DOI: 10.37863/umzh.v74i12.6522

UDC 514.763.2+515.164.17

Yu. Bogdanskii (Nat. Techn. Univ. Ukraine "Igor Sikorsky Kyiv Polytechnic Institute"),
V. Shram[1] (Univ. Bonn, Germany)

DIVERGENCE OF MULTIVECTOR FIELDS ON INFINITE-DIMENSIONAL MANIFOLDS

ДИВЕРГЕНЦІЯ БАГАТОВЕКТОРНИХ ПОЛІВ НА НЕСКІНЧЕННОВИМІРНИХ МНОГОВИДАХ

We study the divergence of multivector fields on Banach manifolds with a Radon measure. We propose an infinite-dimensional version of divergence consistent with the classical divergence from finite-dimensional differential geometry. We then transfer certain natural properties of the divergence operator to the infinite-dimensional setting. Finally, we study the relation between the divergence operator div_M on a manifold M and the divergence operator div_S on a submanifold $S \subset M$.

Досліджується дивергенція багатовекторних полів на банахових многовидах із мірою Радона. Запропоновано нескінченновимірну версію дивергенції, яка узгоджується з класичним оператором дивергенції, що розглядається в скінченновимірній диференціальній геометрії. Низку природних властивостей дивергенції перенесено на нескінченновимірний випадок. Крім того, досліджено зв'язок між оператором дивергенції div_M на многовиді M і оператором дивергенції div_S на підмноговиді $S \subset M$.

1. Classical divergence. Let M be an orientable differentiable real n-dimensional manifold of class C^2. A choice of a volume form Ω on M gives rise to the divergence operator, which is defined as follows. For a vector field $\boldsymbol{X}$ (of class C^1), $\operatorname{div} \boldsymbol{X}$ is the function on M such that

$$\operatorname{div} \boldsymbol{X} \cdot \Omega = \mathrm{d}\, i_{\boldsymbol{X}} \Omega, \tag{1}$$

where $i_{\boldsymbol{X}}$ denotes the interior product of a differential form by a vector field $\boldsymbol{X}$ (namely, $i_{\boldsymbol{X}}\omega(\boldsymbol{Z_1}, \ldots, \boldsymbol{Z_{k-1}}) = \omega(\boldsymbol{X}, \boldsymbol{Z_1}, \ldots, \boldsymbol{Z_{k-1}})$).

For a decomposable m-vector field $\overrightarrow{\boldsymbol{X}} = \boldsymbol{X_1} \wedge \ldots \wedge \boldsymbol{X_m}$ and a differential k-form ω, the interior product $i_{\overrightarrow{\boldsymbol{X}}}\omega = i(\overrightarrow{\boldsymbol{X}})\omega$ of ω by $\overrightarrow{\boldsymbol{X}}$ is given by

$$i_{\overrightarrow{\boldsymbol{X}}}\omega := i_{\boldsymbol{X_m}} \ldots i_{\boldsymbol{X_1}}\omega, \quad \text{if} \quad m \le k, \tag{2}$$

and

$$i_{\overrightarrow{\boldsymbol{X}}}\omega := 0, \quad \text{if} \quad m > k.$$

Throughout this paper, by an m-vector field of class C^p we mean a **linear combination of decomposable m-vector fields** $\sum_i c_i \boldsymbol{Z_1^i} \wedge \ldots \wedge \boldsymbol{Z_m^i}$, where all $\boldsymbol{Z_j^i} \in C^p(M)$. That said, one might notice that some of the definitions and results in the article can also be transferred to multivector fields understood in a broader sense.

In an obvious way the above definition of $i_{\overrightarrow{\boldsymbol{X}}}$ extends to an arbitrary multivector field $\overrightarrow{\boldsymbol{X}}$.

This operation satisfies the following property: for any k-vector field $\overrightarrow{\boldsymbol{X}}$, m-vector field $\overrightarrow{\boldsymbol{Z}}$ and differential $(k+m)$-form ω, one has the equality

[1] Corresponding author, e-mail: shram.vladyslav@gmail.com.

$$\langle i_{\overrightarrow{X}}\omega, \overrightarrow{Z}\rangle = \langle \omega, \overrightarrow{X} \wedge \overrightarrow{Z}\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the natural pairing between differential forms and multivector fields of the same degree.

Then the divergence $\operatorname{div}\overrightarrow{X}$ of a k-vector field $\overrightarrow{X}$ is defined by the identity (see, for example, [6] for an equivalent definition in terms of the Hodge operator)

$$i_{\operatorname{div}\overrightarrow{X}}\Omega = (-1)^{k-1}\,\mathrm{d}\, i_{\overrightarrow{X}}\Omega. \tag{3}$$

***Remark* 1.** In principle, we could define the interior product by a multivector field in a different way, namely $i'_{\boldsymbol{X_1}\wedge\ldots\wedge\boldsymbol{X_m}} = i_{\boldsymbol{X_1}} \circ \ldots \circ i_{\boldsymbol{X_m}}$. In this case, Eq. (3) from the definition of divergence becomes $i'_{\operatorname{div}\overrightarrow{X}}\Omega = \mathrm{d}\, i'_{\overrightarrow{X}}\Omega$. However, in this paper, we always use the definition of interior product $i_{\overrightarrow{X}}$ given by (2).

The existence of $\operatorname{div}\overrightarrow{X}$ for a multivector field $\overrightarrow{X}$ will follow from Proposition 1, and the uniqueness follows from general facts of multilinear algebra (see, for example, [5], Chapter III).

Let M be a manifold of class C^3. Given a $(k+1)$-vector field $\overrightarrow{X}$ of class C^2 and a differential k-form ω of class C^2_0 (that is, $\omega \in C^2(M)$ and is boundedly supported) on M, Stokes' theorem implies $\int_M \mathrm{d}(\omega \wedge i_{\overrightarrow{X}}\Omega) = 0$, which can be written as

$$\int_M \mathrm{d}\,\omega \wedge i_{\overrightarrow{X}}\Omega = (-1)^{k+1}\int_M \omega \wedge \mathrm{d}\, i_{\overrightarrow{X}}\Omega. \tag{4}$$

Lemma 1. *Let ω and $\overrightarrow{X}$ be a differential k-form and a k-vector field on M, respectively. Then the following equality holds*:

$$\omega \wedge i_{\overrightarrow{X}}\Omega = \langle \omega, \overrightarrow{X}\rangle\Omega. \tag{5}$$

Proof. Without loss of generality we may assume that $\overrightarrow{X}$ is decomposable: $\overrightarrow{X} = \boldsymbol{X_1}\wedge\ldots\wedge\boldsymbol{X_k}$. We have

$$\omega \wedge i_{\overrightarrow{X}}\Omega = \omega \wedge (i_{\boldsymbol{X_k}}\ldots i_{\boldsymbol{X_1}}\Omega) = (-1)^{k-1}(i_{\boldsymbol{X_k}}\omega) \wedge (i_{\boldsymbol{X_{k-1}}}\ldots i_{\boldsymbol{X_1}}\Omega) = \ldots$$

$$\ldots = (-1)^{\frac{(k-1)k}{2}}(i_{\boldsymbol{X_1}}\ldots i_{\boldsymbol{X_k}}\omega) \wedge \Omega = (i_{\boldsymbol{X_k}}\ldots i_{\boldsymbol{X_1}}\omega) \wedge \Omega = \langle \omega, \overrightarrow{X}\rangle\Omega.$$

Let μ be a measure on M induced by the volume form Ω (for $f \in C^1(M)$, one has $\int_M f\,d\mu = \int_M f\Omega$). Given a differential k-form ω of class C^2_0 and a $(k+1)$-vector field $\overrightarrow{X}$ of class C^2, by (4) and (5), we get

$$\int_M \langle \mathrm{d}\,\omega, \overrightarrow{X}\rangle d\mu = \int_M \mathrm{d}\,\omega \wedge i_{\overrightarrow{X}}\Omega = (-1)^{k+1}\int_M \omega \wedge \mathrm{d}\, i_{\overrightarrow{X}}\Omega = -\int_M \omega \wedge i_{\operatorname{div}\overrightarrow{X}}\Omega = -\int_M \langle \omega, \operatorname{div}\overrightarrow{X}\rangle d\mu.$$

Thus, (4) is equivalent to

$$\int_M \langle \mathrm{d}\,\omega, \overrightarrow{X}\rangle d\mu = -\int_M \langle \omega, \operatorname{div}\overrightarrow{X}\rangle d\mu. \tag{6}$$

Using the measure μ, one can now view the divergence of a $(k+1)$-vector field $\overrightarrow{\boldsymbol{X}}$ on M as a k-vector field which satisfies (6) for any differential k-form of class C_0^2. For a manifold of class C^3, this leads to a definition of $\operatorname{div}\overrightarrow{\boldsymbol{X}}$ which is equivalent to the original one.

