DOI: 10.37863/umzh.v74i12.6717

UDC 517.53

A. Banerjee, S. Maity[1] (Univ. Kalyani, West Bengal, India)

SUBSEQUENT INVESTIGATIONS OF THE LEAST CARDINALITIES OF UNIQUE RANGE SET FOR TWO MINIMUM WEIGHTS OVER A NON-ARCHIMEDEAN FIELD

ПОДАЛЬШІ ДОСЛІДЖЕННЯ НАЙМЕНШИХ ПОТУЖНОСТЕЙ МНОЖИНИ УНІКАЛЬНОСТІ ЗА ДВОМА МІНІМАЛЬНИМИ ВАГАМИ НАД НЕАРХІМЕДОВИМ ПОЛЕМ

First of all, we indicate a severe error in the analysis of the main results of both Chakraborty [Ukr. Math. J., **72**, № 11, 1794 – 1806 (2021)] and Chakraborty – Chakraborty [Ukr. Math. J., **72**, № 7, 1164 – 1174 (2020)], to show that both these papers cease to be true. Further, pertinent to the results of these two papers, we deal with the unique range set of a meromorphic function over a non-Archimedean field with the smallest possible weights 0 and 1 under the aegis of its most generalized form to improve the existing result.

Насамперед вказано на грубу помилку в аналізі основних результатів, що наведені в статтях Chakraborty [Ukr. Math. J., **72**, № 11, 1794 – 1806 (2021)] та Chakraborty – Chakraborty [Ukr. Math. J., **72**, № 7, 1164 – 1174 (2020)], щоб показати, що обидві статті втрачають силу. Далі, що стосується результатів цих двох статей, розглянуто множину унікальності мероморфної функції над неархімедовим полем з найменшими можливими вагами 0 і 1 під егідою його найбільш загальної форми для того, щоб покращити існуючий результат.

1. Introduction and motivation. At the outset, we would like to mention that though the whole paper has been oriented about uniqueness theory over non-Archimedean field but to enlighten some important facts relevant to the focus of the paper we have to mention some terminologies of value distribution theory over complex field available in the book [12]. Let z be a zero of $f(z)-a=0$, the multiplicity of z is denoted by $w(a,f;z)$. Let $\mathcal{M}(\mathbb{C})$ denotes the collection of all meromorphic functions on $\mathbb{C}$. For $f\in\mathcal{M}(\mathbb{C})$ and $a\in\mathbb{C}\cup\{\infty\}$ we define

$$E_f(a)=\big\{(z,w(a,f;z)): z \text{ is zero of } f(z)-a=0\big\}.$$

In the case of ignoring multiplicities we denote the set by $\overline{E}_f(a)$. Let $f,\ g\in\mathcal{M}(\mathbb{C})$, we say f and g share the value a CM (counting multiplicity) if $E_f(a)=E_g(a)$ and share the value a IM (ignoring multiplicity) if $\overline{E}_f(a)=\overline{E}_g(a)$. Now, for $f\in\mathcal{M}(\mathbb{C})$ and $S\subset\mathbb{C}\cup\{\infty\}$, define

$$E_f(S)=\bigcup_{a\in S}\{(z,w(a,f;z)): z \text{ is zero of } f(z)-a=0\}.$$

In the case of ignoring multiplicities we denote the set by $\overline{E}_f(S)$. Two functions $f,\ g\in\mathcal{M}(\mathbb{C})$ are said to share a set S CM (IM), if $E_f(S)=E_g(S)(\overline{E}_f(S)=\overline{E}_g(S))$.

The notion of weighted sharing of sets, introduced in [15], defined as follows:

[1] Corresponding author, e-mail: sayantanmaity100@gmail.com.

Let $k \in \mathbb{Z}^+ \cup \{\infty\}$, the set of all a-points of f with multiplicity m is counted m times if $m \leq k$ and counted $k+1$ times if $m > k$ is denoted by $E_f^k(a)$. For two functions $f, g \in \mathcal{M}(\mathbb{C})$ if $E_f^k(a) = E_g^k(a)$, then we say f, g share the value a with weight k. We say f, g share the set S with weight k if $E_f^k(S) = E_g^k(S)$ for a set $S \subset \mathbb{C} \cup \{\infty\}$. We write f, g share (S, k) to mean that f, g share the set S with weight k. In particular, if $S = \{a\}$, then we write f, g share (a, k).

Definition 1.1 [5]. *Let f, g be two meromorphic functions over $\mathbb{C}$ and $S \subset \mathbb{C} \cup \{\infty\}$. If $E_f^k(S) = E_g^k(S)$ implies $f \equiv g$, then S is called a unique range set for meromorphic functions with weight k or in brief URSMk.*

In particular, for $k = \infty$ and 0 we write unique range set for meromorphic (entire) functions as URSM (URSE) and URSM-IM (URSE-IM), respectively.

Over complex field, considering a new polynomial Frank – Reinders [10] obtained an URSM with cardinality ≥ 11 as follows:

Theorem A [10]. *Let $n \geq 11$ be an integer and $c \neq 0, 1$ be a complex number. Consider the polynomial*

$$P_{FR}(z) = \frac{(n-1)(n-2)}{2} z^n - n(n-2) z^{n-1} + \frac{n(n-1)}{2} z^{n-2} - c. \tag{1.1}$$

Then $S = \{z \in \mathbb{C} \mid P_{FR}(z) = 0\}$ is a URSM.

Let us denote

$$U_{CM}^M = \{S : S \text{ is } URSM\},$$

$$U_{IM}^M = \{S : S \text{ is } URSM\text{-}IM\},$$

$$\lambda_{CM}^M = \min\{n(S) : S \in U_{CM}^M\},$$

and

$$\lambda_{IM}^M = \min\{n(S) : S \in U_{IM}^M\},$$

where $n(s)$ is the cardinality of S. Analogously for entire functions we can define U_{CM}^E, U_{IM}^E, λ_{CM}^E, λ_{IM}^E. In [17] (see Theorem 9), Li – Yang proved that $\lambda_{CM}^M \geq 5$. Later in 2003, Yang – Yi [20, p. 527] considered the following example to show that $\lambda_{CM}^M \geq 6$.

***Example* 1.1.** Let $S = \{a_j, j = 1, 2, 3, 4, 5\}$ be an arbitrary subset of $\mathbb{C}$ with five distinct elements and let $f(z) = a_5 + \dfrac{1}{e^z + d}$, $g(z) = a_5 + \dfrac{1}{(d - b_1)(d - b_2)e^{-z} + d}$ provided that $b_1 + b_2 \neq b_3 + b_4$, where $b_j = \dfrac{1}{a_j - a_5}$, $j = 1, 2, 3, 4$, and $d = \dfrac{b_3 b_4 - b_1 b_2}{b_3 + b_4 - b_1 - b_2}$. In this case, $E_f^\infty(S) = E_g^\infty(S)$ but $f \not\equiv g$.

