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SOME COEFFICIENT BOUNDS ASSOCIATED WITH TRANSFORMS OF BOUNDED TURNING FUNCTIONS

ДЕЯКІ ОЦІНКИ КОЕФІЦІЄНТІВ, ПОВ'ЯЗАНІ З ПЕРЕТВОРЕННЯМИ ОБМЕЖЕНИХ ПОВОРОТНИХ ФУНКЦІЙ

We present the derivation of an upper bound for the Hankel determinants of certain orders linked with the $k\mathrm{th}$-root transform $[f(z^k)]^{\frac{1}{k}}$ of the holomorphic mapping $f(z)$ whose derivative has a positive real part with normalization, namely, $f(0) = 0$ and $f'(0) = 1$.

Отримано верхню межу для визначників Ганкеля певних порядків, що пов'язані з k-м кореневим перетворенням $[f(z^k)]^{\frac{1}{k}}$ голоморфного відображення $f(z)$, похідна якого має додатну дійсну частину з нормалізацією, а саме $f(0) = 0$ і $f'(0) = 1$.

1. Preliminaries. Let $\mathcal{A}$ represent group of mappings f of the type

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

in $\mathcal{U}_d = \{z \in \mathcal{C} \colon |z| < 1\}$, denotes the open unit disc and S is the subfamily of $\mathcal{A}$, possessing univalent (schlicht) mappings. The $k\mathrm{th}$-root transform for the mapping f in (1.1) is

$$G(z) := \left[f(z^k)\right]^{\frac{1}{k}} = z + \sum_{n=1}^{\infty} b_{kn+1} z^{kn+1}. \tag{1.2}$$

At this sequel, we have introduced and interpreted the concept of Hankel determinant for $G(z)$ for f (1.1) with $q, t, k \in \mathbb{N} = \{1, 2, 3, \ldots\}$ as

$$H_{q,k}(t) = \begin{vmatrix} b_{k(t-1)+1} & b_{kt+1} & \cdots & b_{k(t+q-2)+1} \\ b_{kt+1} & b_{k(t+1)+1} & \cdots & b_{k(t+q-1)+1} \\ \cdots & \cdots & \cdots & \cdots \\ b_{k(t+q-2)+1} & b_{k(t+q-1)+1} & \cdots & b_{k[t+2(q-1)-1]+1} \end{vmatrix}, \quad b_1 = 1. \tag{1.3}$$

In particular, if $k = 1$ in (1.3) it reduces to the Hankel determinant $H_{q,k}(t) = H_q(t)$, given by Pommerenke [10], investigated by several authors. In specific for $q = 2$, $t \in \{1, 2, 3\}$ and $q = 3$, $t = 1$, the Hankel determinant (1.3) has been simplified respectively to

$$H_{2,k}(1) = \begin{vmatrix} b_1 & b_{k+1} \\ b_{k+1} & b_{2k+1} \end{vmatrix} = b_{2k+1} - b_{k+1}^2, \tag{1.4}$$

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$$H_{2,k}(2) = \begin{vmatrix} b_{k+1} & b_{2k+1} \\ b_{2k+1} & b_{3k+1} \end{vmatrix} = b_{k+1}b_{3k+1} - b_{2k+1}^2, \tag{1.5}$$

$$H_{2,k}(3) = \begin{vmatrix} b_{2k+1} & b_{3k+1} \\ b_{3k+1} & b_{4k+1} \end{vmatrix} = b_{2k+1}b_{4k+1} - b_{3k+1}^2, \tag{1.6}$$

and

$$H_{3,k}(1) = \begin{vmatrix} b_1 & b_{k+1} & b_{2k+1} \\ b_{k+1} & b_{2k+1} & b_{3k+1} \\ b_{2k+1} & b_{3k+1} & b_{4k+1} \end{vmatrix}. \tag{1.7}$$

Expanding the determinant in $H_{3,k}(1)$, we have

$$H_{3,k}(1) = \big[b_{4k+1}(b_{2k+1} - b_{k+1}^2) + b_{3k+1}(b_{k+1}b_{2k+1} - b_{3k+1}) +$$

$$+ b_{2k+1}(b_{k+1}b_{3k+1} - b_{2k+1}^2)\big]. \tag{1.8}$$

Ali et al. [1] derived exact estimates for $|b_{2k+1} - \mu b_{k+1}^2|$, represents the generalized Fekete–Szegö functional related to the function $G(z)$, when f is a member of specific subfamilies of S. We mention $H_{2,k}(2)$, $H_{2,k}(3)$ and $H_{3,k}(1)$ respectively the 2nd and 3rd order Hankel determinants for the function f given in (1.2).

