DOI: 10.37863/umzh.v74i12.6578

UDC 517.9

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CONTINUOUS CHARACTERIZATION OF THE BESOV SPACES OF VARIABLE SMOOTHNESS AND INTEGRABILITY

НЕПЕРЕРВНА ХАРАКТЕРИЗАЦІЯ ПРОСТОРІВ БЄСОВА ЗМІННОЇ ГЛАДКОСТІ ТА ІНТЕГРОВНОСТІ

We obtain new equivalent quasinorms of the Besov spaces of variable smoothness and integrability. Our main tools are the continuous version of the Calderón reproducing formula, maximal inequalities, and the variable-exponent technique; however, allowing the parameters to vary from point to point leads to additional difficulties which, in general, can be overcome by imposing regularity assumptions on these exponents.

Отримано нові еквівалентні квазінорми просторів Бєсова змінної гладкості та інтегровності. Наші основні інструменти — це неперервна версія формули відтворення Калдерона, максимальні нерівності та техніка змінної експоненти. Зазначимо, що дозвіл для параметрів змінюватися від точки до точки викликає додаткові труднощі, які, як правило, можна подолати шляхом накладення припущень регулярності на відповідні експоненти.

1. Introduction. Besov spaces of variable smoothness and integrability initially appeared in the paper of A. Almeida and P. Hästö [3], where several basic properties were shown, such as the Fourier analytical characterization. Later the author [9] characterized these spaces by local means and established the atomic characterization. After that, Kempka and Vybíral [14] characterized these spaces by ball means of differences and also by local means. The duality of these function spaces is given in [12, 16].

The interest in these spaces comes not only from theoretical reasons but also from their applications to several classical problems in analysis. For further considerations of PDEs, we refer to [8] and references therein.

The main aim of this paper is to present new equivalent quasinorm of these function spaces, which based on the continuous version of Calderón reproducing formula. Firstly, we define new family of function spaces and prove their basic properties. Secondly, under some suitable assumptions on the parameters we prove that these function spaces are just the Besov spaces of variable smoothness and integrability of Almeida and Hästö. Finally we characterize these function spaces in terms of continuous local means.

This paper needs some notation. As usual, we denote by $\mathbb{N}_0$ the set of all nonnegative integers. The notation $f \lesssim g$ means that $f \leq cg$ for some independent positive constant c (and nonnegative functions f and g), and $f \approx g$ means that $f \lesssim g \lesssim f$. For $x \in \mathbb{R}$, $\lfloor x \rfloor$ stands for the largest integer smaller than or equal to x.

If $E \subset \mathbb{R}^n$ is a measurable set, then $|E|$ stands for the Lebesgue measure of E and χ_E denotes its characteristic function. By c we denote generic positive constants, which may have different values at different occurrences. Although the exact values of the constants are usually irrelevant for our purposes, sometimes we emphasize their dependence on certain parameters (e.g., $c(p)$ means that c depends on p, etc.).

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The symbol $\mathcal{S}(\mathbb{R}^n)$ is used in place of the set of all Schwartz functions on $\mathbb{R}^n$. We define the Fourier transform of a function $f \in \mathcal{S}(\mathbb{R}^n)$ by

$$\mathcal{F}(f)(\xi) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx, \quad \xi \in \mathbb{R}^n.$$

We denote by $\mathcal{S}'(\mathbb{R}^n)$ the dual space of all tempered distributions on $\mathbb{R}^n$. The variable exponents that we consider are always measurable functions p on $\mathbb{R}^n$ with range in $(0, \infty]$. We denote by $\mathcal{P}_0(\mathbb{R}^n)$ the set of such functions bounded away from the origin (i.e., $p^- > 0$). The subset of variable exponents with range in $[1, \infty]$ is denoted by $\mathcal{P}(\mathbb{R}^n)$. We use the standard notation

$$p^- := \operatorname*{ess-inf}_{x\in\mathbb{R}^n} p(x) \qquad \text{and} \qquad p^+ := \operatorname*{ess-sup}_{x\in\mathbb{R}^n} p(x).$$

We put

$$\omega_p(t) = \begin{cases} t^p, & \text{if} \quad p \in (0,\infty) \quad \text{and} \quad t > 0, \\ 0, & \text{if} \quad p = \infty \quad \text{and} \quad 0 < t \le 1, \\ \infty, & \text{if} \quad p = \infty \quad \text{and} \quad t > 1. \end{cases}$$

The variable exponent modular is defined by

$$\varrho_{p(\cdot)}(f) := \int_{\mathbb{R}^n} \omega_{p(x)}(|f(x)|)\, dx.$$

The variable exponent Lebesgue space $L^{p(\cdot)}$ consists of measurable functions f on $\mathbb{R}^n$ such that $\varrho_{p(\cdot)}(\lambda f) < \infty$ for some $\lambda > 0$. We define the Luxemburg (quasi)norm on this space by the formula

$$\|f\|_{p(\cdot)} := \inf\left\{\lambda > 0 \colon \varrho_{p(\cdot)}\left(\frac{f}{\lambda}\right) \le 1\right\}.$$

A useful property is that $\|f\|_{p(\cdot)} \le 1$ if and only if $\varrho_{p(\cdot)}(f) \le 1$ (see Lemma 3.2.4 from [8]).

Let $p, q \in \mathcal{P}_0(\mathbb{R}^n)$. The mixed Lebesgue-sequence space $\ell^{q(\cdot)}(L^{p(\cdot)})$ is defined on sequences of $L^{p(\cdot)}$-functions by the modular

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}((f_v)_v) := \sum_{v=0}^{\infty} \inf\left\{\lambda_v > 0 \colon \varrho_{p(\cdot)}\left(\frac{f_v}{\lambda_v^{1/q(\cdot)}}\right) \le 1\right\}.$$

The (quasi)norm is defined from this as usual:

$$\left\|(f_v)_v\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} := \inf\left\{\mu > 0 \colon \varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\left(\frac{1}{\mu}(f_v)_v\right) \le 1\right\}. \tag{1}$$

If $q^+ < \infty$, then we can replace (1) by a simpler expression

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}((f_v)_v) = \sum_{v=0}^{\infty} \left\| |f_v|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}}.$$

We use this notation even when $q^+ = \infty$. Let $(f_t)_{0<t\le 1}$ be a sequence of measurable functions when t is a continuous variable. We set

$$\varrho_{\widetilde{\ell^{q(\cdot)}(L^{p(\cdot)})}}((f_t)_{0<t\le 1}) := \int\limits_0^1 \inf\left\{\lambda_t : \varrho_{p(\cdot)}\left(\frac{f_t}{\lambda_t^{1/q(\cdot)}}\right) \le 1\right\}\frac{dt}{t}.$$

The (quasi)norm is defined by

$$\left\|(f_t)_{0<t\le 1}\right\|_{\widetilde{\ell^{q(\cdot)}(L^{p(\cdot)})}} := \inf\left\{\mu > 0 : \varrho_{\widetilde{\ell^{q(\cdot)}(L^{p(\cdot)})}}\left(\frac{1}{\mu}(f_t)_{0<t\le 1}\right) \le 1\right\}.$$

We say that a real valued-function g on $\mathbb{R}^n$ is *locally* log-*Hölder continuous* on $\mathbb{R}^n$, abbreviated $g \in C^{\log}_{\mathrm{loc}}(\mathbb{R}^n)$, if there exists a constant $c_{\log}(g) > 0$ such that

$$\left|g(x) - g(y)\right| \le \frac{c_{\log}(g)}{\log(e + 1/|x-y|)} \tag{2}$$

for all $x, y \in \mathbb{R}^n$.

We say that g satisfies the log-*Hölder decay condition*, if there exist two constants $g_\infty \in \mathbb{R}$ and $c_{\log} > 0$ such that

$$\left|g(x) - g_\infty\right| \le \frac{c_{\log}}{\log(e + |x|)}$$

for all $x \in \mathbb{R}^n$. We say that g is *globally* log-*Hölder continuous* on $\mathbb{R}^n$, abbreviated $g \in C^{\log}(\mathbb{R}^n)$, if it is locally log-Hölder continuous on $\mathbb{R}^n$ and satisfies the log-Hölder decay condition. The constants $c_{\log}(g)$ and $c_{\log}$ are called the *locally* log-*Hölder constant* and the log-*Hölder decay constant*, respectively. We note that any function $g \in C^{\log}_{\mathrm{loc}}(\mathbb{R}^n)$ always belongs to L^∞.

We define the following class of variable exponents:

$$\mathcal{P}_0^{\log}(\mathbb{R}^n) := \left\{p \in \mathcal{P}_0(\mathbb{R}^n) : \frac{1}{p} \in C^{\log}(\mathbb{R}^n)\right\},$$

which is introduced in [6] (Section 2). The class $\mathcal{P}^{\log}(\mathbb{R}^n)$ is defined analogously. We define

$$\frac{1}{p_\infty} := \lim_{|x|\to\infty} \frac{1}{p(x)},$$

and we use the convention $\dfrac{1}{\infty} = 0$. Note that although $\dfrac{1}{p}$ is bounded, the variable exponent p itself can be unbounded. We put

$$\Psi(x) := \sup_{|y|\ge|x|} \left|\varphi(y)\right|$$

for $\varphi \in L^1$. We suppose that $\Psi \in L^1$. Then it was proved in [8] (Lemma 4.6.3) that if $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$, then

$$\|\varphi_\varepsilon * f\|_{p(\cdot)} \le c\|\Psi\|_1\|f\|_{p(\cdot)}$$

for all $f \in L^{p(\cdot)}$, where

$$\varphi_\varepsilon := \frac{1}{\varepsilon^n}\varphi\left(\frac{\cdot}{\varepsilon}\right), \quad \varepsilon > 0.$$

We put

$$\eta_{t,m}(x) := t^{-n}(1 + t^{-1}|x|)^{-m}$$

for any $x \in \mathbb{R}^n$, $t > 0$ and $m > 0$. Note that $\eta_{t,m} \in L^1$, when $m > n$, and $\left\|\eta_{t,m}\right\|_1 = c(m)$ is independent of t. If $t = 2^{-v}$, $v \in \mathbb{N}_0$, then we put

$$\eta_{v,m} := \eta_{2^{-v},m}.$$

We refer to the recent monograph [5] for further properties, historical remarks and references on variable exponent spaces.