For entire functions, Li [16] obtained the following example to show that $\lambda_{CM}^E \geq 5$.

***Example* 1.2.** Let $S = \{a_j; j = 1, 2, 3, 4\}$ be an arbitrary subset of $\mathbb{C}$ with four distinct elements and let $f(z) = (d - a_1)e^z + d$ and $g(z) = (d - a_2)e^{-z} + d$ provided that $a_1 + a_2 \neq a_3 + a_4$, where $d = \dfrac{a_3 a_4 - a_1 a_2}{a_3 + a_4 - a_1 - a_2}$. It is easy to show that $E_f^\infty(S) = E_g^\infty(S)$ but $f \not\equiv g$.

But for strictly IM sharing, finding of suitable examples is not so easy rather very complicated. We can find only the following example to show that $\lambda_{IM}^M \geq 3$.

***Example* 1.3.** Let $S = \{0,1\}$ and $f(z) = \dfrac{e^{2z}-1}{e^{2z}+1}$, $g(z) = \dfrac{4e^{2z}}{(e^{2z}+1)^2}$. Notice that $E_f^0(S) = E_g^0(S)$ but $f \not\equiv g$.

In the case of entire functions, if we consider $f(z) = \sin z$, $g(z) = \cos z$ and $S = \{-1,0,1\}$, then $E_f^0(S) = E_g^0(S)$, but $f \not\equiv g$. Thus, $\lambda_{IM}^E \geq 4$.

From the above discussion we see that it is a very interesting question to find the minimum cardinality of a URSM-IM. In this regard, in 1999, Bartels [6] considered a set whose elements are the roots of $P_{FR}(z)$ to obtain the following result.

Theorem B [6]. *Let $n \geq 17$ be an integer and $c \neq 0,1$ be a complex number and consider $P_{FR}(z)$ defined in Theorem* A. *Then $S = \{z \in \mathbb{C} \mid P_{FR}(z) = 0\}$ is a URSM-IM.*

After that many researchers tried to reduce the lower bound of the cardinality of a URSM-IM, but for a long time, nobody succeeded. Recently, Chakraborty [8] reduces the minimum cardinality of the URSM-IM from 17 to 15 as follows:

Theorem C [8]. *Let $S = \{z \in \mathbb{C} \mid P_{FR}(z) = 0\}$, where $P_{FR}(z)$ is defined in* (1.1). *If $n \geq 15$, then S is a URSM-IM.*

***Remark* 1.1.** We see that Theorem C is a huge improvement of Theorem B. But unfortunately there is a serious error in the proof of Theorem C. In [8, p. 1805], the author uses the inequality $N(r,1;F \mid \geq 2) \leq N(r,0;f' \mid f \neq 0)$ (resp., $N(r,1;G \mid \geq 2) \leq N(r,0;g' \mid g \neq 0)$). But the inequality is not true because, if z_0 be an 1-point of F of multiplicity $p(\geq 2)$ then z_0 must be a zero of f' of multiplicity $p-1$. Actually the inequality would have been true had the author made the estimations as $2N(r,0;f' \mid f \neq 0)$ (resp., $2N(r,0;g' \mid g \neq 0)$) instead of $N(r,0;f' \mid f \neq 0)$ (resp., $N(r,0;g' \mid g \neq 0)$). But in that case, the cardinality n of the set S becomes ≥ 19. So by any means the possible corrected version of the main theorem of [8] has no value in comparison to Theorem B.

***Remark* 1.2.** If we closely study the proof of Theorem C, we notice that the Lemma 4.6 of [8] plays an important role in the proof. However it can be noticed that, Lemma 2.2 of [4] exhibits better inequality than Lemma 4.6 of [8]. So in the equation (5.5) of [8], if instead of Lemma 4.6 of [8], one use Lemma 2.2 of [4], then proceeding in a similar manner as done in [8], we get $\dfrac{3}{2}\overline{N}(r,1;F \mid \geq 2)$ (resp., $\dfrac{3}{2}\overline{N}(r,1;G \mid \geq 2)$) in place of $N(r,1;F \mid \geq 2)$ (resp., $N(r,1;G \mid \geq 2)$) in [8, p. 1805]. Calculating similar procedure, we obtain $n \geq 17$, but this was already proved in Theorem B.

Thus we observe that, so far using the existing techniques and methods over $\mathbb{C}$, it is a very challenging question to reduce the minimum cardinality of URSM-IM from 17. Thus the following question comes naturally.

***Question* 1.1.** Can it be possible to find a unique range set with cardinality less than 17, if we increase the weight of the sharing from 0 to 1?

Recently, Chakraborty – Chakraborty [9] answered the above question affirmatively and proved that for $n \geq 13$, there is a URSM1 over $\mathbb{C}$. The result is as follows:

Theorem D [9]. *Suppose that $n(\geq 1)$ is a positive integer. Further, suppose that $S = \{z \in \mathbb{C} \mid P_{FR}(z) = 0\}$, where the polynomial $P_{FR}(z)$ of degree n is defined by* (1.1). *Let f and g be two nonconstant meromorphic functions such that f and g share $(S,1)$ and $n \geq 13$, then $f \equiv g$.*

***Remark* 1.3.** Thus Theorem D says that S is a URSM1 with minimum cardinality 13. But the authors did the same mistake in equation (4.4) (see [9, p. 1170]) as we have already pointed out in Remark 1.1. The error could have been removed had the estimations of the counting functions $N(r,1;F|\geq 2)$ (resp., $N(r,1;G|\geq 2)$) be replace by $\overline{N}(r,1;F|\geq 2)$ (resp., $\overline{N}(r,1;G|\geq 2)$) in equation (4.4) of [9], and in that case one can get $n\geq 13$. However, in 2016, as a consequence of a main result of [3], the first author of this paper proved that the same set S as defined in the last theorem is a URSM1 with cardinality ≥ 12 (see Remark of [3, p. 205]). Thus we see that, long before Chakraborty – Chakraborty's [9] result a better result was already exhibited by Banerjee [3]. Therefore, the result of the corrected version of Chakraborty – Chakraborty [9] is also redundant. Actually, in equation (4.4), instead of $\overline{N}(r,1;F|\geq 2)$ (resp., $\overline{N}(r,1;G|\geq 2)$) of [9] a better estimation $\overline{N}(r,1;F|\geq 3)$ (resp., $\overline{N}(r,1;G|\geq 3)$) can be used to get the cardinality ≥ 12.

So, from the above discussion we see that over $\mathbb{C}$ the least cardinalities of URSM-IM and URSM1 are 17 and 12 respectively and there are no such fruitful method available in the literature to reduce the same. So the next question can appear in one's mind.

***Question* 1.2.** Instead of $\mathbb{C}$ if we work on non-Archimedean field $\mathbb{F}$, can it be possible to reduce the minimum cardinality of URSM-IM and URSM1?