Proposition 1. *Let $\boldsymbol{X}$ and $\overrightarrow{\boldsymbol{Z}}$ be a vector field and a k-vector field of class C^1 on M, respectively. Then one has the formula*

$$\operatorname{div}(\boldsymbol{X}\wedge\overrightarrow{\boldsymbol{Z}}) = \operatorname{div}\boldsymbol{X}\cdot\overrightarrow{\boldsymbol{Z}} - \boldsymbol{X}\wedge\operatorname{div}\overrightarrow{\boldsymbol{Z}} + \mathcal{L}_{\boldsymbol{X}}\overrightarrow{\boldsymbol{Z}}, \tag{7}$$

where $\mathcal{L}_{\boldsymbol{X}}$ denotes the Lie derivation along the field $\boldsymbol{X}$.

Proof. It suffices to prove formula (7) only for a decomposable multivector field $\overrightarrow{\boldsymbol{Z}} = \boldsymbol{Z_1}\wedge\ldots$ $\ldots\wedge\boldsymbol{Z_k}$. We have

$$(-1)^k\,\mathrm{d}\,i_{\boldsymbol{X}\wedge\overrightarrow{\boldsymbol{Z}}}\Omega = \mathrm{d}\,i_{\overrightarrow{\boldsymbol{Z}}\wedge\boldsymbol{X}}\Omega = \mathrm{d}\,i_{\boldsymbol{X}}(i_{\overrightarrow{\boldsymbol{Z}}}\Omega) = -i_{\boldsymbol{X}}\,\mathrm{d}(i_{\overrightarrow{\boldsymbol{Z}}}\Omega) + \mathcal{L}_{\boldsymbol{X}}(i_{\overrightarrow{\boldsymbol{Z}}}\Omega).$$

For the first term on the right-hand side we get

$$-i_{\boldsymbol{X}}\,\mathrm{d}(i_{\overrightarrow{\boldsymbol{Z}}}\Omega) = -(-1)^{k-1}i_{\boldsymbol{X}}i_{\operatorname{div}\overrightarrow{\boldsymbol{Z}}}\Omega = -(-1)^{k-1}i_{\operatorname{div}\overrightarrow{\boldsymbol{Z}}\wedge\boldsymbol{X}}\Omega = -i_{\boldsymbol{X}\wedge\operatorname{div}\overrightarrow{\boldsymbol{Z}}}\Omega.$$

For the second term

$$\mathcal{L}_{\boldsymbol{X}}(i_{\overrightarrow{\boldsymbol{Z}}}\Omega) = \mathcal{L}_{\boldsymbol{X}}(i_{\boldsymbol{Z_k}}\ldots i_{\boldsymbol{Z_1}}\Omega) = i_{\boldsymbol{Z_k}}\mathcal{L}_{\boldsymbol{X}}(i_{\boldsymbol{Z_{k-1}}}\ldots i_{\boldsymbol{Z_1}}\Omega) + i_{\mathcal{L}_{\boldsymbol{X}}\boldsymbol{Z_k}}(i_{\boldsymbol{Z_{k-1}}}\ldots i_{\boldsymbol{Z_1}}\Omega) = \ldots$$

$$\ldots = i_{\boldsymbol{Z_k}}\ldots i_{\boldsymbol{Z_1}}\mathcal{L}_{\boldsymbol{X}}\Omega + \sum_{r=1}^{k} i_{\boldsymbol{Z_k}}\ldots i_{\mathcal{L}_{\boldsymbol{X}}\boldsymbol{Z_r}}\ldots i_{\boldsymbol{Z_1}}\Omega = i_{\overrightarrow{\boldsymbol{Z}}}\,\mathrm{d}\,i_{\boldsymbol{X}}\Omega + \sum_{r=1}^{k} i_{\boldsymbol{Z_1}\wedge\ldots\wedge\mathcal{L}_{\boldsymbol{X}}\boldsymbol{Z_r}\wedge\ldots\wedge\boldsymbol{Z_k}}\Omega =$$

$$= i_{\overrightarrow{\boldsymbol{Z}}}\operatorname{div}\boldsymbol{X}\cdot\Omega + i_{\mathcal{L}_{\boldsymbol{X}}\overrightarrow{\boldsymbol{Z}}}\Omega = i_{\operatorname{div}\boldsymbol{X}\cdot\overrightarrow{\boldsymbol{Z}}}\Omega + i_{\mathcal{L}_{\boldsymbol{X}}\overrightarrow{\boldsymbol{Z}}}\Omega = i_{\operatorname{div}\boldsymbol{X}\cdot\overrightarrow{\boldsymbol{Z}}+\mathcal{L}_{\boldsymbol{X}}\overrightarrow{\boldsymbol{Z}}}\Omega.$$

Putting the two terms together, we obtain the identity (7).

***Corollary* 1.** *The divergence of a k-vector field $($of class $C^p)$ exists and is a $(k-1)$-vector field $($of class $C^{p-1})$.*

Proof. The statement immediately follows from formula (7).

Given a differential k-form ω and a decomposable m-vector field $\overrightarrow{\boldsymbol{X}} = \boldsymbol{X_1}\wedge\ldots\wedge\boldsymbol{X_m}$, one defines the *interior product* $j_\omega\overrightarrow{\boldsymbol{X}} = j(\omega)\overrightarrow{\boldsymbol{X}}$ of $\overrightarrow{\boldsymbol{X}}$ by ω as follows:

$$j_\omega\overrightarrow{\boldsymbol{X}} := \frac{1}{k!(m-k)!}\sum_{\sigma\in S_m}\operatorname{sign}(\sigma)\omega(\boldsymbol{X_{\sigma(1)}},\ldots,\boldsymbol{X_{\sigma(k)}})\boldsymbol{X_{\sigma(k+1)}}\wedge\ldots\wedge\boldsymbol{X_{\sigma(m)}}, \quad \text{if} \quad k\le m,$$

and

$$j_\omega\overrightarrow{\boldsymbol{X}} := 0, \quad \text{if} \quad k>m.$$

In an obvious way this definition then extends to an arbitrary multivector field $\overrightarrow{\boldsymbol{X}}$. For a similar definition, see, for example, [12].

The interior product of a multivector field by a differential form satisfies the following property: for any differential k-form ω, differential m-form η and $(k+m)$-vector field $\overrightarrow{\boldsymbol{X}}$, one has

$$\langle\eta, j_\omega\overrightarrow{\boldsymbol{X}}\rangle = \langle\omega\wedge\eta, \overrightarrow{\boldsymbol{X}}\rangle. \tag{8}$$

One can prove the following generalisation of Lemma 1 (see [6]): for any differential k-form ω and m-vector field $\overrightarrow{\boldsymbol{X}}$, the following relation holds:

$$i_{j(\omega)\overrightarrow{\boldsymbol{X}}}\Omega = (-1)^{k(m+1)}\omega\wedge i_{\overrightarrow{\boldsymbol{X}}}\Omega. \tag{9}$$

Proposition 2. *Let ω and $\overrightarrow{\boldsymbol{X}}$ be a differential k-form and an m-vector field $(k < m)$, respectively. Then the Leibniz rule holds*

$$\operatorname{div}(j(\omega)\overrightarrow{\boldsymbol{X}}) = (-1)^k j(\mathrm{d}\,\omega)\overrightarrow{\boldsymbol{X}} + (-1)^k j(\omega)\operatorname{div}\overrightarrow{\boldsymbol{X}}.$$

Proof. Using (9), we have

$$(-1)^{m-k-1}\,\mathrm{d}\,i_{j(\omega)\overrightarrow{\boldsymbol{X}}}\Omega = (-1)^{m-k-1+k(m+1)}\,\mathrm{d}\,\omega\wedge i_{\overrightarrow{\boldsymbol{X}}}\Omega + (-1)^{m-k-1+k(m+1)+k}\omega\wedge\mathrm{d}\,i_{\overrightarrow{\boldsymbol{X}}}\Omega =$$

$$= (-1)^{km+m-1}\,\mathrm{d}\,\omega\wedge i_{\overrightarrow{\boldsymbol{X}}}\Omega + (-1)^{km+k}\omega\wedge\mathrm{d}\,i_{\operatorname{div}\overrightarrow{\boldsymbol{X}}}\Omega =$$

$$= (-1)^{km+m-1+(k+1)(m+1)} i_{j(\mathrm{d}\,\omega)\overrightarrow{\boldsymbol{X}}}\Omega + (-1)^{km+k+km} i_{j(\omega)\operatorname{div}\overrightarrow{\boldsymbol{X}}}\Omega =$$

$$= (-1)^k i_{j(\mathrm{d}\,\omega)\overrightarrow{\boldsymbol{X}}}\Omega + (-1)^k i_{j(\omega)\operatorname{div}\overrightarrow{\boldsymbol{X}}}\Omega.$$

2. Associated measures on Banach manifolds (see [1, 3]). Let M be a connected Hausdorff real Banach manifold of class C^2 with a model space E.