2. Basic tools. In this section, we present some useful results. The following lemma is proved in [7] (Lemma 6.1) (see also [14], Lemma 19).

Lemma 1. *Let $\alpha \in C^{\log}_{\text{loc}}(\mathbb{R}^n)$, $m \in \mathbb{N}_0$ and $R \geq c_{\log}(\alpha)$, where $c_{\log}(\alpha)$ is the constant from* (2), *for $g = \alpha$. Then there exists a constant $c > 0$ such that*

$$t^{-\alpha(x)}\eta_{t,m+R}(x-y) \leq ct^{-\alpha(y)}\eta_{t,m}(x-y)$$

for any $0 < t \leq 1$ and $x, y \in \mathbb{R}^n$.

The previous lemma allows us to treat the variable smoothness in many cases as if it were not variable at all. Namely, we can move the factor $t^{-\alpha(x)}$ inside the convolution as follows:

$$t^{-\alpha(x)}\eta_{t,m+R} * f(x) \leq c\eta_{t,m} * (t^{-\alpha(\cdot)}f)(x).$$

The following lemma is from [22] (Lemma 3.14).

Lemma 2. *Let $p, q \in \mathcal{P}_0(\mathbb{R}^n)$ and f be a measurable function on $\mathbb{R}^n$. If*

$$\left\||f|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}} \geq 1,$$

then

$$\|f\|^{q^-}_{p(\cdot)} \leq \left\||f|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}}.$$

The next lemma is a Hardy-type inequality, see [15].

Lemma 3. *Let $s > 0$ and $(\varepsilon_t)_{0<t\leq 1}$ be a sequence of positive measurable functions, when t is a continuous variable. Let*

$$\eta_t = t^s \int_t^1 \tau^{-s}\varepsilon_\tau \frac{d\tau}{\tau} \qquad \text{and} \qquad \delta_t = t^{-s}\int_0^t \tau^s \varepsilon_\tau \frac{d\tau}{\tau}.$$

Then there exists a constant $c > 0$ depending only on s such that

$$\int_0^1 \eta_t \frac{dt}{t} + \int_0^1 \delta_t \frac{dt}{t} \leq c\int_0^1 \varepsilon_t \frac{dt}{t}.$$

Lemma 4. *Let $r, N > 0$, $m > n$ and $\theta, \omega \in \mathcal{S}(\mathbb{R}^n)$ with $\operatorname{supp}\mathcal{F}\omega \subset \overline{B(0,1)}$. Then there exists a constant $c = c(r, m, n) > 0$ such that, for all $g \in \mathcal{S}'(\mathbb{R}^n)$, we have*

$$\left|\theta_N * \omega_N * g(x)\right| \leq c\big(\eta_{N,m} * |\omega_N * g|^r(x)\big)^{1/r}, \quad x \in \mathbb{R}^n,$$

where $\theta_N(\cdot) := N^n\theta(N\cdot)$, $\omega_N(\cdot) := N^n\omega(N\cdot)$ and $\eta_{N,m} := N^n(1 + N|\cdot|)^{-m}$.

The proof of this lemma is given in [10] (Lemma 2.2). The following lemma is from A. Almeida and P. Hästö [3] (Lemma 4.7) (we use it, since the maximal operator is in general not bounded on $\ell^{q(\cdot)}(L^{p(\cdot)})$, see [3], Example 4.1).

Lemma 5. *Let $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ and $q \in \mathcal{P}_0(\mathbb{R}^n)$ with $\frac{1}{q} \in C^{\log}_{\mathrm{loc}}(\mathbb{R}^n)$. For $m > n + c_{\log}\left(\frac{1}{q}\right)$, there exists $c > 0$ such that*

$$\left\|(\eta_{v,m} * f_v)_v\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \le c\left\|(f_v)_v\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}.$$

Lemma 6. *Let $0 < \alpha < \beta < \infty, p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ and $q \in \mathcal{P}(\mathbb{R}^n)$ with $\frac{1}{q} \in C^{\log}_{\mathrm{loc}}(\mathbb{R}^n)$. Let*

$$g_t(x) := \int\limits_{\alpha t}^{\beta t} \eta_{\tau,m} * f_\tau(x)\frac{d\tau}{\tau}, \quad t \in (0,1], \quad x \in \mathbb{R}^n.$$

(i) *Assume that $0 < \beta t \le 1$. The inequality*

$$\left\||c g_t|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}} \le \int\limits_{\alpha t}^{\beta t} \left\||f_\tau|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}} \frac{d\tau}{\tau} + t, \quad t \in (0,1],$$

holds for every sequence of functions $(f_t)_{0<t\le 1}$ and constant $m > n + c_{\log}\left(\frac{1}{q}\right)$ such that the first term on right-hand side is at most one, where the constant c independent of t.

(ii) *The inequality*

$$\left\|(g_t)_{0<t\le 1}\right\|_{\widetilde{\ell^{q(\cdot)}(L^{p(\cdot)})}} \le c\left\|(f_t)_{0<t\le 1}\right\|_{\widetilde{\ell^{q(\cdot)}(L^{p(\cdot)})}}$$

holds for every sequence of functions $(f_t)_{0<t\le 1}$ and constant $m > n + c_{\log}\left(\frac{1}{q}\right)$ such that the right-hand side is finite.

Proof. First let us prove (i). The claim can be reformulated as showing that

$$J := \left\|c_1 \delta^{-\frac{1}{q(\cdot)}} g_t\right\|_{p(\cdot)} \le 2^{1-\frac{1}{q^-}} + \ln\frac{\beta}{\alpha}, \quad t \in (0,1],$$

where $c_1 > 0$ and $\delta := \int_{\alpha t}^{\beta t} \left\||f_\tau|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}} \frac{d\tau}{\tau} + t$. Applying Lemma 1, with an appropriate choice of c_1, we get

$$J \le \int\limits_{\alpha t}^{\beta t} \left\|c_1 \delta^{-\frac{1}{q(\cdot)}}(\eta_{\tau,m} * f_\tau)\right\|_{p(\cdot)} \frac{d\tau}{\tau} \le$$

$$\le \int\limits_{\alpha t}^{\beta t} \left\|\eta_{\tau,m-c_{\log}\left(\frac{1}{q}\right)} * c_1 \delta^{-\frac{1}{q(\cdot)}}|f_\tau|\right\|_{p(\cdot)} \frac{d\tau}{\tau} \le$$

$$\leq \int\limits_{\alpha t}^{\beta t} \left\|\delta^{-\frac{1}{q(\cdot)}} f_\tau\right\|_{p(\cdot)} \frac{d\tau}{\tau},$$

since $\delta \in (t, 1+t]$ and the convolution with a radially decreasing L^1-function is bounded in $L^{p(\cdot)}$, since $m > n + c_{\log}\left(\frac{1}{q}\right)$. Write

$$\int\limits_{\alpha t}^{\beta t} \left\|\delta^{-\frac{1}{q(\cdot)}} f_\tau\right\|_{p(\cdot)} \frac{d\tau}{\tau} = \int\limits_{(\alpha t, \beta t]\cap B} \ldots \frac{d\tau}{\tau} + \int\limits_{(\alpha t, \beta t]\cap B^c} \cdots \frac{d\tau}{\tau} = J_{1,t} + J_{2,t},$$

where

$$B := \left\{\tau > 0 \colon \left\||\delta^{-\frac{1}{q(\cdot)}} f_\tau|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}} \geq 1\right\}.$$

By Lemma 2, we have

$$J_{1,t} \leq \int\limits_{(\alpha t, \beta t]\cap B} \left\||\delta^{-\frac{1}{q(\cdot)}} f_\tau|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}}^{\frac{1}{q^-}} \frac{d\tau}{\tau} \leq 2^{1-\frac{1}{q^-}} \delta^{-1} \int\limits_{\alpha t}^{\beta t} \left\||f_\tau|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}} \frac{d\tau}{\tau} \leq 2^{1-\frac{1}{q^-}}$$

and

$$J_{2,t} \leq \int\limits_{\alpha t}^{\beta t} \left\|\delta^{-\frac{1}{q(\cdot)}} f_\tau\right\|_{p(\cdot)} \frac{d\tau}{\tau} \leq \int\limits_{\alpha t}^{\beta t} \frac{d\tau}{\tau} = \ln\frac{\beta}{\alpha}.$$

Now we prove (ii). By the scaling argument, it suffices to consider the case

$$\left\|(f_t)_{0<t\leq 1}\right\|_{\widetilde{\ell^{q(\cdot)}(L^{p(\cdot)})}} = 1$$

and show that the modular of f on the left-hand side is bounded. In particular, we show that

$$\int\limits_0^1 \left\||cg_t|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}} \frac{dt}{t} \leq 2$$

for some positive constant c. Applying Hardy inequality (see Lemma 3 and the property (i)), we obtain the desired result.

Lemma 7. *Let* $0 < r < \infty$ *and* $m > \max\left(n, \frac{n}{r}\right)$. *Let* $\{\mathcal{F}\Phi, \mathcal{F}\varphi\}$ *be a resolution of unity* (*see Section* 3)

$$\mathcal{F}\Phi(\xi) + \int\limits_0^1 \mathcal{F}\varphi(t\xi)\frac{dt}{t} = 1, \quad \xi \in \mathbb{R}^n.$$

(i) *Let* $\theta \in \mathcal{S}(\mathbb{R}^n)$ *be such that* $\operatorname{supp}\mathcal{F}\theta \subset \left\{\xi \in \mathbb{R}^n : |\xi| \leq 2\right\}$. *There exists a constant* $c > 0$ *such that*

$$|\theta * f|^r \leq c\eta_{1,mr} * |\Phi * f|^r + c\int\limits_{1/4}^1 \eta_{1,mr} * |\varphi_\tau * f|^r \frac{d\tau}{\tau}$$

for any $f \in \mathcal{S}'(\mathbb{R}^n)$, *where* $\varphi_\tau = \tau^{-n}\varphi\left(\frac{\cdot}{\tau}\right)$.