To seek the possible answer of Question 1.2 under the most generalized form of the range set is the prime concern of the paper. Before approaching further, we recall some basics of non-Archimedean field.

2. Basis of value distribution theory over non-Archimedean field. Throughout the paper we consider $\mathbb{F}$ to be an algebraically closed non-Archimedean field with characteristic zero such that it is complete for a nontrivial non-Archimedean absolute value. We denote by $\log$ and $\ln$ as the real logarithm of base $p>1$ and e, respectively. Let us denote the collection of all meromorphic functions on $\mathbb{F}$ by $\mathcal{M}(\mathbb{F})$ and $\widetilde{\mathbb{F}}=\mathbb{F}\cup\{\infty\}$. The definition of CM (IM) sharing is similar as complex field. The notion of weighted sharing over $\mathbb{F}$ was introduced by Meng – Liu [18] and it is similar as over $\mathbb{C}$. URSM*k* over $\mathbb{F}$ also can be defined analogously as Definition 1.1.

Definition 2.1. *Let $P(z)$ be a polynomial in $\mathbb{F}$. If for any two nonconstant meromorphic functions f and g, the condition $P(f)\equiv cP(g)$ implies $f\equiv g$, where c is a non-zero constant, then P is called a strong uniqueness polynomial for meromorphic functions or SUPM in brief.*

To find the sufficient condition for a polynomial to be a SUPM, Fujimoto [11] introduced the following definition and called it as "Property H" which was latter well-known as "Critical Injection Property".

Definition 2.2 [5]. *Let $P(z)$ be a polynomial such that $P'(z)$ has l distinct zero namely $z_1,z_2,\ldots,z_l$. If $P(z_i)\neq P(z_j)$ for $i\neq j$, $i,j\in\{1,2,\ldots,l\}$, then $P(z)$ is said to satisfy the critical injection property.*

Over the non-Archimedean field the same definitions of critical injection property can be given.

For basic terminologies of value distribution theory over non-Archimedean field, readers can make a glance on [1, 2, 18]. Here we recall a few of them.

For a real constant ρ such that $0<\rho\leq r$, the counting function $N(r,a;f)$ of $f\in\mathcal{M}(\mathbb{F})$ is defined as follows:

$$N(r,a;f)=\frac{1}{\ln p}\int\limits_{\rho}^{r}\frac{n(t,a;f)}{t}\,dt,$$

where $n(t, a; f)$ is the number of solution (CM) of $f(z) = a$ in the disk $D_t = \{z \in \mathbb{F} : |z| \leq t\}$. For $l \in \mathbb{Z}^+$, define

$$N_l(r, a; f) = \frac{1}{\ln p} \int\limits_{\rho}^{r} \frac{n_l(t, a; f)}{t}\, dt,$$

where $n_l(t, a; f) = \sum_{|z| \leq r} \min\{l, w(a, f; z)\}$. Thus $N_1(r, a; f)$ denotes the counting function of a-points of f where multiplicity is counted only once, in short we call it "reduced counting function". Let us consider a nonconstant entire function f on $\mathbb{F}$ so that f has a power series expansion of the form $f = \sum_{n=0}^{\infty} a_n z^n$. For every $r > 0$, we define

$$|f|_r = \max\{|a_n| r^n : 0 \leq n < \infty\}.$$

Next consider f to be a nonconstant meromorphic function over $\mathbb{F}$. Thus $f = \dfrac{g}{h}$ such that g, h are entire functions in $\mathbb{F}$ and having no common zeros. We define $|f|_r = \dfrac{|g|_r}{|h|_r}$. Define the proximity function of f as follows:

$$m(r, \infty; f) = \log^+ |f|_r = \max\{0, \log |f|_r\} \text{ and } m(r, a; f) = \log^+ \left|\frac{1}{f - a}\right|_r.$$

Note that $w(a, f; z) = w(0, g - ah; z)$, $w(\infty, f; z) = w(0, h; z)$, $N(r, a; f) = N(r, 0; g - ah)$ and $N(r, \infty; f) = N(r, 0; h)$. The Nevanlinna characteristic function is defined as

$$T(r, f) = \max\{N(r, \infty; f), N(r, 0; f)\}.$$

We write simply $m(r, f)$ and $N(r, f)$ instead of $m(r, \infty; f)$ and $N(r, \infty; f)$.

Definition 2.3. *For $a \in \widetilde{\mathbb{F}}$ we denote by $N(r, a; f \mid = 1)$ the counting function of simple a-points of f. For $k \in \mathbb{Z}^+$ we denote by $N(r, a; f \mid \leq k)(N(r, a; f \mid \geq k))$ the counting function of those a-points of f whose multiplicities are not greater(less) than k where each a point is counted according to its multiplicity. $N_1(r, a; f \mid \leq k)(N_1(r, a; f \mid \geq k))$ are defined similarly, where in counting the a-points of f we ignore the multiplicities.*

Definition 2.4. *Let $a \in \widetilde{\mathbb{F}}$, f and g be two nonconstant meromorphic functions such that f and g share the value a IM. Let z_1 be an a-point of f with multiplicity s and an a-point of g with multiplicity t.*

By $N_1^E(r, a; f \mid = 1)$ we mean the reduced counting function of those a-points of f and g where $s = t = 1$.

For $k \in \mathbb{Z}^+$, $N_1^E(r, a; f \mid \geq k)$ denotes the reduced counting function of those a-points of f and g where $s = t \geq k$.

By $N_1^L(r, a; f)$ $(N_1^L(r, a; g))$ we mean the reduced counting function of those a-points of f and g where $s > t$ $(t > s)$.

We denote by $N_1^*(r, a; f, g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g. Note that $N_1^*(r, a; f, g) = N_1^*(r, a; g, f)$ and $N_1^*(r, a; f, g) = N_1^L(r, a; f) + N_1^L(r, a; g)$.

3. Background and main results. In the perspective of Question 1.2, the next theorem states that there is a URSM-IM over non-Archimedean field $\mathbb{F}$ of cardinality 16. Thus over $\mathbb{F}$ it is possible to reduce the minimum cardinality of URSM-IM by 1 than the complex field.