We say that an atlas $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$ on M is *bounded* if there exists a real number $K > 0$ such that, for any pair of charts $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β), the transition map $F_{\beta\alpha} = \varphi_\beta \circ \varphi_\alpha^{-1}$ satisfies the condition

$$\big(x \in \varphi_\alpha(U_\alpha \cap U_\beta)\big) \implies \big(\|F'_{\beta\alpha}(x)\| \le K, \|F''_{\beta\alpha}(x)\| \le K\big).$$

We then say that two bounded atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ are *equivalent* if $\mathcal{A}_1 \cup \mathcal{A}_2$ is again a bounded atlas. A *bounded structure* (of class C^2) on M is defined as an equivalence class of bounded atlases on M.

Let $(M_1, \mathcal{A}_1)$ and $(M_2, \mathcal{A}_2)$ be Banach manifolds M_1 and M_2 of class C^2 modelled on E_1 and E_2 together with bounded atlases $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively. We say that a map $f\colon M_1 \to M_2$ is a *bounded morphism* if there exists a real number $C > 0$ such that for any pair of charts $(U, \varphi) \in \mathcal{A}_1$ and $(V, \psi) \in \mathcal{A}_2$, the following condition is satisfied:

$$(p \in U,\ f(p) \in V) \implies \Big(\big\|(\psi\circ f\circ\varphi^{-1})^{(k)}(\varphi(p))\big\| \le C \text{ for } k = 1, 2\Big).$$

In a natural way one then defines a *bounded isomorphism* between $(M_1, \mathcal{A}_1)$ and $(M_2, \mathcal{A}_2)$.

The property of being a bounded morphism does not depend on the choice of representatives of the corresponding equivalence classes of bounded atlases on M_1 and M_2.

A choice of a bounded atlas on M leads to a well-defined notion of the length $L(\Gamma)$ of a piecewise-smooth curve Γ in M. The corresponding intrinsic metric ρ is consistent with the original topology. A bounded morphism $f\colon (M_1, \mathcal{A}_1) \to (M_2, \mathcal{A}_2)$ is Lipschitz with respect to the corresponding intrinsic metrics.

A choice of a bounded atlas also allows to introduce a norm $|||\cdot_p|||$ on the tangent space T_pM to the manifold M, defined by $|||\xi_p||| := \sup_\alpha \|\xi_{\varphi_\alpha}\|$, where $\{(U_\alpha, \varphi_\alpha)\}$ is the set of charts of the original atlas for which $p \in U_\alpha$, and $\xi_\varphi \in E$ is the representation of a tangent vector ξ in a chart φ. Furthermore, one has the property of *uniform topological isomorphism* of the spaces T_pM and the model space E, namely $\|\xi_\varphi\| \le |||\xi_p||| \le K\|\xi_\varphi\|$, where K is the constant from the definition of a bounded atlas, and φ is a chart at the point $p \in M$.

***Remark* 2.** One can prove that a bounded structure on a manifold is a special case of a Finsler structure (in this case the assignment $\langle p, \xi\rangle \mapsto |||\xi_p|||$ is a continuous function on the tangent bundle TM). However, in order to get the result of Theorem 2 below, it appears that further restrictions on the Finsler structure are needed.

By a differential k-form on M of class C^1 we mean a C^1-section of the bundle $L^k_{\text{alt}}(TM) \to M$, where $L^k_{\text{alt}}(TM)$ is obtained by bundling together the spaces $L^k_{\text{alt}}(T_pM)$ of all bounded alternating k-linear forms on T_pM, so that the space $L^k_{\text{alt}}(T_pM)$ is the fibre at $p \in M$ of this bundle.

On a manifold with a bounded atlas $(M, \mathcal{A})$ one has a well-defined notion of a *bounded* vector field $\boldsymbol{X}$ of class C^1. Namely, $\boldsymbol{X}$ is said to be of class $C^1_b(M)$ if there exists a real number $C > 0$ such that for any chart (U, φ), the local representation $\boldsymbol{X}_\varphi$ of $\boldsymbol{X}$ satisfies $\big\|\boldsymbol{X}_\varphi(\varphi(x))\big\| \leq C$ and $\big\|\boldsymbol{X}'_\varphi(\varphi(x))\big\| \leq C$ for all $x \in U$. Boundedness of a vector field does not depend on the choice of a bounded atlas from the corresponding equivalence class. In the same way one defines differential forms of class $C^1_b(M)$. Finally, in a similar fashion we can also define smooth functions of class C^p_b, $p = 0, 1, 2,\ C_b = C^0_b$. We will use this same notation also in the case when the domain of a field, differential form or a function is a connected open subset V in M, in E or in a surface in M. A vector field (resp., differential form) of class $C^1_b(V)$ is said to be of class $C^1_0(V)$ if its support is bounded and contained in V together with its ε-neighbourhood for some $\varepsilon > 0$.

We say that a bounded atlas $\mathcal{A}$ is *uniform* if there exists a real number $r > 0$ such that for any $p \in M$, there is a chart $(U, \varphi) \in \mathcal{A}$ such that $\varphi(U)$ contains a ball of radius r in E centred at $\varphi(p)$ [1, 7, 11].

An intrinsic metric on M, induced by a uniform atlas, makes M into a complete metric space. Furthermore, if a bounded atlas is equivalent to a uniform one, then the metric induced by this atlas is also complete. If an equivalence class of atlases which defines a bounded structure on M contains a uniform atlas, we call such a structure *uniform*. If manifolds M_1 and M_2 are boundedly isomorphic, then their structures are either both uniform or nonuniform.

The flow $\Phi(t, x)$ of a vector field $\boldsymbol{X}$ of class C^1_b on a manifold M with a uniform structure is defined on $\mathbb{R} \times M$ [11, p. 92].

If V is an open subset of $\mathbb{R}^m$, then, given a manifold with a bounded atlas $(M, \mathcal{A})$, we agree to define a bounded structure on $M \times V$ (with a model space $E \oplus \mathbb{R}^m$) by the atlas $\mathcal{A} \times \text{id} = \big\{(U \times V, \varphi \times \text{id}) \colon (U, \varphi) \in \mathcal{A}\big\}$.

An *elementary surface* $S \subset M$ of codimension m is defined as follows. Let N be a manifold with a bounded structure modelled on a subspace E_1 of E of codimension m (from now on we identify E with $E_1 \oplus \mathbb{R}^m$). Let V be an open neighbourhood of $\overrightarrow{0} \in \mathbb{R}^m$, and $g \colon N \times V \to U \subset M$ be a bounded (straightening) isomorphism onto an open subset U in M. Then, by definition, an elementary surface is $S = g(N \times \{\overrightarrow{0}\})$.

For $\varepsilon > 0$, we define

$$S_{-\varepsilon} := S \cap \big\{x \colon \rho(x, M \setminus U) \geq \varepsilon\big\}.$$

Then $S = \bigcup\limits_{n=1}^{\infty} S_{-\frac{1}{n}}$.

We say that a differential m-form ω of class C^1_b defined on U is an *associated m-form of the embedding* $S \subset M$ if for any $x \in S$, the tangent space T_xS is an associated subspace of the exterior form $\omega(x)$ in T_xM (i.e., $T_xS = \{Y \in T_xM \colon i_Y\omega(x) = 0\}$, where i_Y is the interior product of an exterior form by a vector Y).

If $g \colon N \times V \to U$ is a straightening isomorphism of an elementary surface S, P is a projection of $N \times V$ onto V, and h is a continuously differentiable function on V such that $h(\overrightarrow{0}) \neq 0$, then $\omega = (g^{-1})^* P^*(h\, dt_1 \wedge \ldots \wedge dt_m)$ is an example of an associated m-form of the embedding $S \subset M$. Note that the constructed m-form ω is closed.

Let us now consider a Borel measure μ on M. The associated measure $\sigma = \sigma_{\overrightarrow{\boldsymbol{Y}}}$ on S is constructed as follows.

We first consider a strictly transversal to S system $\overrightarrow{\boldsymbol{Y}} = \{\boldsymbol{Y_1}, \ldots, \boldsymbol{Y_m}\}$ of pairwise commuting vector fields of class C_b^1 defined on U. Strict transversality of $\overrightarrow{\boldsymbol{Y}}$ is understood in the following sense: for each $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x \in S_{-\varepsilon}$, one has $|\omega(\overrightarrow{\boldsymbol{Y}})(x)| = |\omega(\boldsymbol{Y_1}, \ldots, \boldsymbol{Y_m})(x)| \geq \delta$. Existence of such a system of fields was proved in [3].

