DOI: 10.37863/umzh.v74i1.2298

UDC 517.9

A. Kalybay (KIMEP Univ., Almaty, Kazakhstan),
D. Karatayeva (L. N. Gumilyov Eurasian Nat. Univ., Nur-Sultan, Kazakhstan)

OSCILLATION AND NONOSCILLATION CRITERIA FOR A HALF-LINEAR DIFFERENCE EQUATION OF THE SECOND ORDER AND EXTENDED DISCRETE HARDY INEQUALITY *

КРИТЕРІЇ КОЛИВАННЯ ТА НЕКОЛИВАННЯ ДЛЯ НАПІВЛІНІЙНОГО РІЗНИЦЕВОГО РІВНЯННЯ ДРУГОГО ПОРЯДКУ ТА РОЗШИРЕННЯ ДИСКРЕТНОЇ НЕРІВНОСТІ ГАРДІ

We establish the oscillatory properties of a half-linear difference equation of the second order by using a suitable extension of the weighted discrete Hardy inequality.

За допомогою відповідного розширення дискретної нерівності Гарді встановлено коливні властивості напівлінійного різницевого рівняння другого порядку.

1. Introduction. We consider the following second order half-linear difference equation:

$$\Delta\big(\rho_i|\Delta y_i|^{p-2}\Delta y_i\big) + v_i|y_{i+1}|^{p-2}y_{i+1} = 0, \quad i = 0, 1, 2, \ldots, \tag{1.1}$$

where $1 < p < \infty$, $\Delta y_i = y_{i+1} - y_i$. The coefficients $\rho = \{\rho_i\}$ and $v = \{v_i\}$ of equation (1.1) are sequences of real numbers. Moreover, $\rho_i > 0$ for any $i = 0, 1, 2, \ldots$.

Let us list notions and statements required for this paper. Let $m \geq 0$ and $n \geq 0$ be integer numbers. For simplicity we will use the term "interval" meaning "discrete interval".

If there exists a nontrivial solution $y = \{y_i\}$ of equation (1.1) such that $y_m \neq 0$ and $y_m y_{m+1} < 0$, then we say that the solution y has a generalized zero on the interval $(m, m+1]$.

A nontrivial solution y of equation (1.1) is called oscillatory if it has an infinite number of generalized zeros, otherwise it is called nonoscillatory.

Equation (1.1) is called oscillatory if all its nontrivial solutions are oscillatory, otherwise it is called nonoscillatory.

By Sturm's separation theorem [18] (Theorem 3), equation (1.1) is oscillatory if one of its nontrivial solutions is oscillatory.

Equation (1.1) is called disconjugate on the interval $[m, n]$, $0 \leq m < n$, if its any nontrivial solution has no more than one generalized zero on the interval $(m, n+1]$ and its nontrivial solution $\tilde{y}$ with the initial condition $\tilde{y}_m = 0$ has not a generalized zero on the interval $(m, n+1]$, otherwise it is called conjugate on the interval $[m, n]$.

Equation (1.1) is called disconjugate on the interval $[m, \infty)$ if for any $n > m$ it is disconjugate on the interval $[m, n]$.

The investigation of the oscillatory properties of (1.1) is a subject of many works (see, e.g., the papers [1, 6–11, 16–19, 21–23] and references given there). This problem was firstly studied for $p = 2$, when equation (1.1) is the following linear difference equation:

* This paper was supported by the Ministry of Education and Science of the Republic of Kazakhstan (Grant No. AP08856100, in area "Scientific research in the field of natural sciences").

$$\Delta(\rho_i \Delta y_i) + v_i y_{i+1} = 0, \quad i = 0, 1, 2, \ldots, \tag{1.2}$$

that is often written in the form

$$\rho_{n-1} y_{n-1} + \rho_n y_{n+1} + u_n y_n = 0, \quad u_n = v_n - \rho_n - \rho_{n-1}, \quad n = 1, 2, 3, \ldots .$$

Moreover, there are many works devoted to the differential analogues of equations (1.1) and (1.2):

$$\big(\rho(t)|y'(t)|^{p-2} y'(t)\big)' + v(t)|y(t)|^{p-2} y(t) = 0, \quad t > 0, \tag{1.3}$$

$$\big(\rho(t) y'(t)\big)' + v(t) y(t) = 0, \quad t > 0,$$

respectively. The last equation is the famous Sturm equation, the investigation of which was started in 1836 in the work [20] and has been continued up to the present days.

One of the known methods to study the oscillatory properties of equation (1.1) is the "variational method". This method is based on the following (see [18]).