In Section 2, motivated by the results obtained by earlier authors, we estimate an upper bound to $H_{2,k}(3)$ and $H_{3,k}(1)$ for the family of bounded turning functions, denoted by $\Re$, defined below.

Definition 1.1. *The mapping* f (1.1) *belongs to* $\Re$, *if*

$$\mathrm{Re} f'(z) > 0, \quad z \in \mathcal{U}_d.$$

A. E. Livingston [7] established the subfamily $\Re$ of S, and later MacGregor [8] carried a consistent study about the properties of functions belongs to this class.

In deriving our results, the required sharp estimates specified below, given in the form of lemmas, which holds suitable for functions possessing positive real part.

The collection $\mathcal{P}$ of all functions g, each one called as Carathéodory function [4] of the form

$$g(z) = 1 + \sum_{t=1}^{\infty} c_t z^t, \tag{1.9}$$

holomorphic in $\mathcal{U}_d$ and $\mathrm{Re}\, g(z) > 0$ for $z \in \mathcal{U}_d$.

Lemma 1.1 [5]. *If* $g \in \mathcal{P}$, *then* $|c_i - \mu c_j c_{i-j}| \leq 2$ *satisfies for the values* $i, j \in \mathbb{N}$ *with* $i > j$ *and* $\mu \in [0, 1]$.

Lemma 1.2 [7]. *If* $g \in \mathcal{P}$, *then* $|c_i - c_j c_{i-j}| \leq 2$ *holds for the values* $i, j \in \mathbb{N}$ *with* $i > j$.

Lemma 1.3 [9]. *If* $g \in \mathcal{P}$, *then* $|c_t| \leq 2$, *for* $t \in \mathbb{N}$, *equality occurs for the function* $h(z) = \dfrac{1+z}{1-z}$, $z \in \mathcal{U}_d$.

Lemma 1.4 [11]. *If* $g \in \mathcal{P}$, *then* $\left|c_2 c_4 - c_3^2\right| \leq 4 - \dfrac{1}{2}|c_2|^2 + \dfrac{1}{4}|c_2|^3$.

To procure our results, we adopt the procedure framed through Libera and Zlotkiewicz [6].

2. Important outcomes.

Theorem 2.1. *If* $f \in \Re$ *and* G *given in* (1.2) *is the* kth-*root transformation of* f, *then*

$$\left|H_{2,k}(3)\right| \leq \left[\frac{542k^3 - 383k^2 - 30k + 15}{540k^5}\right]$$

and the result is sharp for $k = 1$.

Proof. For $f \in \Re$, according to the Definition 1.1,

$$f'(z) = g(z), \quad z \in \mathcal{U}_d. \tag{2.1}$$

Substitute the values for f and g in (2.1), it simplifies to

$$a_{n+1} = \frac{c_n}{n+1}, \quad n \in \mathbb{N}. \tag{2.2}$$

For the mapping f (1.1), a calculation gives

$$\left[f(z^k)\right]^{\frac{1}{k}} = \left[z^k + \sum_{n=2}^{\infty} a_n z^{nk}\right]^{\frac{1}{k}} =$$

$$= \left[z + \frac{1}{k}a_2 z^{k+1} + \left\{\frac{1}{k}a_3 + \frac{1-k}{2k^2}a_2^2\right\}z^{2k+1} + \right.$$

$$+ \left\{\frac{1}{k}a_4 + \frac{1-k}{k^2}a_2 a_3 + \frac{(1-k)(1-2k)}{6k^3}a_2^3\right\}z^{3k+1} +$$

$$+ \left\{\frac{1}{k}a_5 + \frac{1-k}{2k^2}(a_3^2 + 2a_2 a_4) + \frac{(1-k)(1-2k)}{2k^3}a_2^2 a_3 + \right.$$

$$\left.\left. + \frac{(1-k)(1-2k)(1-3k)}{24k^4}a_2^4\right\}z^{4k+1} + \ldots\right]. \tag{2.3}$$