Theorem E ([13], Theorem 3.47)**.** *Let* $S = \{z \in \mathbb{F} \mid P_{FR}(z) = 0\}$, *where* $P_{FR}(z)$ *is defined in* (1.1). *If* $n \geq 16$, *then* S *is a URSM-IM.*

Now we introduce a new polynomial of degree $m + n + 1$ in the following manner:

$$P(z) = \sum_{j=0}^{n} \binom{n}{j} \frac{(-1)^j}{m+n+1-j} z^{m+n+1-j} a^j +$$

$$+ \sum_{i=1}^{m} \sum_{j=0}^{n} \binom{m}{i} \binom{n}{j} \frac{(-1)^{i+j}}{m+n+1-i-j} z^{m+n+1-i-j} a^j b^i + c =$$

$$= Q(z) + c, \tag{3.1}$$

where a and b be distinct such that $a \in \mathbb{F} \setminus \{0\}$, $b \in \mathbb{F}, c \in \mathbb{F} \setminus \{-Q(a), -Q(b)\}$. It is easy to verify that

$$P'(z) = (z-a)^n (z-b)^m.$$

***Remark* 3.1.** In (3.1), put $a = 1$, $b = 0$ and $n = 2$, the polynomial (3.1) reduces to

$$P_1(z) = \frac{z^{m+3}}{m+3} - 2\frac{z^{m+2}}{m+2} + \frac{z^{m+1}}{m+1} + c,$$

where $c \neq 0, -\dfrac{2}{(m+1)(m+2)(m+3)}$. Multiplying $P_1(z)$ by $\dfrac{(m+1)(m+2)(m+3)}{2}$ and putting $m + 3 = t$, we get

$$P_2(z) = \frac{(t-1)(t-2)}{2} z^t - t(t-2) z^{t-1} + \frac{t(t-1)}{2} z^{t-2} - d,$$

where $d = -c\dfrac{(m+1)(m+2)(m+3)}{2}$. As $c \neq 0, -\dfrac{2}{(m+1)(m+2)(m+3)}$ it follows that $d \neq$ $\neq 0, 1$, and notice that $P_2(z)$ is same as $P_{FR}(z)$. Thus, $P(z)$ as defined in (3.1), is a generalization of the polynomial $P_{FR}(z)$.

***Remark* 3.2.** The set of all zeros of $P'(z)$ is $\{a, b\}$. $P(z)$ have only simple zeros since $c \in$ $\in \mathbb{F} \setminus \{-Q(a), -Q(b)\}$.

***Remark* 3.3.** Notice that $P(z) - P(b) = (z-b)^{m+1} W_1(z)$, where $W_1(b) \neq 0$ and $W_1(z)$ has no multiple zero. Similarly, $P(z) - P(a) = (z-a)^{n+1} W_2(z)$, where $W_2(a) \neq 0$ and $W_2(z)$ has no multiple zero. If possible, let $P(a) = P(b)$, then this implies $(z-a)^{n+1} W_2(z) = (z-b)^{m+1} W_1(z)$. As $a \neq b$ so $W_2(z)$ has a factor $(z-b)^{m+1}$, hence the degree of $P(z)$ is at least $m + n + 2$, which is a contradiction. Thus, $P(a) \neq P(b)$. Therefore, $P(z)$ is a critically injective polynomial.

From Remark 3.1 we see that as $P(z)$ is a generalization of $P_{FR}(z)$, it will be natural to investigate analogous results of Theorem E under $P(z)$. In this respect we have the following result.

Theorem 3.1. *Let* f, g *be two nonconstant meromorphic functions on* $\mathbb{F}$ *and* m, n *be two positive integers such that* $n \geq 2$, $m \geq n + 2$ *and* $m + n \geq 15$. *Consider the polynomial* (3.1), *then the set* $\widetilde{S} = \{z \in \mathbb{F} \mid P(z) = 0\}$ *is URSM-IM.*

From Theorem 3.1 we can deduce the following corollaries.

***Corollary* 3.1.** (i) *Let* $m \geq 13$. *Consider the polynomial*

$$P_1(z) = \frac{z^{m+3}}{m+3} - 2\frac{z^{m+2}}{m+2} + \frac{z^{m+1}}{m+1} + c,$$

where $c \neq 0, -\dfrac{2}{(m+1)(m+2)(m+3)}$. *Then the set* $\widetilde{S}_1 = \{z \in \mathbb{F} \mid P_1(z) = 0\}$ *is a URSM-IM.*

(ii) *Let* $t \geq 16$. *Consider the polynomial*

$$P_2(z) = \frac{(t-1)(t-2)}{2}z^t - t(t-2)z^{t-1} + \frac{t(t-1)}{2}z^{t-2} - d,$$

where $d \neq 0, 1$. *Then the set* $\widetilde{S}_2 = \{z \in \mathbb{F} \mid P_2(z) = 0\}$ *is a URSM-IM.*

***Remark* 3.4.** Theorem 3.1 exhibits a URSM-IM of cardinality ≥ 16.

***Remark* 3.5.** Corollary 3.1(ii) is actually Theorem E. So Theorem 3.1extends Theorem E at a large extent.

In the next theorem instead of IM sharing we increase the weight only by 1 to investigate the least cardinality of the range set, under the periphery of the smallest positive integer weight.

Theorem 3.2. *Let* f, g *be two nonconstant meromorphic functions on* $\mathbb{F}$ *and* m, n *be two positive integers such that* $n \geq 2$, $m \geq n+2$ *and* $m+n \geq 10$. *Consider the polynomial* (3.1), *then the set* $\widetilde{S} = \{z \in \mathbb{F} \mid P(z) = 0\}$ *is URSM1.*

From Theorem 3.2, it is seen that the cardinality of URSM1 is ≥ 11. Thus, for non-Archimedean field cardinality of URSM1 reduces by 1 than complex field.

***Corollary* 3.2.** (i) *Let* $m \geq 8$. *Consider the polynomial* $P_1(z)$ *as in Corollary* 3.1. *Then the set* $\widetilde{S}_1 = \{z \in \mathbb{F} \mid P_1(z) = 0\}$ *is a URSM1.*

(ii) *Let* $t \geq 11$. *Consider the polynomial* $P_2(z)$ *as in Corollary* 3.1. *Then the set* $\widetilde{S}_2 = \{z \in \in \mathbb{F} \mid P_2(z) = 0\}$ *is a URSM1.*

4. Lemmas.

Lemma 4.1 [13]. *Let* $f(z)$ *be a nonconstant meromorphic function on* $\mathbb{F}$ *and* $a_1, a_2, \ldots, a_n \in \widetilde{\mathbb{F}}$ *are distinct points. Then*

$$(n-2)T(r,f) \leq \sum_{i=1}^{n} N_1(r, a_i; f) - N^0\big(r, 0; f'\big) - \log r + O(1),$$

where $N^0(r, 0; f')$ *denotes the counting function of zeros of* f' *which are not* a_i, $i = 1, 2, \ldots, n$, *points of* f.

Lemma 4.2 [7]. *Let* $f(z)$ *be a nonconstant meromorphic function on* $\mathbb{F}$ *and* $Q(z)$ *be a polynomial of degree* n *over* $\mathbb{F}$. *Then* $T(r, Q(f)) = nT(r, f) + O(1)$.

The next lemma follows from the equivalence of (i) and (iv) of Theorem 1 of Wang [19].

Lemma 4.3 [19]. *Let* f, g *be two nonconstant meromorphic functions on* $\mathbb{F}$ *and* $P(z)$ *be a critically injective polynomial such that derivative of* $P(z)$ *is of the form* $(z-\alpha)^m(z-\beta)^n$ *and let* $\min\{m, n\} \geq 2$. *If* $P(f) \equiv P(g)$ *then* $f \equiv g$.