Theorem A. *Let* $0 \le m < n < \infty$. *Equation* (1.1) *is disconjugate on the interval* $[m, n]$ *if and only if*

$$\sum_{i=m}^{n} (\rho_i |\Delta y_i|^p - v_i |y_{i+1}|^p) \ge 0 \tag{1.4}$$

holds for all nontrivial $y = \{y_i\}_{i=m}^{n+1}$, $y_m = 0$ *and* $y_{n+1} = 0$.

Here, we use a statement equivalent to Theorem A, the proof of which is given in [1]. To introduce this equivalent statement we need the definition of the set $\overset{\circ}{Y}(m, n)$ for $0 \le m < n \le \infty$. Denote by $\overset{\circ}{Y}(m, n)$ the set of all nontrivial sequences of real numbers $y = \{y_i\}_{i=0}^{\infty}$ such that $\operatorname{supp} y \subset [m+1, n]$, $n < \infty$, where $\operatorname{supp} y := \{i \ge 0 : y_i \ne 0\}$. When $n = \infty$, we suppose that for any y there exists an integer $k = k(y)$, $m < k < \infty$, such that $\operatorname{supp} y \subset [m+1, k]$.

Theorem B. *Let* $0 \le m < n \le \infty$. *Equation* (1.1) *is disconjugate on the interval* $[m, n]$ ($[m, n] = [m, \infty)$ *for* $n = \infty$) *if and only if*

$$\sum_{i=m}^{n} v_{i-1} |y_i|^p \le \sum_{i=m}^{n} \rho_i |\Delta y_i|^p, \quad y \in \overset{\circ}{Y}(m, n), \tag{1.5}$$

holds, *where* $v_{-1} = 0$.

Let $w = \{w_i\}$ be a fixed sequence of nonnegative real numbers. Let $\rho = \{\rho_i\}$, as before, be a fixed sequence of positive real numbers. For an arbitrary sequence $a = \{a_i\}$ we consider the inequality

$$\left(\sum_{i=1}^{\infty} w_i \left| \sum_{j=1}^{i} a_j \right|^q \right)^{\frac{1}{q}} \le C \left(\sum_{i=1}^{\infty} \rho_i |a_i|^p \right)^{\frac{1}{p}}, \tag{1.6}$$

that is, the known (now classical) weighted discrete Hardy inequality. In the papers [2 – 5] there are criteria for the validity of inequality (1.6) for all relations between p and q. In addition, the work [13] presents the history of the development of this inequality and relative results.

For $y_{i+1} = \sum_{j=1}^{i} a_j$ and $\Delta y_i = y_{i+1} - y_i,\ i = 1, 2, \ldots,$ inequality (1.6) can be rewritten in the difference form

$$\left(\sum_{i=1}^{\infty} w_i |y_{i+1}|^q \right)^{\frac{1}{q}} \leq C \left(\sum_{i=1}^{\infty} \rho_i |\Delta y_i|^p \right)^{\frac{1}{p}}, \tag{1.7}$$

where $y = \{y_i\}$ is an arbitrary sequence of real numbers with $y_1 = 0$.

In view of Theorem B it is easy to see the connection between the difference equation

$$\Delta(\rho_i |\Delta y_i|^{p-2} \Delta y_i) + w_i |y_{i+1}|^{p-2} y_{i+1} = 0, \quad i = 0, 1, 2, \ldots, \tag{1.8}$$

and the Hardy inequality (1.7) for a sequence y from the set $\overset{\circ}{Y}(m,n)$, given in the form

$$\left(\sum_{i=m}^{n} w_{i-1} |y_i|^p \right)^{\frac{1}{p}} \leq C \left(\sum_{i=m}^{n} \rho_i |\Delta y_i|^p \right)^{\frac{1}{p}}, \quad y \in \overset{\circ}{Y}(m,n). \tag{1.9}$$

Indeed, in the paper [12] the oscillation of equation (1.8) was established on the basis of inequality (1.9).

The difference between equations (1.1) and (1.8) is the fact that in (1.8) the sequence w consists of nonnegative real numbers, while in (1.1) the sequence v consists of any real numbers. If in (1.1) the sequence v consists of negative real numbers, then it is obvious that (1.4) holds for all $0 \leq m < n < \infty$. Therefore, equation (1.1) is nonoscillatory. It is naturally to pose a question about the influence of the positive part of the sequence v on the oscillation of equation (1.1). Assume that $v_i^+ = \max(0;\ v_i)$ and $v_i^- = \max(0;\ -v_i)$, then $v_i = v_i^+ - v_i^-$ for all $i = 0, 1, 2, \ldots$. Hence, equation (1.1) and inequality (1.4) have the forms

$$\begin{gathered} \Delta(\rho_i |\Delta y_i|^{p-2} \Delta y_i) - v_i^- |y_{i+1}|^{p-2} y_{i+1} + v_i^+ |y_{i+1}|^{p-2} y_{i+1} = 0, \quad i = 0, 1, 2, \ldots, \\ \sum_{i=m}^{n} (\rho_i |\Delta y_i|^p + v_i^- |y_{i+1}|^p - v_i^+ |y_{i+1}|^p) \geq 0, \end{gathered} \tag{1.10}$$

respectively, i.e., the positive part $\{v_i^+\}$ of the sequence $\{v_i\}$ can be considered as a perturbation of the nonoscillation equation

$$\Delta(\rho_i |\Delta y_i|^{p-2} \Delta y_i) - v_i^- |y_{i+1}|^{p-2} y_{i+1} = 0, \quad i = 0, 1, 2, \ldots .$$

