - 48k^2 - 106k + 24}{16^k}.$$

### 3.6. Series for $\pi$ .

**Example 75**  $\left( \left[ \frac{5}{2}, 2, 2, \frac{3}{4} \right]$  in Theorem 2):

$$\frac{5\pi}{16} = \sum_{k=0}^{\infty} \left[ \begin{array}{c} -\frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, -\frac{3}{8} \\ \frac{1}{2}, \frac{3}{4}, \frac{9}{8}, \frac{13}{8} \end{array} \right]_k \frac{1 - 7k + 40k^2}{16^k}.$$

**Example 76**  $\left( \left[ \frac{5}{2}, 2, 2, \frac{1}{4} \right]$  in Theorem 2):

$$\frac{25\pi}{16} = \sum_{k=0}^{\infty} \left[ \begin{array}{c} -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, -\frac{5}{8} \\ \frac{1}{2}, \frac{9}{4}, \frac{7}{8}, \frac{11}{8} \end{array} \right]_k \frac{120k^2 + 77k + 5}{16^k}.$$

**Example 77**  $\left( \left[ \frac{3}{2}, 2, 1, \frac{1}{4} \right]$  in Theorem 2):

$$\frac{3\pi}{8} = \sum_{k=0}^{\infty} \left[ \begin{array}{c} -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{3}{8} \\ \frac{1}{2}, \frac{5}{4}, \frac{7}{8}, \frac{11}{8} \end{array} \right]_k \frac{1 + 11k + 106k^2 + 240k^3}{16^k}.$$

**Example 78**  $\left( \left[ \frac{3}{2}, 1, 1, \frac{5}{6} \right]$  in Theorem 1):

$$\frac{36\pi}{5\sqrt{3}} = \sum_{k=0}^{\infty} \left[ \begin{array}{c} \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{6}, \frac{5}{6}, \frac{14}{9} \\ \frac{3}{2}, \frac{5}{3}, \frac{5}{4}, \frac{7}{4}, \frac{7}{6}, \frac{5}{9} \end{array} \right]_k \frac{60k^2 + 64k + 13}{16^k}.$$

**Example 79** (Chu and Zhang [21]:  $\left[ \frac{3}{2}, 1, \frac{5}{6}, 1 \right]$  in Theorem 1):

$$\frac{20\pi}{9\sqrt{3}} = \sum_{k=0}^{\infty} \left[ \begin{array}{c} 1, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{8}{5} \\ \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{5}, \frac{7}{6}, \frac{11}{6} \end{array} \right]_k \frac{12k^2 + 15k + 4}{16^k}.$$

**Example 80**  $\left( \left[ \frac{1}{2}, 1, \frac{1}{6}, 1 \right] \right.$  in Theorem 1):

$$\frac{20\pi}{27\sqrt{3}} = \sum_{k=0}^{\infty} \left[ \begin{array}{cccc} 1, & \frac{1}{3}, & \frac{2}{3}, & \frac{3}{4}, \frac{5}{4} \\ \frac{1}{2}, & \frac{1}{2}, & \frac{3}{2}, & \frac{7}{6}, \frac{11}{6} \end{array} \right]_k \frac{4 - 11k - 69k^2 - 60k^3}{16^k}.$$

**Example 81**  $\left( \left[ \frac{3}{2}, 1, \frac{5}{6}, 2 \right] \right.$  in Theorem 1):

$$\frac{140\pi}{27\sqrt{3}} = \sum_{k=0}^{\infty} \left[ \begin{array}{cccc} 2, & \frac{4}{3}, & -\frac{1}{3}, & \frac{3}{4}, \frac{5}{4} \\ \frac{1}{2}, & \frac{3}{2}, & \frac{3}{2}, & \frac{11}{6}, \frac{13}{6} \end{array} \right]_k \frac{60k^3 + 133k^2 + 85k + 13}{16^k}.$$