(ii) *Let* $\omega \in \mathcal{S}(\mathbb{R}^n)$ *be such that* $\operatorname{supp} \mathcal{F}\omega \subset \left\{\xi \in \mathbb{R}^n : \frac{1}{2} \leq |\xi| \leq 2\right\}$. *There exists a constant* $c > 0$ *such that*

$$|\omega_t * f|^r \leq c\eta_{1,mr} * |\Phi * f|^r + c \int\limits_{t/4}^{\min(1,4t)} \eta_{\tau,mr} * |\varphi_\tau * f|^r \frac{d\tau}{\tau}$$

for any $f \in \mathcal{S}'(\mathbb{R}^n)$ *and any* $0 < t \leq 1$, *where* $\omega_t = t^{-n}\omega\left(\frac{\cdot}{t}\right)$.

Proof. We split the proof into two steps. First the case $1 \leq r < \infty$ follows by the Hölder inequality.

Step 1. Proof of (i). Since $\{\mathcal{F}\Phi, \mathcal{F}\varphi\}$ is a resolutions of unity, it follows that

$$\theta * f = \Phi * \theta * f + \int\limits_{1/4}^{1} \theta * \varphi_\tau * f \frac{d\tau}{\tau}.$$

First recall the elementary inequality

$$d^n \eta_{d,m}(y - z) \leq d^{2n} \eta_{d,-m}(y - x)\eta_{d,m}(x - z), \quad d > 0, \quad x, y, z \in \mathbb{R}^n,$$

which together with Lemma 4 implies that

$$|\Phi * \theta * f(y)|^r \lesssim \eta_{1,mr} * |\Phi * f|^r(y) =$$

$$= c \int\limits_{\mathbb{R}^n} \eta_{1,mr}(y - z)|\Phi * f(z)|^r dz \lesssim$$

$$\lesssim \eta_{1,-mr}(y - x)\eta_{1,mr} * |\Phi * f|^r(x)$$

for any $x \in \mathbb{R}^n$ and any $m > \frac{n}{r}$. Furthermore,

$$|\Phi * \theta * f(y)| \leq \int\limits_{\mathbb{R}^n} \eta_{1,N}(y - z)|\theta * f(z)|dz \leq$$

$$\leq \eta_{1,-m}(y - x)\theta_1^{*,m} f(x) \int\limits_{\mathbb{R}^n} \eta_{1,N-m}(y - z)dz \lesssim$$

$$\lesssim \eta_{1,-m}(y - x)\theta_1^{*,m} f(x)$$

for any $N > m + n$, where

$$\theta_1^{*,m} f(x) = \sup_{y \in \mathbb{R}^n} \frac{|\theta * f(y)|}{(1 + |y - x|)^m}, \quad x \in \mathbb{R}^n.$$

Therefore,

$$|\Phi * \theta * f(y)| \lesssim \eta_{1,-m}(y-x)(\theta_1^{*,m} f(x))^{1-r} \eta_{1,mr} * |\Phi * f|^r(x)$$

for any $x \in \mathbb{R}^n$ and any $m > n$. Again from Lemma 4 we conclude that

$$|\theta * \varphi_\tau * f(y)|^r \lesssim \eta_{1,mr} * |\varphi_\tau * f|^r(y) \lesssim (1+|y-x|)^{mr} \eta_{1,mr} * |\varphi_\tau * f|^r(x)$$

and

$$|\theta * \varphi_\tau * f(y)| \lesssim \int\limits_{\mathbb{R}^n} \eta_{\tau,N}(y-z)|\theta * f(z)|dz \lesssim$$

$$\lesssim (1+|y-x|)^m \theta_1^{*,m} f(x), \quad \frac{1}{4} \le \tau \le 1,$$

for any $x \in \mathbb{R}^n$, any $m > n$ and any $N > m+n$. Consequently,

$$\theta_1^{*,m} f(x) \le c(\theta_1^{*,m} f(x))^{1-r} \left(\eta_{1,mr} * |\Phi * f|^r(x) + \int\limits_{1/4}^{1} \eta_{1,mr} * |\varphi_\tau * f|^r(x) \frac{d\tau}{\tau} \right), \tag{3}$$

which implies that

$$|\theta * f(x)|^r \le c\eta_{1,mr} * |\Phi * f|^r(x) + c \int\limits_{1/4}^{1} \eta_{1,mr} * |\varphi_\tau * f|^r(x) \frac{d\tau}{\tau}, \tag{4}$$

when $\theta_1^{*,m} f(x) < \infty$, which is true if $m \ge \frac{n}{r} + N_0$ (order of distribution). We will use the Strömberg and Torchinsky idea [18]. Observe that the right-hand side of (4) decreases as m increases. Therefore, we have (4) for all $m > \frac{n}{r}$ but with $c = c(f)$ depending on f. We can easily check that if the right-hand side of (4), with $c = c(f)$, is finite imply that $\theta_1^{*,m} f(x) < \infty$, otherwise, there is nothing to prove. Returning to (3) and having in mind that now $\theta_1^{*,m} f(x) < \infty$, we obtain the desired estimate (4).

Step 2. Proof of (ii). We have

$$\omega_t * f = \int\limits_{t/4}^{\min(1,4t)} \omega_t * \varphi_\tau * f \frac{d\tau}{\tau} + \begin{cases} 0, & \text{if} \quad 0 < t < \frac{1}{4}, \\ \omega_t * \Phi * f, & \text{if} \quad \frac{1}{4} \le t \le 1. \end{cases}$$

Let

$$g_t(y) := \int\limits_{t/4}^{\min(1,4t)} \omega_t * \varphi_\tau * f(y) \frac{d\tau}{\tau}, \quad y \in \mathbb{R}^n, \quad 0 < t \le 1.$$

It follows from Lemma 4 that

$$|\omega_t * \varphi_\tau * f(y)|^r \lesssim \eta_{t,mr} * |\varphi_\tau * f|^r(y) \lesssim \eta_{\tau,mr} * |\varphi_\tau * f|^r(y) =$$

$$= c \int\limits_{\mathbb{R}^n} \eta_{\tau,mr}(y-z)|\varphi_\tau * f(z)|^r dz \lesssim$$

$$\lesssim (1+\tau^{-1}|y-x|)^{mr} \eta_{\tau,mr} * |\varphi_\tau * f|^r(x)$$

and

$$|\omega_t * \varphi_\tau * f(y)| \lesssim \int\limits_{\mathbb{R}^n} \eta_{\tau,N}(y-z)|\omega_t * f(z)| dz \lesssim$$

$$\lesssim \omega^*_{t,m} f(y) \int\limits_{\mathbb{R}^n} \eta_{\tau,N}(y-z)(1+t^{-1}|y-z|)^m dz \lesssim$$

$$\lesssim \omega_t^{*,m} f(y) \lesssim (1+t^{-1}|y-x|)^m \omega_t^{*,m} f(x)$$

for any $x, y \in \mathbb{R}^n$, any $\dfrac{t}{4} \leq \tau \leq \min(1, 4t),\ 0 < t \leq 1$, and any $N > m+n$, where

$$\omega_t^{*,m} f(x) = \sup_{y \in \mathbb{R}^n} \frac{|\omega_t * f(y)|}{(1+t^{-1}|y-x|)^m}, \quad x, y \in \mathbb{R}^n, \quad 0 < t \leq 1.$$

Therefore, $|g_t(y)|$ can be estimated from above by

$$c(\omega_t^{*,m} f(x))^{1-r}(1+t^{-1}|y-x|)^{m(1-r)} \times$$

$$\times \int\limits_{t/4}^{\min(1,4t)} (1+\tau^{-1}|y-x|)^{mr} \eta_{\tau,mr} * |\varphi_\tau * f|^r(x) \frac{d\tau}{\tau} \lesssim$$

$$\lesssim (1+t^{-1}|y-x|)^m (\omega_t^{*,m} f(x))^{1-r} \int\limits_{t/4}^{\min(1,4t)} \eta_{\tau,mr} * |\varphi_\tau * f|^r(x) \frac{d\tau}{\tau},$$

if $0 < t \leq 1$. Now if $\dfrac{1}{4} \leq t \leq 1$, we easily obtain

$$|\omega_t * \Phi * f(y)| = |\omega_t * \Phi * f(y)|^{1-r} |\omega_t * \Phi * f(y)|^r \lesssim$$

$$\lesssim (1+t^{-1}|y-x|)^{m(1-r)} (\omega_t^{*,m} f(x))^{1-r} \eta_{1,mr} * |\Phi * f|^r(y) \lesssim$$

$$\lesssim (1+t^{-1}|y-x|)^m (\omega_t^{*,m} f(x))^{1-r} \eta_{1,mr} * |\Phi * f|^r(x),$$

which yields that

$$\sup_{y \in \mathbb{R}^n} \frac{|\omega_t * \Phi * f(y)|}{(1+t^{-1}|y-x|)^m} \lesssim (\omega_t^{*,m} f(x))^{1-r} \eta_{1,mr} * |\Phi * f|^r(x).$$

Consequently,

$$|\omega_t * f(x)|^r \lesssim \left(\omega_t^{*,m} f(x)\right)^r \lesssim \eta_{1,mr} * |\Phi * f|^r(x) + \int\limits_{t/4}^{\min(1,4t)} \eta_{\tau,mr} * |\varphi_\tau * f|^r(x) \frac{d\tau}{\tau},$$

when $\omega_t^{*,m} f(x) < \infty, 0 < t \leq 1$ and $x \in \mathbb{R}^n$. Using a combination of the arguments used in (i), we arrive at the desired estimate.