To investigate this problem in a more general situation, we consider the following difference equation:

$$\Delta(\rho_i |\Delta y_i|^{p-2} \Delta y_i) + w_i |y_{i+1}|^{p-2} y_{i+1} - r_i |y_{i+1}|^{p-2} y_{i+1} = 0, \quad i = 0, 1, 2, \ldots, \tag{1.11}$$

where $r = \{r_i\}$, as $w = \{w_i\}$, is a sequence of nonnegative real numbers. If we assume that $w_i = v_i^+$ and $r_i = v_i^-$ for all $i = 0, 1, 2, \ldots,$ then equation (1.11) turns to equation (1.10), i.e., equation (1.1) is a partial case of equation (1.11). For equation (1.11), inequality (1.5) is equivalent to the inequality

$$\sum_{i=m}^{n} w_{i-1} |y_i|^p \leq \sum_{i=m}^{n} \left(\rho_i |\Delta y_i|^p + r_{i-1} |y_i|^p \right), \quad y \in \overset{\circ}{Y}(m,n). \tag{1.12}$$

From Theorem B and the last inequality it follows theorem.

Theorem 1.1. *Let* $0 \le m < n \le \infty$. *Equation* (1.11) *is disconjugate on the interval* $[m,n]$ ($[m,n]=[m,\infty)$ *for* $n=\infty$) *if and only if inequality* (1.12) *holds.*

Summing up, the difference between the previous oscillation results of equation (1.1) and the presented results is that here we study the case when the sequence v is not ultimately positive: we drop this restriction and allow arbitrary real numbers v_i. Moreover, in view of Theorem 1.1, the proofs of the main results are based on the fulfillment of the extended Hardy inequality

$$\sum_{i=m}^{n} w_{i-1}|y_i|^p \le C \sum_{i=m}^{n} \left(\rho_i|\Delta y_i|^p + r_{i-1}|y_i|^p\right), \quad y \in \overset{\circ}{Y}(m,n), \tag{1.13}$$

and estimates of its constant C. The results concerning inequality (1.13) are of independent interest.

This paper is organized as follows. In Section 2, we state and prove necessary and sufficient conditions for the fulfillment of the extended discrete Hardy inequality (1.13). In Section 3, first we present the results devoted to the oscillatory properties of the auxiliary equation (1.11), then we present the oscillatory properties of the main equation (1.1).

We note that the similar problem for the differential equation (1.3) was considered in the paper [15].

2. Extended Hardy inequality. In this section, we consider the extended discrete Hardy inequality (1.13). This inequality has been already considered in [14]. However, in order to apply inequality (1.13) to equation (1.11), it is important not only to find its characterizations, but also to estimate its constant C, which is the purpose of this section.

In the sequel, the sums $\sum_{i=k}^{m}$ for $m < k$ and $\sum_{i\in\Omega}$ for empty Ω are equal to zero. Moreover, $\frac{1}{p}+\frac{1}{p'}=1$. The numbers m, n, α, β, c, d, t, s, x, and z with and without indexes are integers.

We need the following lemma proved in [15]. Here, we give both its statement and proof for a more complete presentation.

Lemma 2.1. *Let* $1 < p < \infty$ *be a real number. Let* g *be a function defined as* $g(\lambda) = \frac{\lambda^p}{\lambda^p-1} - \frac{1}{(\lambda-1)^p}$ *on* $(1,\infty) \subset R$. *Then there exists a number* $\lambda_0 := \lambda_0(p)$ *such that* $1 < \lambda_0 < 2$ *and* $\frac{1}{(\lambda_0-1)^p} = \frac{\lambda_0^p}{\lambda_0^p-1}$, *satisfying the conditions* $g(\lambda) > 0$ *for* $\lambda > \lambda_0$ *and* $g(\lambda) < 0$ *for* $1 < \lambda < \lambda_0$.