Let $\Phi_t^{\boldsymbol{Y_k}}$ denote the flow of $\boldsymbol{Y_k}$. We then define $\Phi_{\vec{t}}^{\overrightarrow{\boldsymbol{Y}}} := \Phi_{t_1}^{\boldsymbol{Y_1}} \ldots \Phi_{t_m}^{\boldsymbol{Y_m}}$. One has the property $\Phi_{\vec{t}+\vec{s}}^{\overrightarrow{\boldsymbol{Y}}} = \Phi_{\vec{t}}^{\overrightarrow{\boldsymbol{Y}}} \Phi_{\vec{s}}^{\overrightarrow{\boldsymbol{Y}}}$.

For Borel sets $W \in \mathcal{B}(\mathbb{R}^m)$ and $A \in \mathcal{B}(M)$, the set $\Phi_W A = \Phi_W^{\overrightarrow{\boldsymbol{Y}}} A := \{\Phi_{\vec{t}}^{\overrightarrow{\boldsymbol{Y}}}(x) \colon \vec{t} \in W, x \in A\}$ is a Borel in M. Furthermore, for each $\varepsilon > 0$, there exists $p > 0$ such that $(A \in \mathcal{B}(S_{-\varepsilon}), W \in \mathcal{B}(B_p)) \implies \big(\Phi_W^{\overrightarrow{\boldsymbol{Y}}} A \in \mathcal{B}(U)\big)$, where $B_p = \{\vec{t} : \|\vec{t}\| < p\} \subset \mathbb{R}^m$. For any set $B \in \mathcal{B}(B_p)$, we define a measure ν_B on $\mathcal{B}(S_{-\varepsilon})$ by $\nu_B(A) := \mu(\Phi_B^{\overrightarrow{\boldsymbol{Y}}} A)$.

Let λ_m denote the Lebesgue measure on $\mathbb{R}^m$. If, for any $A \in \mathcal{B}(S_{-\varepsilon})$, the following limit exists:

$$\sigma(A) = \sigma_{\overrightarrow{\boldsymbol{Y}}}(A) = \lim_{r \to 0} \frac{\nu_{B_r}(A)}{\lambda_m(B_r)}, \tag{10}$$

then Nikodym's theorem implies that the map $\mathcal{B}(S_{-\varepsilon}) \ni A \mapsto \sigma_{\overrightarrow{\boldsymbol{Y}}}(A) \in \mathbb{R}$ is a Borel measure on $S_{-\varepsilon}$. Writing $A \in \mathcal{B}(S)$ in the form $A = \bigcup\limits_{n=1}^{\infty} (A \cap S_{-\frac{1}{n}})$ allows to extend the measure $\sigma_{\overrightarrow{\boldsymbol{Y}}}$ to $\mathcal{B}(S)$.

Sufficient conditions for existence of the limit (10) were established in [3]; the authors suggested to call $\sigma_{\overrightarrow{\boldsymbol{Y}}}$ the *surface measure* on S of the first kind induced by the system of vector fields $\overrightarrow{\boldsymbol{Y}}$.

Throughout the remainder of this paper we always assume that the surface measure exists.

Given $\varepsilon > 0$ and $r > 0$, let σ_r denote the measure on $\mathcal{B}(S_{-\varepsilon})$ defined by

$$\sigma_r(A) := \frac{1}{\lambda_m(B_r)} \mu(\Phi_{B_r} A).$$

Then (10) implies that $\sigma_r(A) \to \sigma(A)$ as $r \to 0$ for any Borel set $A \subset S_{-\varepsilon}$.

The following two lemmas were proved in [2].

Lemma 2. *Suppose that μ is a Radon measure on M. Then for any $\varepsilon > 0$, σ_r and σ are Radon measures on $S_{-\varepsilon}$.*

Lemma 3. *Suppose that μ is a (nonnegative) Radon measure on M, and $u \in C_b(M)$. Then, for any $\varepsilon > 0$ and $A \in \mathcal{B}(S_{-\varepsilon})$, the following equality holds:*

$$\lim_{r \to 0} \frac{1}{\lambda_m(B_r)} \int\limits_{\Phi_{B_r} A} u \, d\mu = \int\limits_A u \, d\sigma.$$

3. Multivector fields and divergence operator. The notion of the divergence of a vector field (as given by formula (1)) was generalized by Daletskii and Maryanin [8] to a certain class of Banach manifolds, resulting in the so-called *divergence with respect to a measure*. In that work the divergence of a vector field $\boldsymbol{X}$ with respect to a measure μ was defined as the logarithmic derivative of μ along the vector field $\boldsymbol{X}$.

In this section, we propose a definition of divergence of multivector fields on a Banach manifold, which generalizes the finite-dimensional divergence as given by formula (3). We then establish some of the properties which this new divergence operator satisfies.

Consider a Banach manifold M with a bounded structure and a (nonnegative) Borel measure μ on M. We say that a k-vector field $\overrightarrow{\boldsymbol{Z}}$ on M is *μ-measurable* if there exists a sequence of continuous

k-vector fields $\vec{Z}_{\boldsymbol{n}}$ such that $\lim_{n\to\infty}\left|\left|\left|\vec{Z}_{\boldsymbol{n}}(p)-\vec{Z}(p)_p\right|\right|\right|=0\ (\mathrm{mod}\mu)$ (here $|||\cdot_p|||$ is the norm on $\bigwedge^k(T_pM)$ induced by the corresponding norm $|||\cdot_p|||$ on T_pM, see Section 2).

For a measurable multivector field $\vec{Z}$, the function $x\mapsto\left|\left|\left|\vec{Z}(x)_x\right|\right|\right|$ is μ-measurable on M. In the case, when this function is integrable on M with respect to μ, we say that $\vec{Z}$ is *integrable*: $\vec{Z}\in L_1(\mu)$ (see [4]). In a similar way one defines multivector fields of class $L_p(\mu)$ for $1<p\le\infty$.

It is easy to check that if vector fields $\boldsymbol{Z_2},\ldots,\boldsymbol{Z_k}$ are measurable and bounded on M, and $\boldsymbol{Z_1}$ is a vector field of class $L_p(\mu)$, then $\boldsymbol{Z_1}\wedge\ldots\wedge\boldsymbol{Z_k}\in L_p(\mu)$. One can also prove that if $\boldsymbol{Z_1}\wedge\ldots\wedge\boldsymbol{Z_k}\in L_p(\mu)$, and ω is a differential form of class $C_b(M)$, then $\langle\omega,\boldsymbol{Z_1}\wedge\ldots\wedge\boldsymbol{Z_k}\rangle\in L_p(\mu)$.

Let $L_p\bigwedge^k(\mu)$ denote the set of all linear combinations of decomposable k-vector fields of class $L_p(\mu)$ (modulo the measure μ).

Definition 1. *Let $\vec{Z}\in L_1\bigwedge^k(\mu)$. We call a $(k-1)$-vector field $\overrightarrow{W}\in L_1\bigwedge^{k-1}(\mu)$ a divergence of $\vec{Z}$ $\big(\overrightarrow{W}=\operatorname{div}\vec{Z};\vec{Z}\in D(\operatorname{div})\big)$ if for any differential $(k-1)$-form $\omega\in C_0^1(M)$, the following equality holds*:

$$\int\limits_M\langle\omega,\overrightarrow{W}\rangle\,d\mu=-\int\limits_M\langle\mathrm{d}\,\omega,\vec{Z}\rangle\,d\mu. \tag{11}$$

Uniqueness of the divergence is provided by the following theorem, which was proved in [2].

Theorem 1. *Suppose that there exists a function of class C^1 on E with nonempty bounded support* (*it suffices to assume that E is reflexive*, *see* [10]), *and μ is a Radon measure. Then*, *given a k-vector field $\vec{Z}\in L_1\bigwedge^k(\mu)$, there cannot exist two distinct elements of $L_1\bigwedge^{k-1}(\mu)$, both of which are divergences of $\vec{Z}$.*

***Remark* 3.** Unlike in the finite-dimensional case, divergence need not exist in general. Thus, one encounters the problem of describing, for a given measure, the class of (multi-)vector fields admitting the divergence.

From now on we always assume that the assumptions of Theorem 1 are satisfied. Let us now prove the infinite-dimensional analogues of Propositions 1 and 2.

***Remark* 4.** Throughout this paper, by a k-vector field of class $C_b^1(M)$ we mean a **linear combination of decomposable k-vector fields** $\sum_i c_i\boldsymbol{Z}^{\boldsymbol{i}}_{\boldsymbol{1}}\wedge\ldots\wedge\boldsymbol{Z}^{\boldsymbol{i}}_{\boldsymbol{k}}$, where all $\boldsymbol{Z}^{\boldsymbol{i}}_{\boldsymbol{j}}\in C_b^1(M)$.