Lemma 7 is proved.

The following lemma is from [17] (Lemma 1).

Lemma 8. *Let $\varrho, \mu \in \mathcal{S}(\mathbb{R}^n)$, and $M \geq -1$ an integer such that*

$$\int\limits_{\mathbb{R}^n} x^\alpha \mu(x) dx = 0$$

for all $|\alpha| \leq M$. Then, for any $N > 0$, there exists a constant $c(N) > 0$ such that

$$\sup_{z \in \mathbb{R}^n} |t^{-n} \mu(t^{-1} \cdot) * \varrho(z)| (1 + |z|)^N \leq c(N) t^{M+1}, \quad 0 < t \leq 1.$$

3. Variable Besov spaces. In this section, we present the definition of Besov spaces of variable smoothness and integrability, and prove the basic properties in analogy to the case of fixed exponents. Select a pair of Schwartz functions Φ and φ satisfying

$$\operatorname{supp} \mathcal{F}\Phi \subset \{x \in \mathbb{R}^n : |x| \leq 2\}, \qquad \operatorname{supp} \mathcal{F}\varphi \subset \left\{x \in \mathbb{R}^n : \frac{1}{2} \leq |x| \leq 2\right\} \tag{5}$$

and

$$\mathcal{F}\Phi(\xi) + \int\limits_0^1 \mathcal{F}\varphi(t\xi) \frac{dt}{t} = 1, \quad \xi \in \mathbb{R}^n. \tag{6}$$

Such a resolution (5) and (6) of unity can be constructed as follows. Let $\mu \in \mathcal{S}(\mathbb{R}^n)$ be such that $|\mathcal{F}\mu(\xi)| > 0$ for $1/2 < |\xi| < 2$. There exists $\eta \in \mathcal{S}(\mathbb{R}^n)$ with

$$\operatorname{supp} \mathcal{F}\eta \subset \left\{x \in \mathbb{R}^n : \frac{1}{2} < |x| < 2\right\}$$

such that

$$\int\limits_0^\infty \mathcal{F}\mu(t\xi)\, \mathcal{F}\eta(t\xi) \frac{dt}{t} = 1, \quad \xi \neq 0,$$

see [4, 11, 13]. We set $\mathcal{F}\varphi = \mathcal{F}\mu\, \mathcal{F}\eta$ and

$$\mathcal{F}\Phi(\xi) = \begin{cases} \displaystyle\int_1^\infty \mathcal{F}\varphi(t\xi)\, \frac{dt}{t}, & \text{if} \quad \xi \neq 0, \\ 1, & \text{if} \quad \xi = 0. \end{cases}$$

Then $\mathcal{F}\Phi \in \mathcal{S}(\mathbb{R}^n)$, and as $\mathcal{F}\eta$ is supported in $\left\{x \in \mathbb{R}^n : \dfrac{1}{2} \leq |x| \leq 2\right\}$, we see that $\operatorname{supp} \mathcal{F}\Phi \subset$ $\subset \{x \in \mathbb{R}^n : |x| \leq 2\}$.

Now we define the spaces under consideration.

Definition 1. *Let* $\alpha\colon \mathbb{R}^n \to \mathbb{R}$ *and* $p,q \in \mathcal{P}_0(\mathbb{R}^n)$. *Let* $\{\mathcal{F}\Phi, \mathcal{F}\varphi\}$ *be a resolution of unity and we put* $\varphi_t = t^{-n}\varphi\left(\dfrac{\cdot}{t}\right)$, $0 < t \leq 1$. *The Besov space* $\mathfrak{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ *is the collection of all* $f \in \mathcal{S}'(\mathbb{R}^n)$ *such that*

$$\|f\|^{\Phi,\varphi}_{\mathfrak{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} := \big\|\Phi * f\big\|_{p(\cdot)} + \big\|(t^{-\alpha(\cdot)}\varphi_t * f)_{0<t\leq 1}\big\|_{\ell^{q(\cdot)}\widetilde{(L^{p(\cdot)})}} < \infty.$$

When $q = \infty$, the Besov space $\mathfrak{B}^{\alpha(\cdot)}_{p(\cdot),\infty}$ consist of all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|^{\Phi,\varphi}_{\mathfrak{B}^{\alpha(\cdot)}_{p(\cdot),\infty}} := \big\|\Phi * f\big\|_{p(\cdot)} + \sup_{t\in(0,1]} \big\|t^{-\alpha(\cdot)}(\varphi_t * f)\big\|_{p(\cdot)} < \infty.$$

One recognizes immediately that $\mathfrak{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ is a quasinormed space and if α, p and q are constants, then

$$\mathfrak{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)} = B^{\alpha}_{p,q},$$

where $B^{\alpha}_{p,q}$ is the usual Besov spaces.

Now, we are ready to show that the definition of these function spaces is independent of the chosen resolution $\{\mathcal{F}\Phi, \mathcal{F}\varphi\}$ of unity. This justifies our omission of the subscript Φ and φ in the sequel.

Theorem 1. *Let* $\{\mathcal{F}\Phi, \mathcal{F}\varphi\}$ *and* $\{\mathcal{F}\Psi, \mathcal{F}\psi\}$ *be two resolutions of unity. Let* $\alpha\colon \mathbb{R}^n \to \mathbb{R}$ *and* $p,q \in \mathcal{P}_0(\mathbb{R}^n)$. *Assume that* $p \in \mathcal{P}^{\log}_0(\mathbb{R}^n)$ *and* $\alpha, \dfrac{1}{q} \in C^{\log}_{\mathrm{loc}}(\mathbb{R}^n)$. *Then*

$$\|f\|^{\Phi,\varphi}_{\mathfrak{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \approx \|f\|^{\Psi,\psi}_{\mathfrak{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

Proof. It is sufficient to show that there exists a constant $c > 0$ such that, for all $f \in \mathfrak{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$, we have

$$\|f\|^{\Phi,\varphi}_{\mathfrak{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \lesssim \|f\|^{\Psi,\psi}_{\mathfrak{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

In view of Lemma 7, the problem can be reduced to the case of $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ and $q \in \mathcal{P}(\mathbb{R}^n)$ with $\dfrac{1}{q} \in C^{\log}_{\mathrm{loc}}(\mathbb{R}^n)$. By the scaling argument, it suffices to consider the case $\|f\|^{\Psi,\psi}_{\mathfrak{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} = 1$ and show that

$$\big\|\Phi * f\big\|_{p(\cdot)} \lesssim 1$$

and

$$\int_0^1 \big\||ct^{-\alpha(\cdot)}(\varphi_t * f)|^{q(\cdot)}\big\|_{\frac{p(\cdot)}{q(\cdot)}} \frac{dt}{t} \leq 1$$

for some positive constant c. Interchanging the roles of (Ψ,ψ) and (Φ,φ), we obtain the desired result. We have

$$\mathcal{F}\Phi(\xi) = \mathcal{F}\Phi(\xi)\mathcal{F}\Psi(\xi) + \int_{1/4}^{1} \mathcal{F}\Phi(\xi)\mathcal{F}\psi(\tau\xi)\frac{d\tau}{\tau}$$

and

$$\mathcal{F}\varphi(t\xi) = \int\limits_{t/4}^{\min(1,4t)} \mathcal{F}\varphi(t\xi)\mathcal{F}\psi(\tau\xi)\frac{d\tau}{\tau} + \begin{cases} 0, & \text{if} \quad 0 < t < \dfrac{1}{4}, \\ \mathcal{F}\varphi(t\xi)\mathcal{F}\Psi(\xi), & \text{if} \quad \dfrac{1}{4} \le t \le 1, \end{cases}$$

for any $\xi \in \mathbb{R}^n$. Then we see that

$$\Phi * f = \Phi * \Psi * f + \int\limits_{1/4}^{1} \Phi * \psi_\tau * f \frac{d\tau}{\tau}$$

and

$$\varphi_t * f = \int\limits_{t/4}^{\min(1,4t)} \varphi_t * \psi_\tau * f \frac{d\tau}{\tau} + \begin{cases} 0, & \text{if} \quad 0 < t < \dfrac{1}{4}, \\ \varphi_t * \Psi * f, & \text{if} \quad \dfrac{1}{4} \le t \le 1. \end{cases}$$

First observe that

$$|\Phi * \psi_\tau * f| \lesssim |\eta_{1,m} * \psi_\tau * f| \lesssim \eta_{1,m} * \tau^{-\alpha(\cdot)}|\psi_\tau * f|, \quad \frac{1}{4} \le \tau \le 1, \quad m > n,$$

and

$$|\Phi * \Psi * f| \lesssim \eta_{1,m} * |\Psi * f|, \quad m > n.$$

Therefore,

$$|\Phi * f| \le \eta_{1,m} * |\Psi * f| + \int\limits_{1/4}^{1} \eta_{1,m} * \tau^{-\alpha(\cdot)}|\psi_\tau * f|\frac{d\tau}{\tau} =$$

$$= \eta_{1,m} * |\Psi * f| + g.$$

Since $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ and the convolution with a radially decreasing L^1-function is bounded on $L^{p(\cdot)}$, we have

$$\big\|\eta_{1,m} * |\Psi * f|\big\|_{p(\cdot)} \lesssim \big\|\Psi * f\big\|_{p(\cdot)} \le 1.$$

Now, for some suitable positive constant c_1,

$$\big\|c_1 g\big\|_{p(\cdot)} \le 1$$

if and only if

$$\big\||c_1 g|^{q(\cdot)}\big\|_{\frac{p(\cdot)}{q(\cdot)}} \le 1,$$

which follows by Lemma 6 (i). Therefore,

$$\big\|\Phi * f\big\|_{p(\cdot)} \lesssim 1.$$

Using the fact that the convolution with a radially decreasing L^1-function is bounded in $L^{p(\cdot)}$, we obtain

$$\big\||c\varphi_t * \Psi * f|^{q(\cdot)}\big\|_{\frac{p(\cdot)}{q(\cdot)}} \le 1,$$

with an appropriate choice of c and any $t \in (0,1]$. Observe that

$$|\varphi_t * f| \lesssim \int\limits_{t/4}^{4t} \eta_{\tau,m} * |\psi_\tau * f| \frac{d\tau}{\tau}, \qquad m > n + c_{\log}\left(\frac{1}{q}\right), \quad t \in \left(0, \frac{1}{4}\right].$$

Applying again Lemma 6 (ii), we find that

$$\int\limits_0^{\frac{1}{4}} \left\| |ct^{-\alpha(\cdot)}(\varphi_t * f)|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \frac{dt}{t} \leq 1$$

for some suitable positive constant c.