**Example 82**  $\left( \left[ \frac{1}{2}, 1, -\frac{1}{6}, 2 \right] \right.$  in Theorem 1):

$$\frac{700\pi}{243\sqrt{3}} = \sum_{k=0}^{\infty} \left[ \begin{array}{cccc} 2, & -\frac{1}{3}, & \frac{4}{3}, & \frac{5}{4}, \frac{7}{4} \\ -\frac{1}{2}, & \frac{1}{2}, & \frac{3}{2}, & \frac{11}{6}, \frac{13}{6} \end{array} \right]_k \frac{25 - 3k - 91k^2 - 60k^3}{16^k}.$$

**Example 83**  $\left( \left[ \frac{3}{2}, 1, \frac{5}{6}, 1 \right] \right.$  in Theorem 2):

$$\frac{4\pi}{9\sqrt{3}} = \frac{2}{3} + \sum_{k=1}^{\infty} \left[ \begin{array}{cccc} 1, & \frac{2}{3}, & -\frac{2}{3}, & \frac{1}{4}, -\frac{1}{4} \\ \frac{1}{2}, & \frac{1}{2}, & \frac{3}{2}, & \frac{5}{6}, \frac{7}{6} \end{array} \right]_k \frac{20k^2 + 7k + 2}{16^k}.$$

**Example 84**  $\left( \left[ \frac{1}{2}, 1, \frac{1}{6}, 1 \right] \right.$  in Theorem 2):

$$\frac{4\pi}{81\sqrt{3}} = \frac{2}{3} + \sum_{k=1}^{\infty} \left[ \begin{array}{cccc} 1, & \frac{2}{3}, & -\frac{2}{3}, & \frac{1}{4}, \frac{3}{4} \\ \frac{1}{2}, & \frac{1}{2}, & -\frac{1}{2}, & \frac{5}{6}, \frac{7}{6} \end{array} \right]_k \frac{2 + 7k - 20k^2}{16^k}.$$

**Example 85**  $\left( \left[ \frac{5}{2}, 1, \frac{5}{6}, 2 \right] \right.$  in Theorem 2):

$$\frac{10\pi}{7\sqrt{3}} = 2 + \sum_{k=1}^{\infty} \left[ \begin{array}{cccc} 2, & \frac{5}{3}, & -\frac{5}{3}, & \frac{1}{4}, -\frac{1}{4} \\ \frac{1}{2}, & \frac{1}{2}, & \frac{5}{2}, & \frac{7}{6}, \frac{11}{6} \end{array} \right]_k \frac{12k^2 + 17k + 8}{16^k}.$$

**Example 86**  $\left( \left[ \frac{3}{2}, 1, 2, \frac{5}{6} \right] \right.$  in Theorem 2):

$$\frac{16\pi}{3\sqrt{3}} = 20 + \sum_{k=1}^{\infty} \left[ \begin{array}{c} -\frac{1}{2}, \frac{4}{3}, -\frac{4}{3}, \frac{1}{6}, \frac{5}{6} \\ \frac{1}{2}, \frac{5}{3}, \frac{1}{4}, \frac{3}{4}, \frac{7}{6} \end{array} \right]_k \frac{56 - 33k - 270k^2}{16^k}.$$

**Example 87**  $\left( \left[ \frac{1}{2}, 1, \frac{1}{6}, 2 \right] \right.$  in Theorem 2):

$$\frac{350\pi}{243\sqrt{3}} = 30 + \sum_{k=1}^{\infty} \left[ \begin{array}{c} 2, \frac{5}{3}, -\frac{5}{3}, \frac{3}{4}, \frac{5}{4} \\ \frac{1}{2}, \frac{1}{2}, -\frac{3}{2}, \frac{7}{6}, \frac{11}{6} \end{array} \right]_k \frac{40 - k - 60k^2}{16^k}.$$

**Example 88**  $\left( \left[ \frac{3}{2}, 1, \frac{1}{6}, 2 \right] \right.$  in Theorem 2):

$$\frac{32\pi}{27\sqrt{3}} = -8 + \sum_{k=1}^{\infty} \left[ \begin{array}{c} 2, -\frac{2}{3}, -\frac{4}{3}, \frac{1}{4}, \frac{3}{4} \\ \frac{1}{2}, -\frac{1}{2}, \frac{3}{2}, \frac{5}{6}, \frac{7}{6} \end{array} \right]_k \frac{k(15k+2)(1-12k)}{16^k}.$$