Proposition 3. *Suppose that a vector field $\boldsymbol{X}$ and a k-vector field $\vec{Z}$ lie in $C_b^1(M)\cap D(\operatorname{div})$. Then $\boldsymbol{X}\wedge\vec{Z}\in C_b^1(M)\cap D(\operatorname{div})$ and the following identity holds*:

$$\operatorname{div}(\boldsymbol{X}\wedge\vec{Z})=\operatorname{div}\boldsymbol{X}\cdot\vec{Z}-\boldsymbol{X}\wedge\operatorname{div}\vec{Z}+\mathcal{L}_{\boldsymbol{X}}\vec{Z}. \tag{12}$$

Proof. Let ω be a differential k-form of class C_0^1 on M. One has the equality

$$\langle\mathrm{d}\,\omega,\boldsymbol{X}\wedge\vec{Z}\rangle=\langle i_{\boldsymbol{X}}\,\mathrm{d}\,\omega,\vec{Z}\rangle=\boldsymbol{X}\langle\omega,\vec{Z}\rangle-\langle\mathrm{d}\,i_{\boldsymbol{X}}\omega,\vec{Z}\rangle-\langle\omega,\mathcal{L}_{\boldsymbol{X}}\vec{Z}\rangle. \tag{13}$$

Now, by combining (11) and (13), we get

$$\int\limits_M\langle\mathrm{d}\,\omega,\boldsymbol{X}\wedge\vec{Z}\rangle\,d\mu=-\int\limits_M\langle\omega,-\operatorname{div}\boldsymbol{X}\cdot\vec{Z}+\boldsymbol{X}\wedge\operatorname{div}\vec{Z}-\mathcal{L}_{\boldsymbol{X}}\vec{Z}\rangle\,d\mu,$$

which proves the proposition.

***Corollary* 2.** *If $\vec{Z}=\boldsymbol{Z_1}\wedge\ldots\wedge\boldsymbol{Z_k}$, and all $\boldsymbol{Z_i}\in C_b^1(M)\cap D(\operatorname{div})$, then $\vec{Z}\in C_b^1(M)\cap D(\operatorname{div})$.*

Proposition 4. *Suppose that an* m*-vector field* $\vec{\boldsymbol{Z}} \in D(\mathrm{div})$, *and let* ω *be a differential* k*-form* ($k < m$) *of class* $C_b^1(M)$. *Then* $j(\omega)\vec{\boldsymbol{Z}} \in D(\mathrm{div})$, *and the following Leibniz rule holds*:

$$\mathrm{div}(j(\omega)\vec{\boldsymbol{Z}}) = (-1)^k j(\mathrm{d}\,\omega)\vec{\boldsymbol{Z}} + (-1)^k j(\omega)\,\mathrm{div}\,\vec{\boldsymbol{Z}}.$$

Proof. For any differential $(m-k-1)$-form η of class $C_0^1(M)$, using identities (8) and (11), we have

$$\int\limits_M \Big(\Big\langle \mathrm{d}\,\eta, j(\omega)\vec{\boldsymbol{Z}}\Big\rangle + \Big\langle \eta, (-1)^k j(\mathrm{d}\,\omega)\vec{\boldsymbol{Z}} + (-1)^k j(\omega)\,\mathrm{div}\,\vec{\boldsymbol{Z}}\Big\rangle\Big) d\mu =$$

$$= \int\limits_M \Big(\langle \omega\wedge \mathrm{d}\,\eta, \vec{\boldsymbol{Z}}\rangle + (-1)^k\langle \mathrm{d}\,\omega\wedge\eta, \vec{\boldsymbol{Z}}\rangle + (-1)^k\langle\omega\wedge\eta, \mathrm{div}\,\vec{\boldsymbol{Z}}\rangle\Big) d\mu =$$

$$= \int\limits_M \Big((-1)^k\langle \mathrm{d}(\omega\wedge\eta), \vec{\boldsymbol{Z}}\rangle + (-1)^k\langle\omega\wedge\eta, \mathrm{div}\,\vec{\boldsymbol{Z}}\rangle\Big) d\mu = 0.$$

4. Divergence on submanifolds. If M is a finite-dimensional (orientable) manifold endowed with a volume form Ω, and U is its open submanifold, then it is natural to take $\Omega\big|_U$ to be the volume form on U. In this case one has the equality

$$\mathrm{div}_U(\vec{\boldsymbol{Z}}\big|_U) = (\mathrm{div}\,\vec{\boldsymbol{Z}})\big|_U, \tag{14}$$

where div_U is the divergence on U induced by the volume form $\Omega\big|_U$.

In the case, when U is an open submanifold of a Banach manifold M, the definition of divergence div_U of a multivector field is obtained from Definition 1 by replacing (11) with

$$\int\limits_U \langle\omega, \vec{\boldsymbol{W}}\rangle\, d\mu = -\int\limits_U \langle \mathrm{d}\,\omega, \vec{\boldsymbol{Z}}\rangle\, d\mu,$$

which now has to hold for any differential form of class $C_0^1(U)$. In this case formula (14) also holds.

Let now M be an orientable manifold of finite dimension n; $S\subset M$ an orientable embedded submanifold of dimension $m = n-p$, which is an elementary surface in the sense of Section 2; α an associated differential p-form of the embedding $S\subset M$; $\vec{\boldsymbol{Y}} = \{\boldsymbol{Y_1}, \ldots, \boldsymbol{Y_p}\}$ a commuting strictly transversal to S system of vector fields of class $C_b^1(U)$, where U is from the definition of an elementary surface.

For any $\varepsilon > 0$, there exists $\gamma = \gamma(\varepsilon) > 0$ such that for each $(\vec{t}, x) \in B_\gamma \times S_{-\varepsilon}$, one has $\Phi_{\vec{t}}x \in U$, and $\langle\alpha, \vec{\boldsymbol{Y}}\rangle(\Phi_{\vec{t}}x) \neq 0$ (here $B_\gamma = \{\vec{t}\in\mathbb{R}^p : \|\vec{t}\| < \gamma\}$).

Without loss of generality we may assume that $\langle\alpha, \vec{\boldsymbol{Y}}\rangle(\Phi_{\vec{t}}x) > 0$. One has that the map $q : \Phi_{B\gamma}S_{-\varepsilon} \ni \Phi_{\vec{t}}x \mapsto x \in S_{-\varepsilon}$ is continuously differentiable.

Let $\Omega = \Omega_S$ be a volume form on S; $\boldsymbol{X}$ a vector field on S; $\widetilde{\boldsymbol{X}}$ the vector field on $\Phi_{B_\gamma}S_{-\varepsilon}$ which is q-related to $\boldsymbol{X}$ ($q_*(\widetilde{\boldsymbol{X}}(\Phi_{\vec{t}}x)) = \boldsymbol{X}(x)$); $\widetilde{\Omega} = q^*\Omega$ a differential p-form on $\Phi_{B_\gamma}S_{-\varepsilon}$.

Suppose that $\vec{\boldsymbol{X}} = \boldsymbol{X_1}\wedge\ldots\wedge\boldsymbol{X_m}$ is a nowhere-vanishing multivector field on $S_{-\varepsilon}$, and let $\beta = \widetilde{\Omega}\wedge\alpha$. Then, for $x\in S_{-\varepsilon}$,

$$\langle\beta, \widetilde{\vec{\boldsymbol{X}}}\wedge\vec{\boldsymbol{Y}}\rangle(x) = \widetilde{\Omega}(\widetilde{\vec{\boldsymbol{X}}})(x)\cdot\alpha(\vec{\boldsymbol{Y}})(x) = (\Omega(\vec{\boldsymbol{X}})\cdot\alpha(\vec{\boldsymbol{Y}}))(x) > 0$$

(here we used $(i_{\boldsymbol{X_j}}\alpha)(x) = 0$). Choosing a smaller $\gamma > 0$ if needed, we conclude that β is a volume form on $\Phi_{B_\gamma}S_{-\varepsilon}\subset M$.