Theorem 1 is proved.

Let $a > 0$, $\alpha \colon \mathbb{R}^n \to \mathbb{R}$ and $f \in \mathcal{S}'(\mathbb{R}^n)$. Then we define the Peetre maximal function as follows:

$$\varphi_t^{*,a} t^{-\alpha(\cdot)} f(x) := \sup_{y \in \mathbb{R}^n} \frac{t^{-\alpha(y)} |\varphi_t * f(y)|}{(1 + t^{-1}|x-y|)^a}, \quad t > 0,$$

and

$$\Phi^{*,a} f(x) := \sup_{y \in \mathbb{R}^n} \frac{|\Phi * f(y)|}{(1 + |x-y|)^a}.$$

We now present a fundamental characterization of the spaces under consideration.

Theorem 2. *Let* α, $\dfrac{1}{q} \in C_{\text{loc}}^{\log}(\mathbb{R}^n)$, $p \in \mathcal{P}_0^{\log}(\mathbb{R}^n)$, $q^- \geq p^-$ *and* $a > \dfrac{n + c_{\log}(1/q)}{p^-} + c_{\log}(\alpha)$. *Then*

$$\|f\|^*_{\mathfrak{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} := \left\| \Phi^{*,a} \right\|_{p(\cdot)} + \left\| (\varphi_t^{*,a} t^{-\alpha(\cdot)} f)_{0<t\leq 1} \right\|_{\widetilde{\ell^{q(\cdot)}(L^{p(\cdot)})}}$$

is an equivalent quasinorm in $\mathfrak{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$.

Proof. It is easy to see that, for any $f \in \mathcal{S}'(\mathbb{R}^n)$ with $\|f\|^*_{\mathfrak{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} < \infty$ and any $x \in \mathbb{R}^n$, we have

$$t^{-\alpha(x)} |\varphi_t * f(x)| \leq \varphi_t^{*,a} t^{-\alpha(\cdot)} f(x).$$

This shows that $\|f\|_{\mathfrak{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \leq \|f\|^*_{\mathfrak{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}$. We will prove that there exists a constant $C > 0$ such that, for every $f \in \mathfrak{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$,

$$\|f\|^*_{\mathfrak{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \leq C \|f\|_{\mathfrak{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}. \tag{7}$$

By Lemmas 1 and 4, the estimate

$$t^{-\alpha(y)} |\varphi_t * f(y)| \leq C_1 t^{-\alpha(y)} \left(\eta_{t,\sigma p^-} * |\varphi_t * f|^{p^-}(y) \right)^{1/p^-} \leq$$

$$\leq C_2 \left(\eta_{t,(\sigma - c_{\log}(\alpha))p^-} * (t^{-\alpha(\cdot)} |\varphi_t * f|)^{p^-}(y) \right)^{1/p^-} \tag{8}$$

is true for any $y \in \mathbb{R}^n$, $\sigma > \dfrac{n + c_{\log}(1/q)}{p^-} + c_{\log}(\alpha)$ and $t > 0$. Now dividing both sides of (8) by $\left(1 + t^{-1}|x-y|\right)^a$, in the right-hand side we use the inequality

$$\big(1+t^{-1}|x-y|\big)^{-a}\leq\big(1+t^{-1}|x-z|\big)^{-a}\big(1+t^{-1}|y-z|\big)^{a},\quad x,y,z\in\mathbb{R}^n,$$

while in the left-hand side we take the supremum over $y\in\mathbb{R}^n$, we find that, for all $f\in\mathfrak{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ any $t>0$ and any $\sigma>\max\bigg(\dfrac{n+c_{\log}(1/q)}{p^-}+c_{\log}(\alpha),a+c_{\log}(\alpha)\bigg)$,

$$\varphi_t^{*,a}t^{-\alpha(\cdot)}f(x)\leq C_2\big(\eta_{t,ap^-}*(t^{-\alpha(\cdot)p^-}|\varphi_t*f|^{p^-})(x)\big)^{1/p^-},$$

where $C_2>0$ is independent of x,t and f. Assume that the right-hand side of (7) is less than or equal 1. We will prove that

$$\big\|\Phi^{*,a}\big\|_{p(\cdot)}+\Big\|\Big(\big(\eta_{t,ap^-}*(t^{-\alpha(\cdot)p^-}|\varphi_t*f|^{p^-})\big)^{1/p^-}\Big)_{0<t\leq 1}\Big\|_{\widetilde{\ell^{q(\cdot)}(L^{p(\cdot)})}}\lesssim 1. \tag{9}$$

Observe that the second quasinorm of the left-hand side of (9) can be rewritten as

$$\Big\|\Big(\eta_{t,ap^-}*(t^{-\alpha(\cdot)p^-}|\varphi_t*f|^{p^-})\Big)_{0<t\leq 1}\Big\|^{\frac{1}{p^-}}_{\widetilde{\ell^{\frac{q(\cdot)}{p^-}}(L^{\frac{p(\cdot)}{p^-}})}}. \tag{10}$$

Let $0<t<\dfrac{1}{4}$. In view the proof of Lemma 7 (ii), we obtain

$$t^{-\alpha(\cdot)p^-}|\varphi_t*f|^{p^-}\lesssim\int\limits_{t/4}^{4t}\tau^{-\alpha(\cdot)p^-}\eta_{\tau,ap^-}*|\varphi_\tau*f|^{p^-}\frac{d\tau}{\tau}\lesssim$$

$$\lesssim\int\limits_{t/4}^{4t}\eta_{\tau,ap^--c_{\log}(\alpha)p^-}*\tau^{-\alpha(\cdot)p^-}|\varphi_\tau*f|^{p^-}\frac{d\tau}{\tau},$$

by Lemma 1. Therefore,

$$\eta_{t,ap^-}*(t^{-\alpha(\cdot)p^-}|\varphi_t*f|^{p^-})\lesssim\int\limits_{t/4}^{4t}\eta_{t,ap^-}*\eta_{\tau,ap^--c_{\log}(\alpha)p^-}*\tau^{-\alpha(\cdot)p^-}|\varphi_\tau*f|^{p^-}\frac{d\tau}{\tau}\lesssim$$

$$\lesssim\int\limits_{t/4}^{4t}\eta_{\tau,ap^--c_{\log}(\alpha)p^-}*\tau^{-\alpha(\cdot)p^-}|\varphi_\tau*f|^{p^-}\frac{d\tau}{\tau}$$

by [7] (Lemma A.3). Applying Lemma 6, we deduce that (10), with $0<t<\dfrac{1}{4}$, is bounded by

$$\Big\|\big(t^{-\alpha(\cdot)p^-}|\varphi_t*f|^{p^-}\big)_{0<t\leq 1}\Big\|^{\frac{1}{p^-}}_{\widetilde{\ell^{\frac{q(\cdot)}{p^-}}(L^{\frac{p(\cdot)}{p^-}})}}\lesssim 1.$$

Now let $\dfrac{1}{4}\leq t\leq 1$. Again, by Lemma 7 (ii), we get

$$|\varphi_t * f|^{p^-} \leq c\eta_{1,ap^-} * |\Phi * f|^{p^-} + c\int\limits_{t/4}^{1} \eta_{\tau,ap^-} * |\varphi_\tau * f|^{p^-} \frac{d\tau}{\tau}.$$

As above, we obtain

$$\eta_{t,ap^-} * \left(t^{-\alpha(\cdot)p^-} |\varphi_t * f|^{p^-}\right) \leq$$

$$\leq c\eta_{1,ap^-} * |\Phi * f|^{p^-} + c\int\limits_{t/4}^{1} \eta_{\tau,ap^- - c_{\log}(\alpha)p^-} * \tau^{-\alpha(\cdot)p^-} |\varphi_\tau * f|^{p^-} \frac{d\tau}{\tau} =$$

$$= c\eta_{1,ap^-} * |\Phi * f|^{p^-} + h_t(x).$$

We need to prove that

$$\left\|\left(\eta_{1,ap^-} * |\Phi * f|^{p^-}\right)_{\frac{1}{4}\leq t\leq 1}\right\|_{\widetilde{\ell^{\frac{q(\cdot)}{p^-}}(L^{\frac{p(\cdot)}{p^-}})}} \lesssim 1 \qquad \text{and} \qquad \left\|(h_t)_{\frac{1}{4}\leq t\leq 1}\right\|_{\widetilde{\ell^{\frac{q(\cdot)}{p^-}}(L^{\frac{p(\cdot)}{p^-}})}} \lesssim 1. \tag{11}$$

Applying Lemma 6, we obtain the second estimate of (11). Let us prove the first one. This is equivalent to

$$\left\|\left|\eta_{1,ap^-} * |\Phi * f|^{p^-}\right|^{\frac{q(\cdot)}{p^-}}\right\|_{\frac{p(\cdot)}{q(\cdot)}} \lesssim 1,$$

which is equivalent to

$$\left\|\eta_{1,ap^-} * |\Phi * f|^{p^-}\right\|_{\frac{p(\cdot)}{p^-}} \lesssim 1.$$

Since $\dfrac{p(\cdot)}{p^-} \in \mathcal{P}^{\log}(\mathbb{R}^n)$ and the convolution with a radially decreasing L^1-function is bounded in $L^{p(\cdot)}$, we have

$$\left\|\eta_{1,ap^-} * |\Phi * f|^{p^-}\right\|_{\frac{p(\cdot)}{p^-}} \lesssim \left\|\left|\Phi * f\right|^{p^-}\right\|_{\frac{p(\cdot)}{p^-}} = c\left\|\Phi * f\right\|_{p(\cdot)}^{p^-} \lesssim 1.$$

The estimate of $\left\|\Phi^{*,a}\right\|_{p(\cdot)}$ follows easily from the fact that

$$\left\|\Phi^{*,a}\right\|_{p(\cdot)} \lesssim \left\|\eta_{1,ap^-} * |\Phi * f|^{p^-}\right\|_{\frac{p(\cdot)}{p^-}}^{\frac{1}{p^-}} \lesssim 1.$$

Theorem 2 is proved.