Lemma 4.4 [14]. *Let f, g be two nonconstant meromorphic functions on $\mathbb{F}$ and $P(z)$ be a polynomial with no multiple zero and derivative of $P(z)$ is of the form $(z-\alpha)^m(z-\beta)^n$, also let* $\min\{m,n\} \geq 2$. *Assume that there exist constants $c_1 \neq 0$ and c_2 such that*

$$\frac{1}{P(f)} = \frac{c_1}{P(g)} + c_2.$$

Then $c_2 = 0$.

Lemma 4.5. *Let $N(r,0;f' \,|\, f \neq 0)$ denotes the counting function of zeros of f' which are not the zeros of f. Then*

$$N(r,0;f' \,|\, f \neq 0) \leq N_1(r,\infty;f) + N_1(r,0;f) + O(1).$$

Proof. Using the lemma of logarithmic derivative, we get

$$N(r,0;f' \,|\, f \neq 0) \leq N\left(r,0;\frac{f'}{f}\right) \leq$$

$$\leq T\left(r,\frac{f'}{f}\right) \leq$$

$$\leq N\left(r,\frac{f'}{f}\right) + m\left(r,\frac{f'}{f}\right) + O(1) \leq$$

$$\leq N_1(r,\infty;f) + N_1(r,0;f) + O(1).$$

Now let us consider two nonconstant meromorphic functions $\mathcal{F}$ and $\mathcal{G}$ on $\mathbb{F}$ such that $\mathcal{F} = P(f)$ and $\mathcal{G} = P(g)$, where $P(z)$ is defined as in (3.1). Besides this we also consider a function $\mathcal{H}$ as follows:

$$\mathcal{H} = \left(\frac{\mathcal{F}''}{\mathcal{F}'} - \frac{2\mathcal{F}'}{\mathcal{F}}\right) - \left(\frac{\mathcal{G}''}{\mathcal{G}'} - \frac{2\mathcal{G}'}{\mathcal{G}}\right). \tag{4.1}$$

Lemma 4.6. *Let $\mathcal{H} \not\equiv 0$ and $\mathcal{F}$, $\mathcal{G}$ share $(0,0)$, then*

$$N^E(r,0;\mathcal{F} \,|\, =1) = N^E(r,0;\mathcal{G} \,|\, =1) \leq N(r,\infty;\mathcal{H}) + O(1).$$

Proof. As $\mathcal{F}$ and $\mathcal{G}$ share $(0,0)$, so each simple zero of $\mathcal{F}$ is also simple zero of $\mathcal{G}$ and vice versa. Now each simple zero of $\mathcal{F}$ (i.e., simple zero of $\mathcal{G}$) is a zero of $\mathcal{H}$. Note that $m(r,\mathcal{H}) = O(1)$. Hence,

$$N^E(r,0;\mathcal{F}| = 1) = N^E(r,0;\mathcal{G}| = 1) \leq N(r,0;\mathcal{H}) \leq T(r,\mathcal{H}) \leq N(r,\infty;\mathcal{H}) + O(1).$$

Lemma 4.7. *Let $\widetilde{S} = \{z \in \mathbb{F} \,|\, P(z) = 0\}$, where $P(z)$ is defined as in* (3.1). *Let $\mathcal{H} \not\equiv 0$ and f, g be any two nonconstant meromorphic functions on $\mathbb{F}$ such that $E_f^0(\widetilde{S}) = E_g^0(\widetilde{S})$. Then*

$$N_1(r,\infty;\mathcal{H}) \leq N_1(r,\infty;f) + N_1(r,\infty;g) + N_1(r,a;f) + N_1(r,a;g) + N_1(r,b;f) +$$

$$+ N_1(r,b;g) + N_1^*(r,0;\mathcal{F},\mathcal{G}) + N_1^0\big(r,0;f'\big) + N_1^0\big(r,0;g'\big),$$

where $N_1^0(r,0;f')$ denotes reduced counting function of those zeros of f' which are not zeros of $\mathcal{F}(f-a)(f-b)$ and $N_1^0(r,0;g')$ denotes similar counting function.

Proof. Note that $\mathcal{F}' = P'(f) = (f-a)^n(f-b)^m f'$. The lemma directly follows by calculating all the possible poles of $\mathcal{H}$ and observe that all poles of $\mathcal{H}$ are simple.

Lemma 4.8. *Let m, n be two positive integers such that $n \geq 2$ and $m \geq n+2$. Consider the polynomial $P(z)$ as defined in* (3.1), *then $P(z)$ is a SUPM.*

Proof. Consider $P(z)$ as defined in (3.1). In view of Remarks 3.2 and 3.3 we see that $P(z)$ is a critically injective polynomial whose all zeros are simple. Let us assume

$$P(f) \equiv AP(g), \tag{4.2}$$

for some constant $A \neq 0$. By Lemma 4.2 and (4.2) we get

$$T(r,f) = T(r,g) + O(1). \tag{4.3}$$

Case 1. Let $P(b) \neq 1$. Now suppose $A \neq 1$.

Subcase 1.1. First assume $A = P(a)$. From (4.2)

$$P(f) - P(a) \equiv P(a)(P(g) - 1). \tag{4.4}$$

Consider the polynomial $P(z) - 1$. Note that $P(a) - 1 \neq 0$ as $A = P(a) \neq 1$ and $P(b) - 1 \neq 0$. So all zeros of $P(z) - 1$ are simple. Let us denote those simple zeros by α_j, $j = 1, 2, \ldots, m+n+1$. On the other hand, $P(z) - P(a) = (z-a)^{n+1}W_2(z)$, where $W_2(z)$ has no multiple zero and $W_2(a) \neq 0$. So $P(z) - P(a)$ has one zero at a with multiplicity $n+1$, and suppose the other simple zeros are β_j, $j = 1, 2, \ldots, m$. So from (4.4), we have

$$N_1(r,a;f) + \sum_{j=1}^{m} N_1(r,\beta_j;f) = \sum_{j=1}^{m+n+1} N_1(r,\alpha_j;g). \tag{4.5}$$

Using Lemma 4.1 and the equation (4.5), we obtain

$$\begin{aligned}(m+n-1)T(r,g) &\leq \sum_{j=1}^{m+n+1} N_1(r,\alpha_j;g) - \log r + O(1) = \\ &= N_1(r,a;f) + \sum_{j=1}^{m} N_1(r,\beta_j;f) - \log r + O(1) \leq \\ &\leq (m+1)T(r,f) - \log r + O(1).\end{aligned} \tag{4.6}$$

Now, by (4.3) and (4.6), we get

$$(n-2)T(r,g) + \log r \leq O(1),$$

this is a contradiction as $n \geq 2$.