Proposition 5. *Let $\boldsymbol{Z}$ be a vector field of class C_b^1 on S, and let $\operatorname{div}_S \boldsymbol{Z}$ be the divergence of $\boldsymbol{Z}$ with respect to the volume form Ω on S. Given $\varepsilon > 0$, let $\widetilde{\boldsymbol{Z}}$ be the vector field on $\Phi_{B_\gamma} S_{-\varepsilon}$ which is q-related to $\boldsymbol{Z}$, and let $\operatorname{div} \widetilde{\boldsymbol{Z}}$ be the divergence of $\widetilde{\boldsymbol{Z}}$ with respect to the volume form β. Suppose that α is closed. Then*

$$\operatorname{div}_S \boldsymbol{Z} = (\operatorname{div} \widetilde{\boldsymbol{Z}})\big|_S. \tag{15}$$

Proof. Take $x \in S_{-\varepsilon}$. The statement follows from the equalities

$$(\operatorname{div} \widetilde{\boldsymbol{Z}} \cdot \beta)(x) = (\mathrm{d}\, i_{\widetilde{\boldsymbol{Z}}}(\widetilde{\Omega} \wedge \alpha))(x) = (\mathrm{d}\, i_{\boldsymbol{Z}} \Omega)(x) \wedge \alpha(x) = (\operatorname{div}_S \boldsymbol{Z} \cdot \beta)(x).$$

***Corollary* 3.** *In the assumptions of Proposition 5, suppose that $\vec{\boldsymbol{Z}}$ is a multivector field of class C_b^1 on S; $\widetilde{\vec{\boldsymbol{Z}}}$ is the multivector field on $V = \Phi_{B_\gamma} S_{-\varepsilon}$ which is q-related to $\vec{\boldsymbol{Z}}$; div_S and div are the divergence operators on (S, Ω) and (V, β), respectively. Then*

$$\operatorname{div}_S \vec{\boldsymbol{Z}} = (\operatorname{div} \widetilde{\vec{\boldsymbol{Z}}})\big|_S. \tag{16}$$

Proof. Formula (16) follows by induction from formula (15); recurrent formula (7), applied to $\operatorname{div}_S(\boldsymbol{X} \wedge \vec{\boldsymbol{Z}})$ and $\operatorname{div}(\widetilde{\boldsymbol{X}} \wedge \widetilde{\vec{\boldsymbol{Z}}})$; equalities $\widetilde{\boldsymbol{X} \wedge \vec{\boldsymbol{Z}}} = \widetilde{\boldsymbol{X}} \wedge \widetilde{\vec{\boldsymbol{Z}}}$ and $\widetilde{\mathcal{L}_{\boldsymbol{X}} \vec{\boldsymbol{Z}}} = \mathcal{L}_{\widetilde{\boldsymbol{X}}} \widetilde{\vec{\boldsymbol{Z}}}$.

Throughout the remainder of this article, M is a Banach manifold with a uniform atlas, modelled on a space E, where E satisfies the assumptions of Theorem 1. Suppose that S is an elementary surface in M of codimension m; μ is a (nonnegative) Radon measure on M, and the corresponding measure $\sigma = \sigma_{\vec{\boldsymbol{Y}}}$ on the surface $S_{-\varepsilon} \subset S$ is constructed as described in Section 2.

It follows from general theory of differential equations in Banach spaces that there exists $\gamma = \gamma(\varepsilon) > 0$ for which one has a well-defined map $q : \Phi_{B_\gamma} S_{-\varepsilon} \ni \Phi_{\vec{t}} x \mapsto x \in S_{-\varepsilon}$ of class C_b^1. Let $\boldsymbol{Z}$ be a vector field of class C_b^1 on S. Then the q-related vector field $\widetilde{\boldsymbol{Z}}$ is defined on $V = \Phi_{B_\gamma} S_{-\varepsilon}$ and is also of class C_b^1.

Theorem 2. *Suppose that $\widetilde{\boldsymbol{Z}}$ admits the divergence $\operatorname{div} \widetilde{\boldsymbol{Z}} \in L_\infty(V, \mu)$. Then $\boldsymbol{Z}$ admits the divergence $\operatorname{div}_S \boldsymbol{Z} \in L_\infty(S, \sigma)$, and for any $\varepsilon > 0$ and a bounded Borel function $u : S_{-\varepsilon} \to \mathbb{R}$, we have the identity*

$$\int\limits_{S_{-\varepsilon}} u \operatorname{div}_S \boldsymbol{Z}\, d\sigma = \lim_{r \to 0} \frac{1}{\lambda_m(B_r)} \int\limits_{\Phi_{B_r} S_{-\varepsilon}} \widehat{u} \operatorname{div} \widetilde{\boldsymbol{Z}}\, d\mu \tag{17}$$

(*here and henceforth* $\widehat{u}(\Phi_{\vec{t}} x) = u(x)$ *for* $(\vec{t}, x) \in B_\gamma \times S_{-\varepsilon}$).

***Proof. Step* 1.** Let $u \in C_0^1(S)$. Then $u \in C_0^1(S_{-\varepsilon})$ for some $\varepsilon > 0$. We shall prove that, for any $r \in (0, \gamma)$, the following holds:

$$\int\limits_{\Phi_{B_r} S_{-\varepsilon}} \widehat{u} \operatorname{div} \widetilde{\boldsymbol{Z}}\, d\mu = -\int\limits_{\Phi_{B_r} S_{-\varepsilon}} \widetilde{\boldsymbol{Z}} \widehat{u}\, d\mu. \tag{18}$$

The function $\widehat{u}$ is not of class $C_0^1(V)$. We will use the fact that $\widetilde{\boldsymbol{Z}}$ is tangent to the surface $\Phi_{\vec{t}} S_{-\varepsilon}$ for each $\vec{t} \in B_\gamma$.

Let us define a sequence of functions $\varphi_n \in C[0, r]$ for $n > 3$ as follows:

$$\varphi_n(s) = \begin{cases} 0, & \text{if} \quad s \in \left[0, \dfrac{n-3}{n}r\right] \cup \left[\dfrac{n-1}{n}r, r\right], \\ -\dfrac{n^2}{r^2}s + \dfrac{n(n-3)}{r}, & \text{if} \quad s \in \left[\dfrac{n-3}{n}r, \dfrac{n-2}{n}r\right], \\ \dfrac{n^2}{r^2}s - \dfrac{n(n-1)}{r}, & \text{if} \quad s \in \left[\dfrac{n-2}{n}r, \dfrac{n-1}{n}r\right]. \end{cases}$$

Then for the sequence of functions $h_n(s) = 1 + \int_0^s \varphi_n(s)\, ds$, one has that the functions $u_n(\Phi_{\vec{t}}x) = h_n\big(\|\vec{t}\,\|\big) \cdot u(x)$ coincide with $\widehat{u}\big(\Phi_{\vec{t}}x\big)$ for $\|\vec{t}\,\| \le \dfrac{n-3}{n}r$, and $u_n \in C_0^1(\Phi_{B_r}S_\varepsilon)$.

Hence, we have

$$\int\limits_{\Phi_{B_r}S_{-\varepsilon}} u_n \operatorname{div} \widetilde{\boldsymbol{Z}}\, d\mu = - \int\limits_{\Phi_{B_r}S_{-\varepsilon}} \widetilde{\boldsymbol{Z}} u_n\, d\mu \tag{19}$$

and

$$(\widetilde{\boldsymbol{Z}}u_n)(\Phi_{\vec{t}}x) = h_n(\|\vec{t}\,\|) \cdot (\widetilde{\boldsymbol{Z}}\widehat{u})(\Phi_{\vec{t}}x) \text{ for } x \in S_{-\varepsilon}.$$

Passing in (19) to the limit as $n \to \infty$, we obtain (18).

Since the function $\widetilde{\boldsymbol{Z}}\widehat{u} \in C_b(\Phi_{B_\gamma}S_{-\varepsilon})$, Lemma 3 implies the existence of the limit

$$\lim_{r \to 0} \frac{1}{\lambda_m(B_r)} \int\limits_{\Phi_{B_r}S_{-\varepsilon}} \widetilde{\boldsymbol{Z}}\widehat{u}\, d\mu = \int\limits_{S_{-\varepsilon}} \boldsymbol{Z}u\, d\sigma.$$

Therefore, using (18), we obtain the equality

$$\lim_{r \to 0} \frac{1}{\lambda_m(B_r)} \int\limits_{\Phi_{B_r}S_{-\varepsilon}} \widehat{u} \operatorname{div} \widetilde{\boldsymbol{Z}}\, d\mu = - \int\limits_{S_{-\varepsilon}} \boldsymbol{Z}u\, d\sigma, \tag{20}$$

that holds for any function $u \in C_0^1(S_{-\varepsilon})$.

***Step* 2.** The model space E_1 of the manifold S has a finite codimension in E and therefore also admits a function of class $C^1(E_1)$ with bounded nonempty support. The argument used in the proof of Theorem 1 also proves that there exists a family of functions $\{u_\alpha\}$ of class $C_0^1(S_{-\varepsilon})$ such that the sets $U_\alpha = \{x : u_\alpha(x) > 0\}$ constitute a base of the topology of $S_{-\varepsilon}$.

For any choice of $u \in \{u_\alpha\}$, let $U = \{x : u(x) > 0\}$ be the corresponding set of this base. Taking a sequence of smooth functions $h_n \in C^1(\mathbb{R})$ that approximate the Heaviside step function χ, we construct a sequence of functions $v_n = h_n \circ u$ for which $\{x : v_n(x) > 0\} = U$; $v_n \nearrow \mathbb{1}_U = \chi \circ u$ and $V_n = \{x : v_n(x) = 1\} \nearrow U$ (where $\mathbb{1}_U$ denotes the indicator function of U and the notation $V_n \nearrow U$ means that for any $n \in \mathbb{N}$, $V_n \subset V_{n+1}$ and $\bigcup_{n\in\mathbb{N}} V_n = U$).