Comparing the coefficients of z^{k+1}, z^{2k+1}, z^{3k+1} and z^{4k+1} in the expressions (1.2) and (2.3), we obtain

$$b_{k+1} = \frac{1}{k}a_2, \qquad b_{2k+1} = \frac{1}{k}a_3 + \frac{1-k}{2k^2}a_2^2,$$

$$b_{3k+1} = \left[\frac{1}{k}a_4 + \frac{1-k}{k^2}a_2 a_3 + \frac{(1-k)(1-2k)}{6k^3}a_2^3\right], \tag{2.4}$$

$$b_{4k+1} = \left[\frac{1}{k}a_5 + \frac{1-k}{2k^2}(a_3^2 + 2a_2 a_4) + \frac{(1-k)(1-2k)}{2k^3}a_2^2 a_3 + \right.$$

$$\left. + \frac{(1-k)(1-2k)(1-3k)}{24k^4}a_2^4\right].$$

Simplifying the expressions in (2.2) and (2.4), we get

$$b_{k+1} = \frac{c_1}{2k}, \qquad b_{2k+1} = \frac{c_2}{3k} - \frac{k-1}{8k^2}c_1^2,$$

$$b_{3k+1} = \left[\frac{c_3}{4k} - \frac{k-1}{6k^2}c_1c_2 + \frac{(k-1)(2k-1)}{48k^3}c_1^3\right], \tag{2.5}$$

$$b_{4k+1} = \left[\frac{c_4}{5k} - \frac{k-1}{18k^2}c_2^2 - \frac{k-1}{8k^2}c_1c_3 + \frac{(k-1)(2k-1)}{24k^3}c_1^2c_2 - \right.$$

$$\left. - \frac{(k-1)(2k-1)(3k-1)}{384k^4}c_1^4\right].$$

Substituting the values namely b_{2k+1}, b_{3k+1} and b_{4k+1} from (2.5) in the functional given in (1.6), we have

$$H_{2,k}(3) = \left[\frac{1}{15k^2}c_2c_4 - \frac{1}{16k^2}c_3^2 - \frac{k-1}{54k^3}c_2^3 + \frac{k-1}{24k^3}c_1c_2c_3 + \frac{k^2-1}{144k^4}c_1^2c_2^2 - \right.$$

$$\left. - \frac{k-1}{40k^3}c_1^2c_4 - \frac{k^2-1}{192k^4}c_1^3c_3 - \frac{(k^2-1)(2k-1)}{1152k^5}c_1^4c_2 + \frac{(k^2-1)(k-1)(2k-1)}{9216k^6}c_1^6\right]. \tag{2.6}$$

On grouping the terms in (2.6), to apply lemmas mentioned in this paper, we get

$$H_{2,k}(3) = \left[\frac{1}{16k^2}\{c_2c_4 - c_3^2\} + \frac{1}{240k^2}c_2c_4 - \frac{k-1}{54k^3}c_2^3 - \frac{k-1}{40k^3}c_1^2\left\{c_4 - \frac{5(k+1)}{18k}c_2^2\right\} - \right.$$

$$\left. - \frac{(k^2-1)(2k-1)}{1152k^5}c_1^4\left\{c_2 - \frac{2k-1}{8k}c_1^2\right\} + \frac{k-1}{24k^3}c_1c_3\left\{c_2 - \frac{k+1}{8k}c_1^2\right\}\right]. \tag{2.7}$$

Applying the triangle inequality in (2.7), we obtain

$$|H_{2,k}(3)| \leq \left[\frac{1}{16k^2}\left|c_2c_4 - c_3^2\right| + \frac{1}{240k^2}|c_2||c_4| + \frac{k-1}{54k^3}|c_2|^3 + \right.$$

$$+ \frac{k-1}{40k^3}|c_1|^2\left|c_4 - \frac{5(k+1)}{18k}c_2^2\right| + \frac{(k^2-1)(2k-1)}{1152k^5}|c_1|^4\left|c_2 - \frac{2k-1}{8k}c_1^2\right| +$$

$$\left. + \frac{k-1}{24k^3}|c_1||c_3|\left|c_2 - \frac{k+1}{8k}c_1^2\right|\right]. \tag{2.8}$$

By using Lemmas 1.1, 1.3 and 1.4 in the inequality (2.8), we have

$$\left|H_{2,k}(3)\right| \leq \left[\frac{542k^3 - 383k^2 - 30k + 15}{540k^5}\right]. \tag{2.9}$$

***Remark* 2.1.** In particular, $k = 1$ in the expression (2.9), then the result coincides with the development obtained by Zaprawa [11] and the respective function, for which the equality holds

$$f(z) = \left[\log(1+z) - \log(1-z)\right] = 2z + \frac{2}{3}z^3 + \frac{2}{5}z^5 + \ldots.$$

Theorem 2.1 is proved.