4. Relation between $\mathfrak{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ and $B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$. In this section, we present the coincidence between the above function spaces and the variable Besov spaces of Almeida and Hästö, where to define these function spaces we first need the concept of a smooth dyadic resolution of unity. Let Ψ be a function in $\mathcal{S}(\mathbb{R}^n)$ satisfying $\Psi(x) = 1$ for $|x| \leq 1$ and $\Psi(x) = 0$ for $|x| \geq 2$. We define ψ_0 and ψ_1 by $\mathcal{F}\psi_0(x) = \Psi(x)$, $\mathcal{F}\psi_1(x) = \Psi\left(\dfrac{x}{2}\right) - \Psi(x)$ and

$$\mathcal{F}\psi_v(x) = \mathcal{F}\psi_1(2^{1-v}x) \qquad \text{for} \quad v = 2, 3, \ldots .$$

Then $\{\mathcal{F}\psi_v\}_{v\in\mathbb{N}_0}$ is a smooth dyadic resolution of unity, $\sum_{v=0}^{\infty}\mathcal{F}\psi_v(x)=1$ for all $x\in\mathbb{R}^n$. Thus we obtain the Littlewood – Paley decomposition

$$f=\sum_{v=0}^{\infty}\psi_v*f$$

for all $f\in\mathcal{S}'(\mathbb{R}^n)$ (convergence in $\mathcal{S}'(\mathbb{R}^n)$).

We state the definition of the spaces $B^{s(\cdot)}_{p(\cdot),q(\cdot)}$, which introduced and investigated in [3].

Definition 2. *Let* $\{\mathcal{F}\psi_v\}_{v\in\mathbb{N}_0}$*be a resolution of unity,* $s\colon\mathbb{R}^n\to\mathbb{R}$ *and* $p,q\in\mathcal{P}_0(\mathbb{R}^n)$*. The Besov space* $B^{s(\cdot)}_{p(\cdot),q(\cdot)}$ *consists of all distributions* $f\in\mathcal{S}'(\mathbb{R}^n)$ *such that*

$$\|f\|_{B^{s(\cdot)}_{p(\cdot),q(\cdot)}}:=\left\|(2^{vs(\cdot)}\psi_v*f)_v\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}<\infty.$$

Taking $s\in\mathbb{R}$ and $q\in(0,\infty]$ as constants we derive the spaces $B^{s}_{p(\cdot),q}$ studied by Xu in [23]. We refer the reader to the recent papers [1, 2, 9, 14] for further details, historical remarks and more references on these function spaces. For any $p,q\in\mathcal{P}_0^{\log}(\mathbb{R})^n$ and $s\in C^{\log}_{\mathrm{loc}}$, the space $B^{s(\cdot)}_{p(\cdot),q(\cdot)}$ does not depend on the chosen smooth dyadic resolution of unity $\{\mathcal{F}\psi_v\}_{v\in\mathbb{N}_0}$ (in the sense of equivalent quasinorms) and

$$\mathcal{S}(\mathbb{R}^n)\hookrightarrow B^{s(\cdot)}_{p(\cdot),q(\cdot)}\hookrightarrow\mathcal{S}'(\mathbb{R}^n).$$

Moreover, if p,q,s are constants, we reobtain the usual Besov spaces $B^{s}_{p,q}$, studied in detail in [20, 21], see also [19].

Theorem 3. *Let* $\alpha\colon\mathbb{R}^n\to\mathbb{R}$ *and* $p,q\in\mathcal{P}_0(\mathbb{R}^n)$*. Assume that* $p\in\mathcal{P}_0^{\log}(\mathbb{R}^n)$ *and* $\alpha,\dfrac{1}{q}\in$ $\in C^{\log}_{\mathrm{loc}}(\mathbb{R}^n)$*. Then*

$$\mathfrak{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}=B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)},$$

in the sense of equivalent quasinorms.

Proof. *Step* 1. We will prove that

$$\mathfrak{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}\hookrightarrow B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}.$$

From Lemma 7, we only consider the case $p\in\mathcal{P}^{\log}(\mathbb{R}^n)$ and $q\in\mathcal{P}(\mathbb{R}^n)$ with $\dfrac{1}{q}\in C^{\log}_{\mathrm{loc}}(\mathbb{R}^n)$. Let $\{\mathcal{F}\Phi,\mathcal{F}\varphi\}$ and $\{\mathcal{F}\psi_j\}_{j\in\mathbb{N}_0}$ be two resolutions of unity and let $f\in\mathfrak{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ with

$$\|f\|_{\mathfrak{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}\le 1.$$

We have

$$\psi_v*f=\int\limits_{2^{-v-2}}^{\min(1,2^{2-v})}\psi_v*\varphi_t*f\frac{dt}{t}+\begin{cases}0, & \text{if}\quad v\ge 2,\\ \psi_v*\Phi*f, & \text{if}\quad v=0,1.\end{cases}$$

Since the convolution with a radially decreasing L^1-function is bounded in $L^{p(\cdot)}$, we obtain

$$\left\||c\psi_v*\Phi*f|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}}\le 1,\quad v=0,1,$$

for some suitable positive constant c. Applying Lemma 6, we get

$$\left\| |c_1 2^{v\alpha(\cdot)} \psi_v * f|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \leq \int\limits_{2^{-v-2}}^{\min(1,2^{2-v})} \left\| |t^{\alpha(\cdot)} \varphi_t * f|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \frac{dt}{t} + 2^{-v}, \quad v \geq 2,$$

with an appropriate choice of c_1. Taking the sum over $v \geq 2$, we have $\|f\|_{B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \lesssim 1$.

Step 2. We will prove that

$$B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)} \hookrightarrow \mathfrak{B}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}.$$

Let $\{\mathcal{F}\Phi, \mathcal{F}\varphi\}$ and $\{\mathcal{F}\psi_v\}_{v\in\mathbb{N}_0}$ be two resolutions of unity and let $f \in B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ with

$$\|f\|_{B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \leq 1.$$

We have

$$\varphi_t * f = \sum_{v=0}^{\infty} \varphi_t * \psi_v * f =$$

$$= \sum_{v=\lfloor \log_2(\frac{1}{2t}) \rfloor}^{\lfloor \log_2(\frac{4}{t}) \rfloor + 1} \varphi_t * \psi_v * f + \begin{cases} 0, & \text{if} \quad 0 < t \leq \dfrac{1}{4}, \\ \psi_0 * \Phi * f, & \text{if} \quad t > \dfrac{1}{4}, \end{cases}$$

and

$$\Phi * f = \sum_{v=0}^{2} \Phi * \psi_v * f.$$

Notice that if $v < 0$ then we put $\psi_v * f = 0$. Since the convolution with a radially decreasing L^1-function is bounded in $L^{p(\cdot)}$, we obtain

$$\left\| |c\psi_v * \Phi * f|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \leq 1, \quad v = 0, 1, 2,$$

which yields

$$\left\| c|\Phi * f|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \leq 1$$

for some suitable positive constant c. Let $t \in (2^{-i}, 2^{-i+1}]$, $i \in \mathbb{N}$. We have

$$t^{-\alpha(\cdot)} |\varphi_t * f| \lesssim \sum_{v=\lfloor \log_2(\frac{1}{2t}) \rfloor}^{\lfloor \log_2(\frac{4}{t}) \rfloor + 1} t^{-\alpha(\cdot)} \eta_{t,m} * |\psi_v * f| \lesssim$$

$$\lesssim \sum_{v=i-3}^{i-1} 2^{(i-v)\alpha^-} \eta_{v,m-c_{\log}(\alpha)} * 2^{v\alpha(\cdot)} |\psi_v * f| \leq$$

$$\leq c \sum_{j=-3}^{-1} \eta_{j+i,m-c_{\log}(\alpha)} * 2^{(j+i)\alpha(\cdot)} |\psi_{j+i} * f|,$$

where $m > n + c_{\log}(\alpha) + c_{\log}\left(\frac{1}{q}\right)$. Now observe that

$$\int_0^1 \left\| |ct^{-\alpha(\cdot)}\varphi_t * f|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \frac{dt}{t} =$$

$$= \sum_{i=0}^{\infty} \int_{2^{-i}}^{2^{1-i}} \left\| |t^{-\alpha(\cdot)}\varphi_t * f|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \frac{dt}{t} \leq$$

$$\leq \sum_{i=0}^{\infty} \left\| \left(c \sum_{j=-3}^{-1} \eta_{j+i, m - c_{\log}(\alpha)} * 2^{(j+i)\alpha(\cdot)} |\psi_{j+i} * f| \right)^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}}$$

for some suitable positive constant c. The desired estimate follows by Lemma 5.

Theorem 3 is proved.