Nikodym's theorem implies the uniform in $r \in (0, \gamma)$ convergence

$$\sigma_r(U \setminus V_n) = \frac{1}{\lambda_m(B_r)} \mu(\Phi_{B_r}(U \setminus V_n)) \to 0, \quad n \to \infty.$$

Since $\operatorname{div} \widetilde{\boldsymbol{Z}} \in L_\infty(\mu)$, one also has the uniform in $r \in (0, \gamma)$ convergence

$$\frac{1}{\lambda_m(B_r)} \int\limits_{\Phi_{B_r}S_{-\varepsilon}} \left| (\widehat{v_n} - \widehat{\mathbb{1}_U}) \operatorname{div} \widetilde{\boldsymbol{Z}} \right| d\mu \to 0, \quad n \to \infty.$$

This uniform convergence and the convergence (20), together with the inequality

$$\left|\frac{1}{\lambda_m(B_r)}\int\limits_{\Phi_{B_r}U}\operatorname{div}\widetilde{\boldsymbol{Z}}d\mu-\frac{1}{\lambda_m(B_s)}\int\limits_{\Phi_{B_s}U}\operatorname{div}\widetilde{\boldsymbol{Z}}d\mu\right|\leq$$

$$\leq\frac{1}{\lambda_m(B_r)}\int\limits_{\Phi_{B_r}S_{-\varepsilon}}\left|(\widehat{v_n}-\widehat{\mathbb{1}_U})\operatorname{div}\widetilde{\boldsymbol{Z}}\right|d\mu+$$

$$+\frac{1}{\lambda_m(B_s)}\int\limits_{\Phi_{B_s}S_{-\varepsilon}}\left|(\widehat{v_n}-\widehat{\mathbb{1}_U})\operatorname{div}\widetilde{\boldsymbol{Z}}\right|d\mu+$$

$$+\left|\frac{1}{\lambda_m(B_r)}\int\limits_{\Phi_{B_r}S_{-\varepsilon}}\widehat{v_n}\cdot\operatorname{div}\widetilde{\boldsymbol{Z}}\,d\mu-\frac{1}{\lambda_m(B_s)}\int\limits_{\Phi_{B_s}S_{-\varepsilon}}\widehat{v_n}\cdot\operatorname{div}\widetilde{\boldsymbol{Z}}d\mu\right|$$

allow us to conclude that the following limit exists:

$$\lim_{r\to 0}\frac{1}{\lambda_m(B_r)}\int\limits_{\Phi_{B_r}U}\operatorname{div}\widetilde{\boldsymbol{Z}}\,d\mu. \tag{21}$$

***Step* 3.** Let K be a compact subset of $S_{-\varepsilon}$. Then there is a sequence of sets $U_n\in\{U_\alpha\}$ such that $U_n\searrow K$ (i.e., for any $n\in\mathbb{N}$, $U_n\supset U_{n+1}$ and $\bigcap_{n\in\mathbb{N}}U_n=K$).

Again, using Nikodym's theorem and the fact that $\operatorname{div}\widetilde{\boldsymbol{Z}}\in L_\infty(\mu)$, we obtain uniform in $r\in\in(0,\gamma)$ convergence

$$\lim_{n\to\infty}\frac{1}{\lambda_m(B_r)}\int\limits_{\Phi_{B_r}(U_n\setminus K)}\left|\operatorname{div}\widetilde{\boldsymbol{Z}}\right|d\mu=0.$$

From this uniform convergence and the convergence (21), together with the next inequality (here $r,s\in(0,\gamma)$)

$$\left|\frac{1}{\lambda_m(B_r)}\int\limits_{\Phi_{B_r}K}\operatorname{div}\widetilde{\boldsymbol{Z}}d\mu-\frac{1}{\lambda_m(B_s)}\int\limits_{\Phi_{B_s}K}\operatorname{div}\widetilde{\boldsymbol{Z}}d\mu\right|\leq$$

$$\leq\frac{1}{\lambda_m(B_r)}\int\limits_{\Phi_{B_r}(U_n\setminus K)}\left|\operatorname{div}\widetilde{\boldsymbol{Z}}\right|d\mu+\frac{1}{\lambda_m(B_s)}\int\limits_{\Phi_{B_s}(U_n\setminus K)}\left|\operatorname{div}\widetilde{\boldsymbol{Z}}\right|d\mu+$$

$$+\left|\frac{1}{\lambda_m(B_r)}\int\limits_{\Phi_{B_r}U_n}\operatorname{div}\widetilde{\boldsymbol{Z}}d\mu-\frac{1}{\lambda_m(B_s)}\int\limits_{\Phi_{B_s}U_n}\operatorname{div}\widetilde{\boldsymbol{Z}}d\mu\right|,$$

we conclude that the following limit exists:

$$\lim_{r\to 0}\frac{1}{\lambda_m(B_r)}\int\limits_{\Phi_{B_r}K}\operatorname{div}\widetilde{\boldsymbol{Z}}\,d\mu. \tag{22}$$

***Step* 4.** Let A be an arbitrary Borel subset of $S_{-\varepsilon}$. Let K_n be a non decreasing sequence of compact subsets of A satisfying $\sigma(A \setminus K_n) < \dfrac{1}{n}$. Then, for $C = \bigcap_{n=1}^{\infty}(A \setminus K_n)$, one has $\sigma(C) = 0$, and therefore

$$\lim_{r\to 0} \frac{1}{\lambda_m(B_r)} \int\limits_{\Phi_{B_r}C} \left|\operatorname{div} \widetilde{\boldsymbol{Z}}\right| d\mu = 0. \tag{23}$$

Analogously to Step 3, we first obtain a uniform in $r \in (0, \gamma)$ convergence

$$\lim_{n\to\infty} \frac{1}{\lambda_m(B_r)} \int\limits_{\Phi_{B_r}((A\setminus C)\setminus K_n)} \left|\operatorname{div} \widetilde{\boldsymbol{Z}}\right| d\mu = 0,$$

and then use (23) and the existence of the limit (22) in order to conclude that the following limit exists:

$$\lim_{r\to 0} \frac{1}{\lambda_m(B_r)} \int\limits_{\Phi_{B_r}A} \operatorname{div} \widetilde{\boldsymbol{Z}}\, d\mu. \tag{24}$$

Let now τ_r denote the measure on $\mathcal{B}(S_{-\varepsilon})$ defined by

$$\tau_r(A) := \frac{1}{\lambda_m(B_r)} \int\limits_{\Phi_{B_r}A} \operatorname{div} \widetilde{\boldsymbol{Z}}\, d\mu.$$

Existence of the limit (24) means that for any Borel set $A \in \mathcal{B}(S_{-\varepsilon})$, there exists a limit $\lim_{r\to 0} \tau_r(A) =: \tau(A)$. Since $\operatorname{div} \widetilde{\boldsymbol{Z}} \in L_\infty(\mu)$, the measure τ is absolutely continuous with respect to σ, and, additionally, $g_\varepsilon = \dfrac{d\tau}{d\sigma} \in L_\infty(S_{-\varepsilon}, \sigma)$, and

$$\|g_\varepsilon\|_{L_\infty(\sigma)} \le \|\operatorname{div} \widetilde{\boldsymbol{Z}}\|_{L_\infty(\mu)}. \tag{25}$$

For any bounded Borel function u on $S_{-\varepsilon}$, one has

$$\lim_{r\to 0} \frac{1}{\lambda_m(B_r)} \int\limits_{\Phi_{B_r}S_{-\varepsilon}} \widehat{u} \operatorname{div} \widetilde{\boldsymbol{Z}}\, d\mu = \lim_{r\to 0} \int\limits_{S_{-\varepsilon}} u d\tau_r = \int\limits_{S_{-\varepsilon}} u \cdot g_\varepsilon d\sigma. \tag{26}$$

Since (26) holds for any bounded Borel function on $S_{-\varepsilon}$, it follows that $g_{\varepsilon_1} = g_{\varepsilon_2}\big|_{S_{-\varepsilon_1}}$ for $\varepsilon_2 \in (0, \varepsilon_1)$ and, hence, there exists a Borel function g, defined on the whole of S, such that $g_\varepsilon = g\big|_{S_{-\varepsilon}}$ for any $\varepsilon > 0$; moreover, by (25), $g \in L_\infty(S, \sigma)$.

In particular, by (20), for any function $u \in C_0^1(S)$, one has

$$-\int\limits_S \boldsymbol{Z} u d\sigma = \int\limits_S u \cdot g d\sigma.$$

Therefore, there exists $\operatorname{div}_S \boldsymbol{Z} = g$ on S; $\operatorname{div}_S \boldsymbol{Z} \in L_\infty(\sigma)$, and for any bounded Borel function u, defined on $S_{-\varepsilon}$ for some $\varepsilon > 0$, equality (17) holds.