Theorem 2.2. *If $f \in \Re$ and G given in* (1.2) *is the kth-root transformation of f, then*

$$\left|H_{3,k}(1)\right| \leq \left[\frac{90k^4 + 414k^3 + 15k^2 - 135k - 15}{540k^5}\right].$$

Proof. Substituting b_{k+1}, b_{2k+1}, b_{3k+1} and b_{4k+1} values from (2.5) in (1.8), it simplifies to

$$H_{3,k}(1) = \left[\frac{1}{15k^2}c_2c_4 - \frac{k+1}{54k^3}c_2^3 - \frac{1}{16k^2}c_3^2 - \frac{k+1}{40k^3}c_1^2c_4 + \right.$$
$$+\frac{k+1}{24k^3}c_1c_2c_3 + \frac{k^2-1}{144k^4}c_1^2c_2^2 - \frac{(k^2-1)(2k+1)}{1152k^5}c_1^4c_2 -$$
$$\left. -\frac{k^2-1}{192k^4}c_1^3c_3 + \frac{(k^2-1)(k+1)(2k+1)}{9216k^6}c_1^6\right]. \tag{2.10}$$

On grouping the terms in (2.10), to apply lemmas given in this paper, then

$$H_{3,k}(1) = \left[\frac{1}{20k^2}c_4\left\{c_2 - \frac{20k^2(k+1)}{40k^3}c_1^2\right\} - \frac{1}{16k^2}c_3\left\{c_3 - \frac{16k^2}{16k^3}c_1c_2\right\} + \right.$$
$$+\frac{1}{27k^2}c_2\left\{c_4 - \frac{27k^2(k+1)}{54k^3}c_2^2\right\} - \frac{1}{48k}c_2\left\{c_4 - \frac{48k(2k-1)}{48k^3}c_1c_3\right\} +$$
$$+\frac{(k^2-1)(2k+1)}{1152k^5}c_1^4\left\{c_2 - \frac{1152k^5(k+1)}{9216k^6}c_1^2\right\} - \frac{k^2-1}{192k^4}c_1^3c_3 + \frac{k^2-1}{144k^4}c_1^2c_2^2 +$$
$$\left. +\left\{-\frac{1}{20k^2} - \frac{1}{27k^2} + \frac{1}{48k} + \frac{1}{15k^2}\right\}c_2c_4\right].$$

Further, we have

$$H_{3,k}(1) = \left[\frac{1}{20k^2}c_4\left\{c_2 - \frac{k+1}{2k}c_1^2\right\} - \frac{1}{16k^2}c_3\left\{c_3 - \frac{1}{k}c_1c_2\right\} + \right.$$
$$+\frac{1}{27k^2}c_2\left\{c_4 - \frac{k+1}{2k}c_2^2\right\} - \frac{1}{48k}c_2\left\{c_4 - \frac{2k-1}{k^2}c_1c_3\right\} +$$
$$+\frac{(k^2-1)(2k+1)}{1152k^5}c_1^4\left\{c_2 - \frac{k+1}{8k}c_1^2\right\} - \frac{k^2-1}{192k^4}c_1^3c_3 + \frac{k^2-1}{144k^4}c_1^2c_2^2 +$$
$$\left. +\left\{\frac{45k-44}{2160k^2}\right\}c_2c_4\right]. \tag{2.11}$$

Applying the triangle inequality in the expression (2.11), we obtain

$$|H_{3,k}(1)| \le \left[\frac{1}{20k^2}|c_4|\left|c_2 - \frac{k+1}{2k}c_1^2\right| + \frac{1}{16k^2}|c_3|\left|c_3 - \frac{1}{k}c_1c_2\right| + \right.$$
$$+\frac{1}{27k^2}|c_2|\left|c_4 - \frac{k+1}{2k}c_2^2\right| + \frac{1}{48k}|c_2|\left|c_4 - \frac{2k-1}{k^2}c_1c_3\right| +$$
$$+\frac{(k^2-1)(2k+1)}{1152k^5}|c_1^4|\left|c_2 - \frac{k+1}{8k}c_1^2\right| +$$
$$\left. +\frac{k^2-1}{192k^4}|c_1^3||c_3| + \frac{k^2-1}{144k^4}|c_1^2||c_2^2| + \left\{\frac{45k-44}{2160k^2}\right\}|c_2||c_4|\right]. \tag{2.12}$$

Upon applying Lemmas 1.1, 1.2 and 1.3 in the inequality (2.12), it simplifies to

$$\left|H_{3,k}(1)\right| \leq \left[\frac{90k^4 + 414k^3 + 15k^2 - 135k - 15}{540k^5}\right]. \tag{2.13}$$

***Remark* 2.2.** In particular, if $k = 1$ in the expression (2.13), then the result coincides with the result of Zaprawa [12].

Theorem 2.2 is proved.

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