Subcase 1.2. Next assume $A \neq P(a)$. From (4.2)

$$P(f) - AP(b) \equiv A(P(g) - P(b)). \tag{4.7}$$

Now we consider the following two subcases.

Subcase 1.2.1. First assume $P(a) \neq AP(b)$. Consider the polynomial $P(z) - AP(b)$. Now as $c \neq -Q(b)$, i.e., $P(b) \neq 0$ it follows that $P(b) - AP(b) = P(b)(1 - A) \neq 0$. So all zeros of $P(z) - AP(b)$ are simple and denote them by $\zeta_j,\ j = 1, 2, \ldots, m + n + 1$. On the other hand, from the discussion of Remark 3.3 we have $P(z) - P(b)$ has one zero at b with multiplicity $m + 1$ and other simple zeros are $\eta_j,\ j = 1, 2, \ldots, n$. Now proceeding similarly as (4.5) and (4.6), we get $(m - 2)T(r, f) + \log r \leq O(1)$, but this is a contradiction as $n \geq 2$ and $m \geq n + 2$.

Subcase 1.2.2. Next we assume $P(a) = AP(b)$. Consider the polynomial $P(z) - AP(b)$. Note that a is a zero of $P(z) - AP(b)$ with multiplicity $n + 1$ and other zeros are simple say λ_j, $j = 1, 2, \ldots, m$. Next for the polynomial $P(z) - P(b)$, b is a zero of multiplicity $m + 1$ and all other zeros are simple namely $\eta_j,\ j = 1, 2, \ldots, n$. Again proceeding in a similar manner as in (4.5) and (4.6), we deduce $(m - n - 2)T(r, g) + \log r \leq O(1)$, is a contradiction as $m \geq n + 2$.

Thus, from Case 1, we conclude that $A = 1$. Hence, from (4.2), we obtain $P(f) \equiv P(g)$. Now applying Lemma 4.3, we get $f \equiv g$.

Case 2. Let $P(b) = 1$. Now suppose $A \neq 1$.

Subcase 2.1. First assume $A = P(a)$. From (4.2)

$$P(f) - 1 \equiv P(a)\left(P(g) - \frac{1}{P(a)}\right). \tag{4.8}$$

Consider the polynomial $P(z) - \dfrac{1}{P(a)}$. Note that $P(a) \neq 1$.

Subcase 2.1.1. Let $P(a) \neq -1$. Hence $P(a) - \dfrac{1}{P(a)} \neq 0$ as $P(a) \neq 1, -1$. Also $P(b) - \dfrac{1}{P(a)} = 1 - \dfrac{1}{P(a)} \neq 0$. So all the zeros of $P(z) - \dfrac{1}{P(a)}$ are simple. Let us denote them by γ_j, $j = 1, 2, \ldots, m + n + 1$. On the other hand, as $P(b) = 1$, $P(z) - 1$ has only one multiple zero at b with multiplicity $m + 1$ and remaining all zeros are simple namely $\delta_j,\ j = 1, 2, \ldots, n$. Now using the similar steps as in (4.5) and (4.6), we obtain $(m - 2)T(r, g) + \log r \leq O(1)$, which is a contradiction as $n \geq 2$ and $m \geq n + 2$.

Subcase 2.1.2. Let $P(a) = -1$. As $P(a) - \dfrac{1}{P(a)} = 0$, so a is a zero of $P(z) - \dfrac{1}{P(a)}$ with multiplicity $n + 1$, and other zeros are simple say $\theta_j,\ j = 1, 2, \ldots, m$. On the other hand, as $P(b) = 1$, $P(z) - 1$ has only one multiple zero at b with multiplicity $m + 1$ and remaining all zeros are simple say $\delta_j,\ j = 1, 2, \ldots, n$. Now proceeding similarly as in (4.5) and (4.6), we obtain $(m - n - 2)T(r, g) + \log r \leq O(1)$, which is also contradiction as $m \geq n + 2$.

Subcase 2.2. Let $A \neq P(a)$. From (4.2)

$$P(f) - A \equiv A(P(g) - 1). \tag{4.9}$$

Note that, by the assumption of Case 2, $A \neq P(a)$ and we also have $A \neq 1$, so $P(a) - A \neq 0$ and $P(b) - A = 1 - A \neq 0$. Thus, all zeros of $P(z) - A$ are simple namely $\mu_j,\ j = 1, 2, \ldots, m + n + 1$. On the other hand, $P(z) - 1 = P(z) - P(b)$ has a multiple zero at b with multiplicity $m + 1$ and rest all zeros are simple namely $\eta_j,\ j = 1, 2, \ldots, n$. Again by the same argument as used in (4.5) and (4.6), we get $(m - 2)T(r, f) + \log r \leq O(1)$, this is a contradiction as $n \geq 2$ and $m \geq n + 2$.

Thus, from Case 2, it is clear that $A = 1$. Therefore, from (4.2), we get $P(f) \equiv P(g)$. Now applying Lemma 4.3, we conclude $f \equiv g$.

Therefore, from both Cases 1 and 2, we get that $P(z)$ is a SUPM.

Lemma 4.8 is proved.

5. Proofs of the theorems.

***Proof of Theorem* 3.1.** Let $\widetilde{S} = \{z \in \mathbb{F} \,|\, P(z) = 0\}$. Consider two functions $\mathcal{F} := P(f)$ and $\mathcal{G} := P(g)$.

Case 1. First assume $\mathcal{H} \not\equiv 0$. As $E_f^0(\widetilde{S}) = E_g^0(\widetilde{S})$, $\mathcal{F}$ and $\mathcal{G}$ share $(0,0)$. By using Lemmas 4.1, 4.2, 4.6, 4.7 and Definition 2.4, we deduce

$$(m+n+2)T(r,f) \leq$$
$$\leq N_1(r,\infty;f) + N_1(r,0;\mathcal{F}) + N_1(r,a;f) + N_1(r,b;f) - N^0\big(r,0;f'\big) - \log r + O(1) \leq$$
$$\leq N_1(r,\infty;f) + N_1^E(r,0;\mathcal{F}\,|\,=1) + N_1^L(r,0;\mathcal{F}) + N_1^L(r,0;\mathcal{G}) + N_1^E(r,0;\mathcal{F}\,|\,\geq 2) +$$
$$+ N_1(r,a;f) + N_1(r,b;f) - N^0\big(r,0;f'\big) - \log r + O(1) \leq$$
$$\leq N_1(r,\infty;f) + N_1(r,\infty;\mathcal{H}) + N_1^L(r,0;\mathcal{F}) + N_1^L(r,0;\mathcal{G}) + N_1^E(r,0;\mathcal{F}\,|\,\geq 2) +$$
$$+ N_1(r,a;f) + N_1(r,b;f) - N^0\big(r,0;f'\big) - \log r + O(1) \leq$$
$$\leq 2N_1(r,\infty;f) + N_1(r,\infty;g) + 2N_1(r,a;f) + 2N_1(r,b;f) + N_1(r,a;g) + N_1(r,b;g) +$$
$$+ N_1^0\big(r,0;g'\big) + N_1^E(r,0;\mathcal{F}\,|\,\geq 2) + N_1^*(r,0;\mathcal{F},\mathcal{G}) + N_1^L(r,0;\mathcal{F}) + N_1^L(r,0;\mathcal{G}) -$$
$$- \log r + O(1) \leq$$
$$\leq 2N_1(r,\infty;f) + N_1(r,\infty;g) + 2N_1(r,a;f) + 2N_1(r,b;f) + N_1(r,a;g) + N_1(r,b;g) +$$
$$+ N_1^0\big(r,0;g'\big) + N_1^E(r,0;\mathcal{F}\,|\,\geq 2) + 2N_1^L(r,0;\mathcal{F}) + 2N_1^L(r,0;\mathcal{G}) - \log r + O(1). \tag{5.1}$$

