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AN AMBARZUMIAN TYPE THEOREM ON GRAPHS WITH ODD CYCLES [2]

ТЕОРЕМА ТИПУ АМБАРЦУМЯНА ДЛЯ ГРАФІВ З НЕПАРНИМИ ЦИКЛАМИ

We consider an inverse problem for Schrödinger operators on a connected equilateral graph G with standard matching conditions. The graph G consists of at least two odd cycles glued together at a common vertex. We prove an Ambarzumian-type result, i.e., if a specific part of the spectrum is the same as in the case of zero potential, then the potential must be equal to zero.

Розглянуто обернену задачу для операторів Шредінгера на зв'язному рівносторонньому графі G зі стандартними умовами узгодження. Граф G складається принаймні з двох непарних циклів, що склеєні в спільній вершині. Доведено результат типу Амбарцумяна, тобто якщо певна частина спектра така ж сама, як і у випадку нульового потенціалу, то потенціал повинен бути нульовим.

1. Introduction. The addressed problem originates from a work of Ambarzumian [2] on reconstruction of a differential operator from its eigenvalues. Another source of the problem is the so-called *quantum graphs*, i.e., differential operators on graphs [3, 21, 26]. From the classical theory of Sturm – Liouville equations we refer to [7, 17], for special Ambarzumian type inverse problems see [10, 18]. Previous results for graphs are [6, 8, 12, 20, 22, 25]. Both in forward and in inverse problems on graphs a usual ingredient is the calculation of *spectral determinant*s (or alternatively *functional determinant*s or *characteristic function*s) [1, 5, 9, 11, 13 – 16, 19, 23, 24, 27]. For a more detailed discussion of these results see the introduction in [20].

2. Results and discussion. Let $r \geq 2$ and consider r cycle graphs $C_1, C_2, \ldots, C_r$ with odd cycle lengths $n_1, n_2, \ldots, n_r$ $(n_j = 1$ is also possible$)$. Let the vertices of C_j be $v_{j0}, \ldots, v_{jn_j} = v_{j0}$, and let us form the graph G as the union of C_j's, identifying the vertex v_{j0} for all j. We shall say that G is a graph consisting of $r \geq 2$ odd cycles glued together at a common vertex. The edge of G between $v_{j\,k-1}$ and v_{jk} is sometimes denoted by e_{jk}; however, when the particular location of the edges are not important, we shall refer to them as $e_1, e_2, \ldots, e_{|E|}$.

Choosing an arbitrary orientation, we parametrize each edge with $x \in [0, 1]$, and consider a Schrödinger operator with potential $q_j(x) \in L^1(0, 1)$ on the edge e_j and with Neumann (or Kirchhoff) boundary conditions (sometimes called standard matching conditions), i.e., solutions are required to be continuous at the vertices and, in the local coordinate pointing outward, the sum of derivatives is zero. More formally, consider the eigenvalue problem

$$-y'' + q_j(x)y = \lambda y \tag{2.1}$$

on e_j for all j with the conditions

$$y_j(\kappa_j) = y_k(\kappa_k) \tag{2.2}$$

if e_j and e_k are incident edges attached to a vertex v where $\kappa = 0$ for outgoing edges, $\kappa = 1$ for incoming edges (and can be both 0 or 1 for loops); and in every vertex v

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$$\sum_{e_j \text{leaves} v} y_j'(0) = \sum_{e_j \text{enters} v} y_j'(1) \tag{2.3}$$

(loops are counted on both sides).

Theorem 2.1. *Consider the eigenvalue problem* (2.1)–(2.3). *Let G be a graph consisting of $r \geq 2$ odd cycles glued together at a common vertex. If $\lambda = 0$ is the smallest eigenvalue and for infinitely many $k \in \mathbb{Z}^+$ there are $r-1$ eigenvalues* (*counting multiplicities*) *such that $\lambda = (2k+1)^2\pi^2 + o(1)$, then $q = 0$ a.e. on G.*

If the lengths of the odd cycles are all 1, i.e., the cycles are all loops, then the statement reduces to that of Theorem 2.1 in [28], which states the following:

Suppose G is a flower-like graph, i.e., a single vertex attached r loops of length 1. For $k = 1, 2, \ldots,$ let m_k be a sequence of integers with $\lim m_k = +\infty$. If eigenvalues are nonnegative, $\lambda_k = (2m_k + 1)^2\pi^2$ are eigenvalues with multiplicities $(r-1)$, where m_k is a strictly ascending infinite sequence of positive integers, then $q_j(x) = 0$ a.e. on $[0, 1]$, for each $j = 1, 2, \ldots, r$. We have to require $r \geq 2$ for the consequence to hold.