In order to formulate the main result of this section, let us consider $k_0, k \in \mathcal{S}(\mathbb{R}^n)$ and $S \geq -1$ an integer such that for an $\varepsilon > 0$

$$|\mathcal{F}k_0(\xi)| > 0 \quad \text{for} \quad |\xi| < 2\varepsilon, \tag{12}$$

$$|\mathcal{F}k(\xi)| > 0 \quad \text{for} \quad \frac{\varepsilon}{2} < |\xi| < 2\varepsilon \tag{13}$$

and

$$\int_{\mathbb{R}^n} x^{\alpha} k(x) dx = 0 \quad \text{for any} \quad |\alpha| \leq S. \tag{14}$$

Here, (12) and (13) are Tauberian conditions, while (14) states that moment conditions on k. We recall the notation

$$k_t(x) := t^{-n} k(t^{-1}x) \quad \text{for} \quad t > 0.$$

For any $a > 0$, $f \in \mathcal{S}'(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, we denote

$$k_t^{*,a} t^{-\alpha(\cdot)} f(x) := \sup_{y \in \mathbb{R}^n} \frac{t^{-\alpha(y)} |k_t * f(y)|}{(1 + t^{-1}|x - y|)^a}, \quad j \in \mathbb{N}_0.$$

We are now able to state the so called local mean characterization of $B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ spaces, which is a more general form of Theorem 2.

Theorem 4. *Let* $\alpha, \frac{1}{q} \in C^{\log}_{\text{loc}}(\mathbb{R}^n)$, $p \in \mathcal{P}^{\log}_0(\mathbb{R}^n)$, $a > \frac{n}{p^-}$ *and* $\alpha^+ < S + 1$. *Then*

$$\|f\|'_{B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} := \left\| k_0^{*,a} f \right\|_{p(\cdot)} + \left\| (k_t^{*,a} t^{-\alpha(\cdot)} f)_{0 < t \leq 1} \right\|_{\ell^{q(\cdot)}\widetilde{(L^{p(\cdot)})}}$$

is an equivalent quasinorm on $B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$.

Proof. The idea of the proof is from V. S. Rychkov [17]. The proof is divided into three steps.

Step 1. Let $\varepsilon > 0$. Take any pair of functions φ_0 and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ such that

$$|\mathcal{F}\varphi_0(\xi)| > 0 \qquad \text{for} \quad |\xi| < 2\varepsilon,$$

$$|\mathcal{F}\varphi(\xi)| > 0 \qquad \text{for} \quad \frac{\varepsilon}{2} < |\xi| < 2\varepsilon.$$

We prove that there exists a constant $c > 0$ such that, for any $f \in B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$,

$$\|f\|'_{B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \leq c\big\|\varphi_0^{*,a} f\big\|_{p(\cdot)} + \Big\|\big(\varphi_j^{*,a} 2^{j\alpha(\cdot)} f\big)_{j\geq 1}\Big\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}. \tag{15}$$

Let $\Lambda,\ \lambda \in \mathcal{S}(\mathbb{R}^n)$ such that

$$\operatorname{supp}\mathcal{F}\Lambda \subset \{\xi \in \mathbb{R}^n : |\xi| < 2\varepsilon\}, \qquad \operatorname{supp}\mathcal{F}\lambda \subset \{\xi \in \mathbb{R}^n : \varepsilon/2 < |\xi| < 2\varepsilon\}$$

and

$$\mathcal{F}\Lambda(\xi)\mathcal{F}\varphi_0(\xi) + \sum_{j=1}^{\infty} \mathcal{F}\lambda(2^{-j}\xi)\mathcal{F}\varphi(2^{-j}\xi) = 1, \quad \xi \in \mathbb{R}^n.$$

In particular, for any $f \in B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$, the following identity is true:

$$f = \Lambda * \varphi_0 * f + \sum_{j=1}^{\infty} \lambda_j * \varphi_j * f,$$

where

$$\varphi_j := 2^{jn}\varphi(2^j\cdot) \qquad \text{and} \qquad \lambda_j := 2^{jn}\lambda(2^j\cdot), \quad j \in \mathbb{N}.$$

Hence we can write

$$k_t * f = k_t * \Lambda * \varphi_0 * f + \sum_{j=1}^{\infty} k_t * \lambda_j * \varphi_j * f, \quad t \in (0,1].$$

Let $2^{-i} < t \leq 2^{1-i}$, $i \in \mathbb{N}$. First, let $j < i$. Writing, for any $z \in \mathbb{R}^n$,

$$k_t * \lambda_j(z) = 2^{jn} k_{2^j t} * \lambda(2^j z),$$

we deduce from Lemma 8 that, for any $N > 0$, there exists a constant $c > 0$, independent of t and j, such that

$$|k_t * \lambda_j(z)| \leq c\big(2^j t\big)^{S+1} \eta_{j,N}(z), \quad z \in \mathbb{R}^n.$$

This together with Lemma 1 yield that

$$t^{-\alpha(y)}|k_t * \lambda_j * \varphi_j * f(y)|,$$

can be estimated from above by

$$c2^{(j-i)(S+1-\alpha^+)} \varphi_j^{*,a} 2^{j\alpha(\cdot)} f(y) \int\limits_{\mathbb{R}^n} \eta_{j,N-c_{\log}(\alpha)-a}(y-z)dz \lesssim 2^{(j-i)(S+1-\alpha^+)} \varphi_j^{*,a} 2^{j\alpha(\cdot)} f(y)$$

for any $N > n + a + c_{\log}(\alpha)$, any $y \in \mathbb{R}^n$ and any $j < i$.

Next, let $j \geq i$. Then, again by Lemma 8, we have, for any $z \in \mathbb{R}^n$ and any $L > 0$,

$$|k_t * \lambda_j(z)| = t^{-n}\left|k * \lambda_{\frac{1}{2^j t}}\left(\frac{z}{t}\right)\right| \leq c\left(\frac{1}{2^j t}\right)^{M+1} \eta_{t,L}(z),$$

where an integer $M \geq -1$ is taken arbitrarily large, since $D^\beta \mathcal{F}\lambda(0) = 0$ for all β. Hence, again with Lemma 1,

$$t^{-\alpha(y)}|k_t * \lambda_j * \varphi_j * f(y)| \leq$$

$$\leq t^{-\alpha(y)} \int\limits_{\mathbb{R}^n} |k_t * \lambda_j(y-z)||\varphi_j * f(z)|dz \lesssim$$

$$\lesssim 2^{(i-j)(M+1+\alpha^-)-jn} \varphi_j^{*,a} 2^{j\alpha(\cdot)} f(y) \int\limits_{\mathbb{R}^n} \eta_{j,-c_{\log}(\alpha)-a}(y-z)\eta_{i,L}(y-z)dz.$$

We have, for any $j \geq i$,

$$\big(1 + 2^j|z|\big)^{c_{\log}(\alpha)+a} \leq 2^{(j-i)(c_{\log}(\alpha)+a)}\big(1 + 2^i|z|\big)^{c_{\log}(\alpha)+a}.$$

Then, by taking $L > n + a + c_{\log}(\alpha)$, we get

$$t^{-\alpha(y)}|k_t * \lambda_j * \varphi_j * f(y)| \lesssim 2^{(i-j)(M+1+\alpha^- - c_{\log}(\alpha)-a)} \varphi_j^{*,a} 2^{j\alpha(\cdot)} f(y).$$

Let us take $M > c_{\log}(\alpha) - \alpha^- + 2a$ to estimate the last expression by

$$c2^{(i-j)(a+1)} \varphi_j^{*,a} 2^{j\alpha(\cdot)} f(y),$$

where $c > 0$ is independent of i, j and f. Using the fact that for any $z \in \mathbb{R}^n$ and any $N > 0$

$$|k_t * \Lambda(z)| \leq ct^{S+1}\eta_{1,N}(z),$$

we obtain by the similar arguments that for any $2^{-i} \leq t \leq 2^{-i+1}$, $i \in \mathbb{N}$,

$$\sup_{y\in\mathbb{R}^n} \frac{t^{-\alpha(y)}|k_t * \Lambda * \varphi_0 * f(y)|}{(1 + t^{-1}|x-y|)^a} \leq C2^{-i(S+1-\alpha^+)} \varphi_0^{*,a} f(x).$$

Further, note that, for all $x, y \in \mathbb{R}^n$ all $2^{-i} \leq t \leq 2^{1-i}$, $i \in \mathbb{N}$, and any $j \in \mathbb{N}_0$,

$$\varphi_j^{*,a} 2^{j\alpha(\cdot)} f(y) \leq \varphi_j^{*,a} 2^{j\alpha(\cdot)} f(x)(1 + 2^j|x-y|)^a \leq$$

$$\leq \varphi_j^{*,a} 2^{j\alpha(\cdot)} f(x) \max(1, 2^{(j-i)a})(1 + 2^i|x-y|)^a.$$

Hence

$$\sup_{y\in\mathbb{R}^n} \frac{t^{-\alpha(y)}|k_t * \lambda_j * \varphi_j * f(y)|}{(1 + t^{-1}|x-y|)^a} \leq C\varphi_j^{*,a} 2^{j\alpha(\cdot)} f(x) \begin{cases} 2^{(j-i)(S+1-\alpha^+)}, & \text{if} \quad j < i, \\ 2^{i-j}, & \text{if} \quad j \geq i. \end{cases}$$