Now we deduce

$$N_1^0\big(r,0;g'\big) + N_1^E(r,0;\mathcal{F}\,|\,\geq 2) + 2N_1^L(r,0;\mathcal{F}) + 2N_1^L(r,0;\mathcal{G}) \leq$$
$$\leq N_1^0\big(r,0;g'\big) + N_1^E(r,0;\mathcal{G}\,|\,\geq 2) + 2N_1^L(r,0;\mathcal{G}) + 2N_1^L(r,0;\mathcal{F}) \leq$$
$$\leq N_1^0\big(r,0;g'\big) + N_1(r,0;\mathcal{G}\,|\,\geq 2) + N_1^L(r,0;\mathcal{G}) + 2N_1^L(r,0;\mathcal{F}) \leq$$
$$\leq N(r,0;g'\,|\,g\neq 0) + N_1^L(r,0;\mathcal{G}) + 2N_1^L(r,0;\mathcal{F}). \tag{5.2}$$

By using Lemma 4.5, we get

$$N_1^L(r,0;\mathcal{G}) \leq N_1(r,0;\mathcal{G}\,|\,\geq 2) \leq$$
$$\leq N(r,0;g'\,|\,g\neq 0) \leq$$
$$\leq N_1(r,\infty;g) + N_1(r,0;g) + O(1)$$

and similarly $N_1^L(r,0;\mathcal{F}) \leq N_1(r,\infty;f) + N_1(r,0;f) + O(1)$ holds. Thus, from (5.2),

$$N_1^0\big(r,0;g'\big) + N_1^E(r,0;\mathcal{F}\,|\,\geq 2) + 2N_1^L(r,0;\mathcal{F}) + 2N_1^L(r,0;\mathcal{G}) \leq$$
$$\leq 2\{N_1(r,\infty;g) + N_1(r,0;g) + N_1(r,\infty;f) + N_1(r,0;f)\} + O(1). \tag{5.3}$$

Combining (5.1) and (5.3), we obtain

$$(m+n+2)T(r,f) \leq$$

$$\leq 4N_1(r,\infty;f) + 3N_1(r,\infty;g) + 2N_1(r,a;f) + 2N_1(r,b;f) + N_1(r,a;g) + N_1(r,b;g) +$$

$$+ 2N_1(r,0;f) + 2N_1(r,0;g) - \log r + O(1) \leq$$

$$\leq 10T(r,f) + 7T(r,g) - \log r + O(1). \tag{5.4}$$

Similarly, we can have

$$(m+n+2)T(r,g) \leq 10T(r,g) + 7T(r,f) - \log r + O(1). \tag{5.5}$$

Thus, adding (5.4) and (5.5), we get

$$(m+n-15)(T(r,f)+T(r,g)) + 2\log r \leq O(1),$$

which is a contradiction as $m+n \geq 15$.

Case 2. Now consider the case $\mathcal{H} \equiv 0$. Integrating (4.1), we obtain

$$\frac{1}{\mathcal{F}} \equiv \frac{c_1}{\mathcal{G}} + c_2,$$

where $c_1 \neq 0$, c_2 are two constants. From Lemma 4.4 we get $c_2 = 0$. This implies $P(f) \equiv \dfrac{1}{c_1}P(g)$. Now by Lemma 4.8 we have that $P(z)$ is a SUPM, therefore $f \equiv g$.

Theorem 3.1 is proved.

***Proof of Corollary* 3.1.** (i) From Remark 3.1 we know that putting $a = 1,\ b = 0$ and $n = 2$ the polynomial (3.1) reduces to

$$P_1(z) = \frac{z^{m+3}}{m+3} - 2\frac{z^{m+2}}{m+2} + \frac{z^{m+1}}{m+1} + c,$$

where $c \neq 0, -\dfrac{2}{(m+1)(m+2)(m+3)}$. By Theorem 3.1 we get, when $m \geq 13$, then the set $\widetilde{S}_1$ becomes a URSM-IM. Therefore, for $m \geq 13$ and $c \neq 0, -\dfrac{2}{(m+1)(m+2)(m+3)}$, the set $\widetilde{S}_1$ is URSM-IM.

(ii) In the proof of (i), assuming $m+3 = t$, we obtain

$$P_1(z) = \frac{2}{t(t-1)(t-2)}\left[\frac{(t-1)(t-2)}{2}z^t - t(t-2)z^{t-1} + \frac{t(t-1)}{2}z^{t-2} - d\right] =$$

$$= \frac{2}{t(t-1)(t-2)}P_2(z),$$

where $d = -c\dfrac{t(t-1)(t-2)}{2}$. Noticing the fact that $t \geq 16$, from (i) we see that if $c \neq 0$, $-\dfrac{2}{t(t-1)(t-2)}$, $P_1(z)$ is a SUPM and this implies, whenever $d \neq 0, 1;$ $P_2(z)$ is also a SUPM. Therefore, for $t \geq 16$ and $d \neq 0, 1$, the set $\widetilde{S}_2$ is URSM-IM.

Corollary 3.1 is proved.

***Proof of Theorem* 3.2.** Let $\widetilde{S} = \{z \in \mathbb{F} \mid P(z) = 0\}$. Consider two functions $\mathcal{F}$ and $\mathcal{G}$ as Theorem 3.1.