3. Calculation of the spectral determinant. Denote by $c_j(x, \lambda)$ the solution of (2.1) which satisfies the conditions $c_j(0, \lambda) - 1 = c_j'(0, \lambda) = 0$ and by $s_j(x, \lambda)$ the solution of (2.1) which satisfies the conditions $s_j(0, \lambda) = s_j'(0, \lambda) - 1 = 0$. Each $y_j(x, \lambda)$ may be written as a linear combination

$$y_j(x, \lambda) = A_j(\lambda)c_j(x, \lambda) + \sqrt{\lambda}B_j(\lambda)s_j(x, \lambda).$$

Then $y_j(0, \lambda) = A_j(\lambda)$ is the same on each outgoing edge; hence, as in [20], we index the functions $A(\lambda)$ by vertices, and then

$$y_j(x, \lambda) = A_v(\lambda)c_j(x, \lambda) + \sqrt{\lambda}B_j(\lambda)s_j(x, \lambda),$$

if e_j starts from v. If the eigenfunctions are normalized, i.e., $\sum_j \|y_j(x, \lambda)\|_2^2 = 1$, then $A_v(\lambda) = B_j(\lambda) = O(1)$ [8, 20, 28]. The coefficients A_v and B_j form a $\big(|V| + |E|\big)$-dimensional vector, which satisfies $|V|$ Kirchhoff conditions at the vertices and $|E|$ continuity conditions at the incoming ends of edges, namely, for all $v \in V(G)$,

$$\sum_{e_j\,:\,\ldots\to v} \underbrace{\frac{1}{\sqrt{\lambda}}A_{v_j}(\lambda)c_j'(1, \lambda) + B_j(\lambda)s_j'(1, \lambda)}_{\frac{1}{\sqrt{\lambda}}y_j'(1,\lambda)} - \sum_{e_j\,:\,v\to\ldots} \underbrace{B_j(\lambda)}_{\frac{1}{\sqrt{\lambda}}y_j'(0,\lambda)} = 0,$$

where in the first sum v_j denotes the starting point of e_j; and, for all $e_j \in E(G)$,

$$A_u(\lambda)c_j(1, \lambda) + \sqrt{\lambda}B_j(\lambda)s_j(1, \lambda) - A_v(\lambda) = 0,$$

if e_j points from u to v (see equations (2.3) and (2.4) in [20]).

The matrix M of this homogeneous linear system of equations has a special structure. For the convenience of the reader we repeat its description from [20]. Namely, $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, where

A is a $|V|$ by $|V|$ matrix, $a_{vu} = \dfrac{1}{\sqrt{\lambda}} \sum c_j'(1, \lambda)$, the sum is taken on edges pointing from u to v; in the zero-potential case A is $-\sin\sqrt{\lambda}$ times the (transpose of the) directed adjacency matrix of G;

B and C are like incidence matrices;

$$b_{vj} = \begin{cases} s_j'(1,\lambda), & \text{if } \; e_j \; \text{ ends in } \; v, \\ -1 & \text{if } \; e_j \; \text{ starts from } \; v, \\ s_j'(1,\lambda)-1, & \text{if } \; e_j \; \text{ is a loop in } \; v, \\ 0, & \text{otherwise} \end{cases}$$

and

$$c_{jv} = \begin{cases} -1, & \text{if } \; e_j \; \text{ ends in } \; v, \\ c_j(1,\lambda), & \text{if } \; e_j \; \text{ starts from } \; v, \\ -1+c_j(1,\lambda), & \text{if } \; e_j \; \text{ is a loop in } \; v, \\ 0, & \text{otherwise;} \end{cases}$$

D is an $|E|$ by $|E|$ diagonal matrix, $d_{jj} = \sqrt{\lambda} s_j(1,\lambda)$.

The determinant of the matrix M is the so-called spectral determinant of the problem (2.1) – (2.3).