**Example 89**  $\left( \left[ \frac{5}{2}, 1, 2, \frac{5}{6} \right] \right.$  in Theorem 1):

$$\frac{270\pi}{7\sqrt{3}} = \sum_{k=0}^{\infty} \left[ \begin{array}{c} \frac{1}{2}, \frac{1}{3}, -\frac{1}{3}, \frac{1}{6}, \frac{5}{6} \\ \frac{5}{2}, \frac{8}{3}, \frac{5}{4}, \frac{7}{4}, \frac{7}{6} \end{array} \right]_k \frac{70 + 409k + 627k^2 + 270k^3}{16^k}.$$

**Example 90**  $\left( \left[ \frac{5}{2}, 1, 2, \frac{1}{6} \right] \right.$  in Theorem 1):

$$\frac{756\pi}{275\sqrt{3}} = \sum_{k=0}^{\infty} \left[ \begin{array}{c} \frac{1}{2}, -\frac{1}{3}, -\frac{2}{3}, \frac{1}{6}, -\frac{1}{6}, \frac{11}{6} \\ \frac{5}{2}, \frac{10}{3}, \frac{5}{4}, \frac{7}{4}, \frac{5}{6}, \frac{5}{6} \end{array} \right]_k \frac{5 + 92k + 258k^2 + 135k^3}{16^k}.$$

**3.7. BBP-series.** In 1995, Simon Plouffe discovered the following amazing BBP-formula (named after Bailey – Borwein – Plouffe [2] (Theorem 1)):

$$\pi = \sum_{k=0}^{\infty} \left( \frac{1}{16} \right)^k \left\{ \frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right\}$$

that provides a digit-extraction algorithm for  $\pi$  in base 10. By decomposing the factorial fraction in the summand into partial fractions, we can show that the next five series are all equivalent to the above BBP-formula.

**Example 91**  $\left( \left[ \frac{3}{2}, 1, 1, \frac{3}{4} \right] \right.$  in Theorem 1):

$$15\pi = \sum_{k=0}^{\infty} \left[ \begin{array}{cccc} \frac{1}{2}, \frac{3}{4}, \frac{1}{8}, \frac{5}{8} \\ \frac{3}{2}, \frac{7}{4}, \frac{9}{8}, \frac{13}{8} \end{array} \right]_k \frac{120k^2 + 151k + 47}{16^k}.$$

**Example 92**  $\left( \left[ \frac{5}{2}, 1, 2, \frac{3}{4} \right] \right.$  in Theorem 1):

$$\frac{63\pi}{2} = \sum_{k=0}^{\infty} \left[ \begin{array}{cccc} \frac{1}{2}, \frac{3}{4}, \frac{1}{8}, -\frac{3}{8} \\ \frac{5}{2}, \frac{11}{4}, \frac{9}{8}, \frac{13}{8} \end{array} \right]_k \frac{120k^2 + 235k + 99}{16^k}.$$

**Example 93**  $\left( \left[ \frac{3}{2}, 1, 2, -\frac{1}{4} \right] \right.$  in Theorem 2):

$$\frac{21\pi}{8} = 7 + \sum_{k=1}^{\infty} \left[ \begin{array}{cccc} -\frac{1}{2}, -\frac{1}{4}, -\frac{3}{8}, -\frac{7}{8} \\ \frac{1}{2}, \frac{7}{4}, \frac{5}{8}, \frac{9}{8} \end{array} \right]_k \frac{480k^2 - 172k - 9}{16^k}.$$

**Example 94**  $\left( \left[ \frac{5}{2}, 1, 2, \frac{3}{4} \right] \right.$  in Theorem 2):

$$\frac{21\pi}{10} = 7 + \sum_{k=1}^{\infty} \left[ \begin{array}{cccc} -\frac{1}{2}, -\frac{1}{4}, -\frac{3}{8}, -\frac{7}{8} \\ \frac{5}{2}, \frac{3}{4}, \frac{5}{8}, \frac{9}{8} \end{array} \right]_k \frac{23 + 10k - 240k^2}{16^k}.$$