Case 1. First assume $\mathcal{H} \not\equiv 0$. Since $E^1_f(\widetilde{S}) = E^1_g(\widetilde{S})$, then $\mathcal{F}$ and $\mathcal{G}$ share $(0,1)$, this implies $N^E(r,0;\mathcal{F}\,|\,=1) = N(r,0;\mathcal{F}\,|\,=1)$. By using Lemmas 4.1, 4.2, 4.6, 4.7 and Definition 2.4, we get

$$(m+n+2)T(r,f) \leq$$
$$\leq N_1(r,\infty;f) + N_1(r,0;\mathcal{F}) + N_1(r,a;f) + N_1(r,b;f) - N^0\big(r,0;f'\big) - \log r + O(1) \leq$$
$$\leq N_1(r,\infty;f) + N_1(r,0;\mathcal{F}\,|\,=1) + N_1(r,0;\mathcal{F}\,|\,\geq 2) + N_1(r,a;f) + N_1(r,b;f) -$$
$$- N^0\big(r,0;f'\big) - \log r + O(1) \leq$$
$$\leq N_1(r,\infty;f) + N_1(r,\infty;\mathcal{H}) + N_1(r,0;\mathcal{G}\,|\,\geq 2) + N_1(r,a;f) + N_1(r,b;f) -$$
$$- N^0\big(r,0;f'\big) - \log r + O(1) \leq$$
$$\leq 2N_1(r,\infty;f) + N_1(r,\infty;g) + 2N_1(r,a;f) + 2N_1(r,b;f) + N_1(r,a;g) + N_1(r,b;g) +$$
$$+ N^0_1\big(r,0;g'\big) + N_1(r,0;\mathcal{G}\,|\,\geq 2) + N^*_1(r,0;\mathcal{F},\mathcal{G}) - \log r + O(1). \tag{5.6}$$

Now using Lemma 4.5, we deduce

$$N^0_1\big(r,0;g'\big) + N_1(r,0;\mathcal{G}\,|\,\geq 2) + N^*_1(r,0;\mathcal{F},\mathcal{G}) \leq$$
$$\leq N^0_1\big(r,0;g'\big) + N_1(r,0;\mathcal{G}\,|\,\geq 2) + N_1(r,0;\mathcal{G}\,|\,\geq 3) + N_1(r,0;\mathcal{F}\,|\,\geq 3) \leq$$
$$\leq N(r,0;g'\,|\,g\neq 0) + \frac{1}{2}N(r,0;f'\,|\,f\neq 0) \leq$$
$$\leq N_1(r,\infty;g) + N_1(r,0;g) + \frac{1}{2}\{N_1(r,\infty;f) + N_1(r,0;f)\} + O(1). \tag{5.7}$$

Combining (5.6) and (5.7), we have

$$(m+n+2)T(r,f) \leq$$
$$\leq \frac{5}{2}N_1(r,\infty;f) + 2N_1(r,\infty;g) + 2N_1(r,a;f) + 2N_1(r,b;f) + N_1(r,a;g) + N_1(r,b;g) +$$
$$+ \frac{1}{2}N_1(r,0;f) + N_1(r,0;g) - \log r + O(1) \leq$$
$$\leq 7T(r,f) + 5T(r,g) - \log r + O(1). \tag{5.8}$$

Similarly, we can obtain

$$(m+n+2)T(r,g) \leq 7T(r,g) + 5T(r,f) - \log r + O(1). \tag{5.9}$$

Adding (5.8) and (5.9), we get

$$(m+n-10)(T(r,f)+T(r,g)) + 2\log r \leq O(1),$$

this is a contradiction as $m+n\geq 10$.

Case 2. Similar as Case 2 of Theorem 3.1 we get $f \equiv g$.

Theorem 3.2 is proved.

***Proof of Corollary* 3.2.** We omit the proof as the same can be carried out in the line of proof of Corollary 3.1.

6. Acknowledgements. Abhijit Banerjee is thankful to DST-PURSE II programme for financial assistance. Sayantan Maity is thankful to Council of Scientific and Industrial Research (India) for their financial support under File No. 09/106(0191)/2019-EMR-I.

References

1. V. H. An, H. H. Khoai, *Value sharing problems for p-adic meromorphic functions and their difference polynomials*, Ukranian Math. J., **64**, № 2, 147 – 164 (2012).
2. V. H. An, P. N. Hoa, H. H. Khoai, *Value sharing problems for differential and difference polynomials of meromorphic function in a non-Archimedean field*, p-Adic Number, Ultrametric Anal. and Appl., **9**, № 1, 1 – 14 (2017).
3. A. Banerjee, *Fujimoto's theorem — a further study*, J. Contemp. Math. Anal., **51**, № 4, 199 – 207 (2016).
4. A. Banerjee, *Uniqueness of meromorphic functions sharing two sets with finite weight II*, Tamkang J. Math., **41**, № 4, 379 – 392 (2010).
5. A. Banerjee, I. Lahiri, *A uniqueness polynomial generating a unique range set and vise versa*, Comput. Methods Funct. Theory, **12**, № 2, 527 – 539 (2012).
6. S. Bartels, *Meromorphic functions sharing a set with 17 elements ignoring multiplicities*, Complex Var. and Elliptic Equat., **39**, № 1, 85 – 92 (1999).
7. K. Boussaf, A. Escassut, J. Ojeda, *P-adic meromorphic functions $f'P'(f)$, $g'P'(g)$ sharing a small function*, Bull. Sci. Math., **136**, 172 – 200 (2012).
8. B. Chakraborty, *On the cardinality of a reduced unique range set*, Ukranian Math. J., **72**, № 11, 1794 – 1806 (2021).
9. B. Chakraborty, S. Chakraborty, *On the cardinality of unique range sets with weight one*, Ukranian Math. J., **72**, № 7, 1164 – 1174 (2020).
10. G. Frank, M. Reinders, *A unique range set for meromorphic functions with 11 elements*, Complex Var., Theory and Appl., **37**, № 1, 185 – 193 (1998).
11. H. Fujimoto, *On uniqueness of meromorphic functions sharing finite sets*, Amer. J. Math., **122**, 1175 – 1203 (2000).
12. W. K. Hayman, *Meromorphic functions*, Clarendon Press, Oxford (1964).
13. P. C. Hu, C. C. Yang, *Meromorphic functions over non-Archimedean fields*, Kluwer Acad. Publ. (2000).
14. H. H. Khoai, V. H. An, *URS and biURS for meromorphic functions in a non-Archimedean field*, p-Adic Number, Ultrametric Anal. and Appl., **12**, № 4, 276 – 284 (2020).
15. I. Lahiri, *Weighted sharing and uniqueness of meromorphic functions*, Nagoya Math. J., **161**, 193 – 206 (2001).
16. P. Li, *Uniqueness and value sharing of meromorphic functions*, Thesis, The Hong Kong Univ. Sci. Tech. (1996).
17. P. Li, C. C. Yang, *Some further results on the unique range sets of meromorphic functions*, Kodai Math. J., **18**, 437 – 450 (1995).
18. C. Meng, G. Liu, *Uniqueness for the difference monomials of p-adic entire functions*, Tbilisi Math. J., **11**, № 2, 67 – 76 (2018).
19. J. T-Y. Wang, *Uniqueness polynomials and bi-unique range sets for rational functions and non-Archimedean meromorphic functions*, Acta Arith., **104**, № 2, 183 – 200 (2002).
20. C. C. Yang, H. X. Yi, *Uniqueness theory of meromorphic functions*, Sci. Press and Kluwer Acad. Publ. (2003).

Received 02.05.21