Example. Consider a flower-like graph, i.e., a single vertex with r loops. Then

$$M = M_1 = \left[\begin{array}{c|ccc} \dfrac{1}{\sqrt{\lambda}} \displaystyle\sum_{k=1}^{r} c_k'(1,\lambda) & s_1'(1,\lambda)-1 & \ldots & s_r'(1,\lambda)-1 \\ \hline -1+c_1(1,\lambda) & \sqrt{\lambda} s_1(1,\lambda) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -1+c_r(1,\lambda) & 0 & \ldots & \sqrt{\lambda} s_r(1,\lambda) \end{array}\right]$$

with determinant

$$\det M_1 = \lambda^{\frac{r-1}{2}} \left(\sum_{k=1}^{r} c_k'(1,\lambda) \prod_{j=1}^{r} s_j(1,\lambda) - \sum_{k=1}^{r} (s_k'(1,\lambda)-1)(-1+c_k(1,\lambda)) \prod_{j\neq k} s_j(1,\lambda) \right)$$

corresponding to formula (2.9) in [28].

For $\lambda = (2k+1)^2\pi^2 + d + o(1)$ the elements of M have the following asymptotics, independent of k (see [8], equation (2.3), or [22], Lemma 3.1):

$$\frac{1}{\sqrt{\lambda}} c_j'(1,\lambda) = \frac{1}{2\sqrt{\lambda}} \left(d - \int_0^1 q_j \right) + o\left(\frac{1}{\sqrt{\lambda}}\right), \tag{3.1}$$

$$s_j'(1,\lambda) = -1 + o\left(\frac{1}{\sqrt{\lambda}}\right), \tag{3.2}$$

$$c_j(1,\lambda) = -1 + o\left(\frac{1}{\sqrt{\lambda}}\right), \tag{3.3}$$

$$\sqrt{\lambda} s_j(1,\lambda) = \frac{1}{2\sqrt{\lambda}} \left(\int_0^1 q_j - d \right) + o\left(\frac{1}{\sqrt{\lambda}}\right). \tag{3.4}$$

Remark. Using these asymptotics, we get

$$\det M_1 = -4 \sum_{k=1}^{r} \prod_{j\neq k} \sqrt{\lambda} s_j(1,\lambda) + o(\lambda^{-\frac{r}{2}}).$$

This is a special case of (4.1) and of (4.2) below.

4. Proofs of main result.

Lemma 4.1. *The total multiplicities of the eigenvalues* $\lambda = (2k+1)^2\pi^2 + O(1)$ *are exactly* $|E| - |V|$.

Proof. If $q = 0$, $\det M$ is a polynomial of $\cos\sqrt{\lambda}$ and $\sin\sqrt{\lambda}$, hence its zeros are 2π-periodic in $\sqrt{\lambda}$. $\lambda = (2k+1)^2\pi^2 + O(1)$ with periodicity implies $\sqrt{\lambda} = (2k+1)\pi$ exactly. For $\lambda = (2k+1)^2\pi^2$ A and D are zero matrices, thus the rank of M is $2|V|$, and its nullspace is exactly $(|E| - |V|)$-dimensional. Let us write for a moment $\lambda = \lambda(q)$ to denote the dependency of the eigenvalues on the potential. If we arrange the eigenvalues in an increasing sequence $\lambda_1, \ldots, \lambda_n, \ldots$, then there is a constant c depending only on the graph and the L^1-norm of the potential and not depending on n, the particular index of the eigenvalue, such that $|\lambda_n(q) - \lambda_n(0)| \le c$ for all n [20]. Hence, the total multiplicity of eigenvalues $\lambda = (2k+1)^2\pi^2 + O(1)$ is the same for all $q \in L^1$.

Lemma 4.2. *The determinant of* M *for* $\lambda = (2k+1)^2\pi^2 + O(1)$ *is* $O(\lambda^{-\frac{1}{2}(|E|-|V|)})$.

Proof. Each term in the Leibniz formula for the determinant must contain at least $(|E| - |V|)$ factors from A and D having a magnitude of $O\left(\frac{1}{\sqrt{\lambda}}\right)$.

Lemma 4.3. *Assume that a graph has the same number of edges as vertices. Then the determinant of its* (*unoriented*) *incidence matrix is zero*, *except if there is no even cycles in the graph and every component contains exactly one* (*odd*) *cycle. In that case the determinant is* $\pm 2^\kappa$ *where* κ *denotes the number of components.*

Proof. If the graph contains an even cycle, then the corresponding rows are dependent. If a component contains no cycles, then the corresponding columns are dependent. The number of cycles (including loops) is equal to the number of components, hence, if a component contains more than one cycles or loops, then there must be another component without cycles. In all of these cases the determinant of the incidence matrix is zero. Otherwise it is enough to prove the statement for connected graphs, as the incidence matrix is a direct sum of that of the components. It is true for odd cycles as well as for a single vertex with a loop. If the graph is not a single cycle or a single loop, there is at least one vertex with only one incident edge. Removing this vertex (and its edge) from the graph does not change the absolute value of the determinant of the incidence matrix. This can be repeated until we reach a single cycle or a single loop.