Theorem 2 is proved.

***Remark* 5.** Analogously to Lemma 3, one can prove that

$$\int\limits_{S_{-\varepsilon}} u \operatorname{div}_S \boldsymbol{Z}\, d\sigma = \lim_{r\to 0} \frac{1}{\lambda_m(B_r)} \int\limits_{\Phi_{B_r} S_{-\varepsilon}} u \operatorname{div} \widetilde{\boldsymbol{Z}}\, d\mu$$

for any function $u \in C_b(M)$.

For a differential k-form α of class C_b^1 on S, we define $\widehat{\alpha} := q^*\alpha$. For each $\varepsilon > 0$, the form $\widehat{\alpha}$ is defined on $\Phi_{B_{\gamma(\varepsilon)}} S_{-\varepsilon}$.

***Corollary* 4.** *Let* $\overrightarrow{\boldsymbol{Z}} = \boldsymbol{Z_1} \wedge \ldots \wedge \boldsymbol{Z_{k+1}}$ *be a decomposable multivector field of class* C_b^1 *on* S. *Given* $\varepsilon > 0$, *let* $\widetilde{\overrightarrow{\boldsymbol{Z}}} = \widetilde{\boldsymbol{Z_1}} \wedge \ldots \wedge \widetilde{\boldsymbol{Z_{k+1}}}$ *be the* q*-related multivector field on* $\Phi_{B_\gamma} S_{-\varepsilon}$, *and suppose that*, *for each* $i \in \{1, \ldots, k+1\}$, *there exists* $\operatorname{div} \widetilde{\boldsymbol{Z_i}} \in L_\infty(\mu)$. *Then* $\overrightarrow{\boldsymbol{Z}} \in D(\operatorname{div}_S)$ *and* $\operatorname{div}_S \boldsymbol{Z_i} \in L_\infty(\sigma)$ *for each* $i \in \{1, \ldots, k+1\}$. *Moreover*, *for any* $\varepsilon > 0$ *and differential* k*-form* α *of class* $C_0^1(S)$, *the following equality holds*:

$$\int\limits_{S_{-\varepsilon}} \langle \alpha, \operatorname{div}_S \overrightarrow{\boldsymbol{Z}} \rangle\, d\sigma = \lim_{r\to 0} \frac{1}{\lambda_m(B_r)} \int\limits_{\Phi_{B_r} S_{-\varepsilon}} \langle \widehat{\alpha}, \operatorname{div} \widetilde{\overrightarrow{\boldsymbol{Z}}} \rangle\, d\mu.$$

Proof. Induction on k. Theorem 2 constitutes the basis of the induction. The induction step is based on formula (12).

Let $\overrightarrow{\boldsymbol{Z}} = \boldsymbol{X} \wedge \overrightarrow{\boldsymbol{Y}}$, where $\overrightarrow{\boldsymbol{Y}}$ is a k-vector fiel‘ $\widetilde{\overrightarrow{\boldsymbol{Z}}} = \widetilde{\boldsymbol{X}} \wedge \widetilde{\overrightarrow{\boldsymbol{Y}}}$ and $\langle \widehat{\alpha}, \operatorname{div} \widetilde{\overrightarrow{\boldsymbol{Z}}} \rangle = \operatorname{div} \widetilde{\boldsymbol{X}} \cdot \langle \widehat{\alpha}, \widetilde{\overrightarrow{\boldsymbol{Y}}} \rangle - \langle i_{\widetilde{\boldsymbol{X}}} \widehat{\alpha}, \operatorname{div} \widetilde{\overrightarrow{\boldsymbol{Y}}} \rangle + \langle \widehat{\alpha}, \mathcal{L}_{\widetilde{\boldsymbol{X}}} \widetilde{\overrightarrow{\boldsymbol{Y}}} \rangle$.

Since $\langle \widehat{\alpha}, \widetilde{\overrightarrow{\boldsymbol{Y}}} \rangle = \widehat{\langle \alpha, \overrightarrow{\boldsymbol{Y}} \rangle}$, Theorem 2 implies that

$$\int\limits_{S_{-\varepsilon}} \operatorname{div}_S \boldsymbol{X} \cdot \langle \alpha, \overrightarrow{\boldsymbol{Y}} \rangle\, d\sigma = \lim_{r\to 0} \frac{1}{\lambda_m(B_r)} \int\limits_{\Phi_{B_r} S_{-\varepsilon}} \operatorname{div} \widetilde{\boldsymbol{X}} \cdot \langle \widehat{\alpha}, \widetilde{\overrightarrow{\boldsymbol{Y}}} \rangle\, d\mu.$$

Since one has $i_{\widetilde{\boldsymbol{X}}} \widehat{\alpha} = \widehat{i_{\boldsymbol{X}} \alpha}$, the equality

$$\int\limits_{S_{-\varepsilon}} \langle i_{\boldsymbol{X}} \alpha, \operatorname{div}_S \overrightarrow{\boldsymbol{Y}} \rangle\, d\sigma = \lim_{r\to 0} \frac{1}{\lambda_m(B_r)} \int\limits_{\Phi_{B_r} S_{-\varepsilon}} \langle i_{\widetilde{\boldsymbol{X}}} \widehat{\alpha}, \operatorname{div} \widetilde{\overrightarrow{\boldsymbol{Y}}} \rangle\, d\mu$$

follows from the induction hypothesis.

We have $\langle \widehat{\alpha}, \mathcal{L}_{\widetilde{\boldsymbol{X}}} \widetilde{\overrightarrow{\boldsymbol{Y}}} \rangle = \widehat{u}$, where $u = \langle \alpha, \mathcal{L}_{\boldsymbol{X}} \overrightarrow{\boldsymbol{Y}} \rangle$ is a function of class $C_b(S_{-\varepsilon})$, and therefore the identity

$$\int\limits_{S_{-\varepsilon}} \langle \alpha, \mathcal{L}_{\boldsymbol{X}} \overrightarrow{\boldsymbol{Y}} \rangle\, d\sigma = \lim_{r\to 0} \frac{1}{\lambda_m(B_r)} \int\limits_{\Phi_{B_r} S_{-\varepsilon}} \langle \widehat{\alpha}, \mathcal{L}_{\widetilde{\boldsymbol{X}}} \widetilde{\overrightarrow{\boldsymbol{Y}}} \rangle\, d\mu$$

is a direct consequence of Lemma 3.

Applying now formula (12) to $\operatorname{div}_S(\boldsymbol{X} \wedge \overrightarrow{\boldsymbol{Y}})$, we obtain the statement of the corollary.

References

1. Yu. Bogdanskii, *Banach manifolds with bounded structure and Gauss – Ostrogradskii formula*, Ukr. Math. J., **64**, № 10, 1299 – 1313 (2012).
2. Yu. Bogdanskii, *Stokes' formula on Banach manifolds*, Ukr. Math. J., **72**, № 11, 1455 – 1468 (2020).

3. Yu. Bogdanskii, E. Moravetskaya, *Surface measures on Banach manifolds with uniform structure*, Ukr. Math. J., **69**, № 8, 1030 – 1048 (2017).
4. Yu. Bogdanskii, A. Potapenko, *Laplacian with respect to a measure on a Riemannian manifold and Dirichlet problem. I*, Ukr. Math. J., **68**, № 7, 897 – 907 (2016).
5. N. Bourbaki, *Éléments de mathématique. Algèbre*, chap. I-III, Hermann (1970); chap. IV-VII, Masson (1981); chap. VIII-X, Springer (2007, 2012).
6. N. Broojerdian, E. Peyghan, A. Heydari, *Differentiation along multivector fields*, Iran. J. Math. Sci. and Inform., **6**, № 1, 79 – 96 (2011); DOI 10.7508/ijmsi.2011.01.007.
7. Yu. Daletskii, Ya. Belopolskaya, *Stochastic equations and differential geometry*, Vyshcha Shkola, Kyiv (1989).
8. Yu. Daletskii, B. Maryanin, *Smooth measures on infinite-dimensional manifolds*, Dokl. Akad. Nauk SSSR, **285**, № 6, 1297 – 1300 (1985).
9. D. H. Fremlin, *Measurable functions and almost continuous functions*, Manuscripta Math., **33**, № 3-4, 387 – 405 (1981); DOI 10.1007/BF01798235.
10. R. Fry, S. McManus, *Smooth bump functions and the geometry of Banach spaces: a brief survey*, Expo. Math., **20**, № 2, 143 – 183 (2002); DOI 10.1016/S0723-0869(02)80017-2.
11. S. Lang, *Fundamentals of differential geometry*, Springer-Verlag, New York etc. (1999).
12. C.-M. Marle, *Schouten – Nijenhuis bracket and interior products*, J. Geom. and Phys., **23**, № 3-4, 350 – 359 (1997); DOI 10.1016/S0393-0440(97)80009-5.

Received 12.01.21