Therefore, for all $f \in B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$, any $x \in \mathbb{R}^n$ and any $2^{-i} \le t \le 2^{1-i}$, $i \in \mathbb{N}_0$, we get

$$k_t^{*,a} t^{-\alpha(\cdot)} f(x) \lesssim 2^{-i(S+1-\alpha^+)} \varphi_0^{*,a} f(x) +$$

$$+ C \sum_{j=1}^{\infty} \min\Big(2^{(j-i)(S+1-\alpha^+)}, 2^{i-j}\Big) \varphi_j^{*,a} 2^{j\alpha(\cdot)} f(x) =$$

$$= C \sum_{j=0}^{\infty} \min\Big(2^{(j-i)(S+1-\alpha^+)}, 2^{i-j}\Big) \varphi_j^{*,a} 2^{j\alpha(\cdot)} f(x) =$$

$$= C\Psi_i(x).$$

Assume that the right-hand side of (15) is less than or equal one. Then we have

$$\int_0^1 \Big\| |k_t^{*,a} t^{-\alpha(\cdot)} f|^{q(\cdot)} \Big\|_{\frac{p(\cdot)}{q(\cdot)}} \frac{dt}{t} = \sum_{i=0}^{\infty} \int_{2^{-i}}^{2^{1-i}} \Big\| |k_t^{*,a} t^{-\alpha(\cdot)} f|^{q(\cdot)} \Big\|_{\frac{p(\cdot)}{q(\cdot)}} \frac{dt}{t} \le$$

$$\le \sum_{i=0}^{\infty} \Big\| |c\Psi_i|^{q(\cdot)} \Big\|_{\frac{p(\cdot)}{q(\cdot)}}$$

for some positive constant c. The last term on the right-hand side is less than or equal one if and only if

$$\Big\| (c_1 \Psi_i)_i \Big\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \le 1$$

for some suitable positive constant c_1, which follows by Lemma 8 of [14] and the fact that $\alpha^+ < < S + 1$. Also we have, for any $z \in \mathbb{R}^n$, any $N > 0$ and any integer $M \ge -1$,

$$|k_0 * \lambda_j(z)| \le c 2^{-j(M+1)} \eta_{j,N}(z) \qquad \text{and} \qquad |k_0 * \Lambda(z)| \le c \eta_{1,N}(z).$$

As before, we get, for any $x \in \mathbb{R}^n$,

$$k_0^{*,a} f(x) \le C \varphi_0^{*,a} f(x) + C \sum_{j=1}^{\infty} 2^{-j} \varphi_j^{*,a} 2^{j\alpha(\cdot)} f(x). \tag{16}$$

In (16) taking the $L^{p(\cdot)}$-quasinorm and using the embedding $\ell^{q(\cdot)}(L^{p(\cdot)}) \hookrightarrow \ell^{\infty}(L^{p(\cdot)})$ we get (15).

Step 2. Let $\{\mathcal{F}\varphi_j\}_{j \in \mathbb{N}_0} \subset \mathcal{S}(\mathbb{R}^n)$ be such that

$$\operatorname{supp} \mathcal{F}\varphi \subset \big\{\xi \in \mathbb{R}^n : \varepsilon/2 \le |\xi| \le 2\varepsilon\big\}$$

and

$$\operatorname{supp} \mathcal{F}\varphi_0 \subset \big\{\xi \in \mathbb{R}^n : |\xi| \le 2\varepsilon\big\}, \quad \varepsilon > 0,$$

with $\varphi_j = 2^{jn}\varphi(2^j \cdot)$, $j \in \mathbb{N}$. We will prove that

$$\big\|\varphi_0 * f\big\|_{p(\cdot)} + \Big\| \big(2^{j\alpha(\cdot)}(\varphi_j * f)\big)_{j \ge 1} \Big\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \lesssim \|f\|'_{B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}. \tag{17}$$

Let $\Lambda,\ \lambda \in \mathcal{S}(\mathbb{R}^n)$ such that

$$\operatorname{supp}\mathcal{F}\Lambda \subset \{\xi \in \mathbb{R}^n : |\xi| < 2\varepsilon\}, \qquad \operatorname{supp}\mathcal{F}\lambda \subset \{\xi \in \mathbb{R}^n : \varepsilon/2 < |\xi| < 2\varepsilon\},$$

$$\mathcal{F}\Lambda(\xi)\mathcal{F}k_0(\xi) + \int_0^1 \mathcal{F}\lambda(\tau\xi)\mathcal{F}k(\tau\xi)\frac{d\tau}{\tau} = 1, \quad \xi \in \mathbb{R}^n.$$

In particular, for any $f \in B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$, the following identity is true:

$$f = \Lambda * k_0 * f + \int_0^1 \lambda_\tau * k_\tau * f\frac{d\tau}{\tau}.$$

Hence we can write

$$\varphi_j * f = \int_0^1 \varphi_j * \lambda_\tau * k_\tau * f\frac{d\tau}{\tau} = \int_{2^{-j-2}}^{2^{-j+2}} \varphi_j * \lambda_\tau * k_\tau * f\frac{d\tau}{\tau}, \quad j \geq 2.$$

Using the fact that

$$\max(|k_\tau * \lambda_\tau(z)|, |\varphi_j * \lambda_\tau(z)|) \lesssim \eta_{j,N}(z), \quad z \in \mathbb{R}^n, \qquad 2^{-j-2} \leq \tau \leq 2^{-j+2}, \quad j \in \mathbb{N},$$

and Lemma 1, with $N > 0$ large enough, we easily obtain

$$2^{j\alpha(y)}|\varphi_j * \lambda_\tau * k_\tau * f(y)| \lesssim \min(k^{*,a}_\tau \tau^{-\alpha(\cdot)} f(y), \varphi^{*,a}_j 2^{j\alpha(y)} f(y))$$

for any $y \in \mathbb{R}^n$ and any $2^{-j+2} \leq \tau \leq 2^{-j-2}$, $j \in \mathbb{N}$. Therefore,

$$2^{j\alpha(y)}|\varphi_j * f(y)| \lesssim \left(\varphi^{*,a}_j 2^{j\alpha(\cdot)} f(y)\right)^{1-r} \int_{2^{-j-2}}^{2^{-j+2}} \left(k^{*,a}_\tau \tau^{-\alpha(\cdot)} f(y)\right)^r \frac{d\tau}{\tau}, \quad 0 < r < 1,$$

which yields that

$$\varphi^{*,a}_j 2^{j\alpha(\cdot)} f(x) \lesssim \left(\varphi^{*,a}_j 2^{j\alpha(\cdot)} f(x)\right)^{1-r} \int_{2^{-j-2}}^{2^{-j+2}} \left(k^{*,a}_\tau \tau^{-\alpha(\cdot)} f(x)\right)^r \frac{d\tau}{\tau}.$$

This estimate gives

$$\left(\varphi^{*,a}_j 2^{j\alpha(\cdot)} f(x)\right)^r \lesssim \int_{2^{-j-2}}^{2^{-j+2}} \left(k^{*,a}_\tau \tau^{-\alpha(\cdot)} f(x)\right)^r \frac{d\tau}{\tau}$$

and

$$2^{j\alpha(x)r}|\varphi_j * f(x)|^r \lesssim \int_{2^{-j-2}}^{2^{-j+2}} \left(k^{*,a}_\tau \tau^{-\alpha(\cdot)} f(x)\right)^r \frac{d\tau}{\tau}, \quad x \in \mathbb{R}^n, \tag{18}$$

but if $\varphi_j^{*,a} 2^{j\alpha(\cdot)} f(x) < \infty$. Using a combination of the arguments used in Lemma 7, we get (18) for all $0 < r < 1,\ a > 0$ and all $f \in B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$. Similarly, we obtain

$$|\varphi_j * f(x)|^r \lesssim \left(k_0^{*,a} f(x)\right)^r + \int\limits_{\frac{1}{8}}^{1} \left(k_\tau^{*,a} \tau^{-\alpha(\cdot)} f(x)\right)^r \frac{d\tau}{\tau}, \quad j = 0, 1,$$

for any $0 < r < 1, a > 0$ and any $f \in B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$.

Let $\theta > 0$ be such that $\max\left(1, \dfrac{(1/p)^+}{(1/q)^-}\right) < \theta < \dfrac{q^-}{r}$. Hölder's and Minkowski's inequalities yield

$$\left\| |c 2^{j\alpha(\cdot)} (\varphi_j * f)|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \leq \left(\int\limits_{2^{-j-2}}^{2^{-j+2}} \left\| |k_\tau^{*,a} \tau^{-\alpha(\cdot)} f|^{q(\cdot)} \right\|^{\frac{1}{\theta}}_{\frac{p(\cdot)}{q(\cdot)}} \frac{d\tau}{\tau} \right)^{\theta} \leq$$

$$\leq \int\limits_{2^{-j-2}}^{2^{-j+2}} \left\| |k_\tau^{*,a} \tau^{-\alpha(\cdot)} f|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \frac{d\tau}{\tau}.$$

We get

$$\sum_{j=2}^{\infty} \left\| |c 2^{j\alpha(\cdot)} (\varphi_j * f)|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \leq 1,$$

with an appropriate choice of $c > 0$ such that the left-hand side of (18) it at most one. Similarly, we have

$$\left\| |c \varphi_j * f|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} \leq 1, \quad j = 0, 1.$$

The desired estimate follows by the scaling argument.

Step 3. We will prove that, for all $f \in B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$, the following estimates are true:

$$\|f\|'_{B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \lesssim \|f\|_{B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \lesssim \|f\|'_{B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}. \tag{19}$$

Let $\{\mathcal{F}\varphi_j\}_{j\in\mathbb{N}_0}$ be a resolution of unity. The first inequality follows by the chain of the estimates

$$\|f\|'_{B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \lesssim \left\|\varphi_0^{*,a} f\right\|_{p(\cdot)} + \left\| \left(\varphi_j^{*,a} 2^{j\alpha(\cdot)} f\right)_{j\geq 1} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \lesssim \|f\|_{B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}},$$

where the first inequality is (15), and the second inequality is obvious (see [9]). Now the second inequality in (19) can be obtained by the following chain of the estimates:

$$\|f\|_{B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \lesssim \left\|\varphi_0 * f\right\|_{p(\cdot)} + \left\| \left(2^{j\alpha(\cdot)} (\varphi_j * f)\right)_{j\geq 1} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \lesssim \|f\|'_{B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}},$$

where the first inequality is obvious and the second inequality is (17).

Theorem 4 is proved.

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Received 10.02.21