Lemma 4.4. *If* $\lambda = (2k+1)^2\pi^2 + O(1)$, *then the determinant of a* $|V| \times |V|$ *submatrix of* C (*and of* B) *is* $\pm 2^\kappa + o\left(\frac{1}{\sqrt{\lambda}}\right)$ *if the indices of the rows in* C (*the columns in* B) *correspond to the edges of a subgraph which has no even cycles and every component contains exactly one* (*odd*) *cycle. Otherwise the determinant is* $o\left(\frac{1}{\sqrt{\lambda}}\right)$.

Proof. Leaving out the $o\left(\frac{1}{\sqrt{\lambda}}\right)$ terms from the submatrix we make only $o\left(\frac{1}{\sqrt{\lambda}}\right)$ error in its determinant. What we get is the negative of an incidence matrix of a subgraph with $|V|$ vertices and $|V|$ edges. Then the statement follows from the previous lemma.

Theorem 4.1. *If* $\lambda = (2k+1)^2\pi^2 + O(1)$, *then*

$$\det M = (-1)^{|V|} \sum_{\tau} 4^{\kappa(\tau)} \prod_{e_j \notin \tau} \sqrt{\lambda} s_j(1,\lambda) + O(\lambda^{-\frac{|E|-|V|+1}{2}}), \tag{4.1}$$

where the sum is taken for such subgraphs τ of G which have $|V|$ vertices and their incidence matrix is nonsingular (*i.e.*, *τ has no even cycles*, *has κ components, each of which contains exactly one* (*odd*) *cycle*).

Proof. The main terms in the Leibniz formula for the determinant are those which contain exactly $(|E|-|V|)$ elements from D. The product of a fixed set of $(|E|-|V|)$ elements in D is weighted by the determinant of the respective minor, with all other elements of D substituted by zero. The remaining rows in C and columns in B look like an unordered incidence matrix of the graph τ spanned by the remaining $|V|$ edges. Then the determinant of the minor is $(-1)^{|V|}$ times the square of the determinant of the incidence matrix of τ.

***Corollary* 4.1.** *If $\lambda=(2k+1)^2\pi^2+O(1)$ and the graph G consists of r odd cycles of length $n_1,\ldots,n_r$, glued together at a common vertex*, *then*

$$\det M=-4\sum_{i=1}^{r}\prod_{j\neq i}\sum_{l=1}^{n_j}\sqrt{\lambda}s(1,\lambda,q_{jl})+O(\lambda^{-\frac{r}{2}}),\tag{4.2}$$

where q_{jl} is the potential on the lth edge of the jth cycle.

Proof. The incidence matrix of a subgraph of G is nonsingular if and only if we leave out one edge from every but one cycle. Note also that $|V|$ is odd and $|E|-|V|=r-1$. Then the statement follows from (4.1).

Substituting the asymptotics (3.1) – (3.4), we get the following corollary.

***Corollary* 4.2.** *If $\lambda=(2k+1)^2\pi^2+d+o(1)$ and the graph G consists of r odd cycles of length $n_1,\ldots,n_r$, glued together at a common vertex*, *then*

$$\det M=-4\left(\frac{-1}{2\sqrt{\lambda}}\right)^{r-1}p(d)+o(\lambda^{-\frac{r-1}{2}}),$$

where

$$p(d)=\sum_{i=1}^{r}\prod_{j\neq i}\left(n_jd-\sum_{l=1}^{n_j}\int\limits_0^1 q_{jl}\right).\tag{4.3}$$

Lemma 4.5. *Under the assumptions of Theorem* 2.1, $p(d)=\sum_{i=1}^{r}\prod_{j\neq i}n_jd^{r-1}$.