**Example 95**  $\left( \left[ \frac{3}{2}, 1, 2, -\frac{5}{4} \right] \right.$  in Theorem 2):

$$\frac{77\pi}{8} = -\frac{55}{3} + \sum_{k=1}^{\infty} \left[ \begin{array}{cccc} -\frac{1}{2}, -\frac{5}{4}, -\frac{7}{8}, -\frac{11}{8} \\ \frac{1}{2}, \frac{11}{4}, \frac{1}{8}, \frac{5}{8} \end{array} \right]_k \frac{160k^2 - 36k - 13}{16^k}.$$

There is another BBP-formula disguised in the article by Adamchik–Wagon [1]

$$2\pi = \sum_{k=0}^{\infty} \left( \frac{1}{16} \right)^k \left\{ \frac{8}{8k+2} + \frac{4}{8k+3} + \frac{4}{8k+4} - \frac{1}{8k+7} \right\}.$$

Then the same approach of partial fractions can show that it has the following different infinite series representations.

**Example 96**  $\left( \left[ \frac{3}{2}, 1, 1, \frac{1}{4} \right] \right.$  in Theorem 2):

$$\frac{5\pi}{9} = \frac{5}{3} + \sum_{k=1}^{\infty} \left[ \begin{array}{cccc} -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, -\frac{5}{8} \\ \frac{3}{2}, \frac{5}{4}, \frac{3}{8}, \frac{7}{8} \end{array} \right]_k \frac{7 - 6k - 80k^2}{16^k}.$$

**Example 97**  $\left( \left[ \frac{5}{2}, 1, 2, \frac{1}{4} \right] \right.$  in Theorem 2):

$$\frac{15\pi}{14} = 3 + \sum_{k=1}^{\infty} \left[ \begin{array}{cccc} -\frac{1}{2}, \frac{1}{4}, -\frac{5}{8}, -\frac{9}{8} \\ \frac{5}{2}, \frac{9}{4}, \frac{3}{8}, \frac{7}{8} \end{array} \right]_k \frac{19 - 62k - 80k^2}{16^k}.$$

**Example 98**  $\left( \left[ \frac{3}{2}, 1, 2, \frac{1}{4} \right] \right.$  in Theorem 2):

$$\frac{15\pi}{8} = 5 + \sum_{k=1}^{\infty} \left[ \begin{array}{cccc} -\frac{1}{2}, -\frac{3}{4}, -\frac{1}{8}, -\frac{5}{8} \\ \frac{1}{2}, \frac{1}{4}, \frac{7}{8}, \frac{11}{8} \end{array} \right]_k \frac{160k^2 - 108k + 21}{16^k}.$$

**Example 99**  $\left( \left[ \frac{3}{2}, 1, 2, -\frac{3}{4} \right] \right.$  in Theorem 2):

$$\frac{45\pi}{8} = 16 + \sum_{k=0}^{\infty} \left[ \begin{array}{cccc} -\frac{1}{2}, -\frac{3}{4}, -\frac{5}{8}, -\frac{9}{8} \\ \frac{1}{2}, \frac{9}{4}, \frac{3}{8}, \frac{7}{8} \end{array} \right]_k \frac{11 + 260k - 480k^2}{16^k}.$$

**Example 100**  $\left( \left[ \frac{3}{2}, 1, 1, \frac{5}{4} \right] \right.$  in Theorem 1):

$$21\pi = \sum_{k=0}^{\infty} \left[ \begin{array}{cccc} \frac{1}{2}, \frac{1}{4}, \frac{3}{8}, \frac{7}{8} \\ \frac{3}{2}, \frac{5}{4}, \frac{11}{8}, \frac{15}{8} \end{array} \right]_k \frac{65 + 413k + 812k^2 + 480k^3}{16^k}.$$

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Received 22.02.21