Proof. λ is an eigenvalue of the eigenvalue problem (2.1) – (2.3) if and only if $\det M(\lambda)=0$. Let the distinct roots of $p(d)$ be $d_1,\ldots,d_l$. By the previous corollary for $\lambda=(2k+1)^2\pi^2+O(1)$ the distinct roots of $\det M(\lambda)$ are exactly of the form $\lambda=(2k+1)^2\pi^2+d_j+o(1)$, $1\le j\le l$. By Lemma 4.1 the total multiplicity of these eigenvalues is $|E|-|V|$. In Theorem 2.1 we assumed that there are the same number of eigenvalues such that $\lambda=(2k+1)^2\pi^2+o(1)$, hence $d_j=0$ for all j and then $p(d)$ is a constant multiple of d^{r-1}. The principal coefficient is given by (4.3).

***Proof of Theorem* 2.1.** Let us introduce $Q_j=\sum_{l=1}^{n_j}\int_0^1 q_{jl}$.

For a fixed m substituting $d=\dfrac{Q_m}{n_m}$ to (4.3), we get

$$\sum_{i=1}^{r}\frac{1}{n_i}\left(\frac{Q_m}{n_m}\right)^{r-1}=\frac{1}{\prod_{j=1}^{r}n_j}p\left(\frac{Q_m}{n_m}\right)=\frac{1}{n_m}\prod_{j\neq m}\left(\frac{Q_m}{n_m}-\frac{Q_j}{n_j}\right).$$

Introducing $h_j = \dfrac{Q_j}{n_j}$, we have

$$\sum_{i=1}^{r} \frac{1}{n_i} h_m^{r-1} = \frac{1}{n_m} \prod_{j \neq m} (h_m - h_j), \quad m = 1, 2, \ldots, r.$$

We can assume $h_1 \geq h_2 \geq \ldots \geq h_r$. Then, for $m = 2$, the left-hand side is nonpositive, hence $h_2 \leq 0$. Similarly, $h_{r-1} \geq 0$. Hence, for $m = 1$,

$$\frac{1}{n_1} h_1^{r-2}(h_1 - h_r) = \sum_{i=1}^{r} \frac{1}{n_i} h_1^{r-1}.$$

If $h_r = 0$, then $h_1 = 0$ as n_j's are positive. Similarly, $h_1 = 0$ implies $h_r = 0$. If neither of them is zero, then $\dfrac{1}{n_1}(h_1 - h_r) = \sum_{i=1}^{r} \dfrac{1}{n_i} h_1$ and $\dfrac{1}{n_r}(h_r - h_1) = \sum_{i=1}^{r} \dfrac{1}{n_i} h_r$. Subtracting, we get

$$\sum_{i=2}^{r-1} \frac{1}{n_i}(h_1 - h_r) = 0.$$

As n_j's are positive, if $r > 2$ then $h_1 = h_2 = \ldots = h_r = 0$, while for $r = 2$, $h_1 n_1 + h_2 n_2 = 0$. In both cases,

$$\sum_j Q_j = \sum_{e_{jl} \in G} \int_0^1 q_{jl} = \int_G q = 0.$$

Let us denote the operator of the eigenvalue problem (2.1)–(2.3) by L. $\dfrac{\langle \varphi, L\varphi \rangle}{\langle \varphi, \varphi \rangle} \geq \lambda_0 = 0$ and equality holds if and only if φ is an eigenfunction of L. It follows that the constant 1 must be an eigenfunction corresponding to the eigenvalue 0. Substituting this to (2.1) gives $q(x) = 0$.

Theorem 2.1 is proved.

5. Glossary. A *walk* W in a graph is an alternating sequence of vertices and edges, say $X_0, e_1, \ldots, e_l, X_l$ where $e_i = X_{i-1}X_i$, $0 < i < l$. The length of W is l. This walk W is called a *trail* if all its edges are distinct. A *path* is a walk with distinct vertices. A trail whose end vertices coincide (a closed trail) is called a *circuit*. To be precise, a circuit is a closed trail without distinguished endvertices and direction, so that, for example, two triangles sharing a single vertex give rise to precisely two circuits with six edges. If a walk $W = X_0, e_1, \ldots, e_l, X_l$ is such that $l \geq 3$, $X_0 = X_l$, and the vertices $X_i, 0 < i < l$, are distinct from each other and X_0, then W is said to be a *cycle* [4, p. 5].

The *incidence matrix* of a graph has a row for each vertex and a column for each edge, and is defined as

$$R = (r_{ij}), \qquad r_{ij} = \begin{cases} 0, & \text{if } e_j \text{ is not incident to } v_i, \\ 1, & \text{if } e_j \text{ is not a loop and incident to } v_i, \\ 2, & \text{if } e_j \text{ is a loop at } v_i. \end{cases}$$

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