Proof. It is obvious that $g(2) > 0$ and $\lim_{\lambda\to 1+} \frac{\lambda^p(\lambda-1)^p}{\lambda^p-1} = 0$. Using the definition of limit there exists a number $\delta > 0$ for $\varepsilon = 1$ such that $\frac{\widetilde{\lambda}^p(\widetilde{\lambda}-1)^p}{\widetilde{\lambda}^p-1} < 1$ or $\frac{\widetilde{\lambda}^p}{\widetilde{\lambda}^p-1} < \frac{1}{(\widetilde{\lambda}-1)^p}$ for every $\widetilde{\lambda} \in (1, 1+\delta)$. Thus, $g(\widetilde{\lambda}) < 0$. Since the function g is continuous on $(1,\infty)$ there exists a number $\lambda_0 \in (1,2)$ such that $g(\lambda_0) = 0$, i.e.,

$$\frac{1}{(\lambda_0-1)^p} = \frac{\lambda_0^p}{\lambda_0^p-1} \qquad \text{or} \qquad \lambda_0^p(\lambda_0-1)^p = \lambda_0^p - 1.$$

We define the functions $g_1(\lambda) := \frac{1}{(\lambda-1)^p}$ and $g_2(\lambda) := \frac{\lambda^p}{\lambda^p-1}$, which are strongly decreasing on $(1,\infty)$. Then $g_2(\lambda) > g_1(\lambda)$ for $\lambda > \lambda_0$ and $g_1(\lambda) > g_2(\lambda)$ for $1 < \lambda < \lambda_0$.

Lemma 2.1 is proved.

Assume that

$$\varphi_r^-(m,d) = \inf_{m<c\le d}\left\{\left(\sum_{i=c}^{d}\rho_i^{1-p'}\right)^{1-p} + \sum_{i=c}^{d-1} r_i\right\},$$

$$\varphi_r^+(d,n) = \inf_{d\le c<n}\left\{\left(\sum_{i=d}^{c}\rho_i^{1-p'}\right)^{1-p} + \sum_{i=d}^{c-1} r_i\right\},$$

$$B_{r,w} := B_{r,w}(m,n) = \sup_{m<t\le s<n}\left(\sum_{i=t}^{s-1} w_i\right)\left(\varphi_r^-(m,t) + \sum_{i=t}^{s-1} r_i + \varphi_r^+(s,n)\right)^{-1}.$$

Theorem 2.1. *Let* λ_0 *and* λ *be defined as in Lemma* 2.1. *Let* $0 \le m < n \le \infty$ *and* $1 < p < \infty$. *Inequality* (1.13) *holds if and only if* $B_{r,w}(m,n) < \infty$. *Moreover, the least constant in* (1.13) *satisfies*

$$B_{r,w} \le C \le 2\gamma_p\, B_{r,w}, \tag{2.1}$$

where

$$\gamma_p = \inf_{1<\lambda<\lambda_0} \frac{\lambda^p(\lambda^p - 1)}{(\lambda-1)^p}.$$

Proof. *Necessity.* Suppose that inequality (1.13) holds for all sequences $y \in \overset{\circ}{Y}(m,n)$ with the least constant $C > 0$. Let α, t, s and β be integers satisfying the condition $m < \alpha \le t \le s \le \beta < n$.

We introduce a test sequence $y = \{y_k\}$ as the following:

$$y_k = \begin{cases} \sum_{i=\alpha-1}^{k-1}\rho_i^{1-p'}\left(\sum_{i=\alpha-1}^{t-1}\rho_i^{1-p'}\right)^{-1}, & \alpha \le k \le t,\\ 1, & t \le k \le s,\\ \sum_{i=k}^{\beta}\rho_i^{1-p'}\left(\sum_{i=s}^{\beta}\rho_i^{1-p'}\right)^{-1}, & s \le k \le \beta,\\ 0, & m \le k < \alpha \quad \text{or} \quad \beta < k \le n. \end{cases}$$

It is obvious that $y \in \overset{\circ}{Y}(m,n)$.

Let us calculate Δy_k:

$$\Delta y_k = \begin{cases} \rho_k^{1-p'}\left(\sum_{i=\alpha-1}^{t-1}\rho_i^{1-p'}\right)^{-1}, & \alpha - 1 \le k \le t-1,\\ 0, & t \le k \le s,\\ -\rho_k^{1-p'}\left(\sum_{i=s}^{\beta}\rho_i^{1-p'}\right)^{-1}, & s \le k \le \beta,\\ 0, & m \le k < \alpha - 1 \quad \text{or} \quad \beta < k \le n. \end{cases}$$

Then we have

$$\sum_{i=m}^{n} \rho_i |\Delta y_i|^p = \left(\sum_{i=\alpha-1}^{t-1} \rho_i^{1-p'} \right)^{1-p} + \left(\sum_{i=s}^{\beta} \rho_i^{1-p'} \right)^{1-p}. \tag{2.2}$$

Moreover, simple calculations give that

$$\sum_{i=m}^{n} w_{i-1} |y_i|^p \geq \sum_{i=t}^{s} w_{i-1} = \sum_{i=t-1}^{s-1} w_i \tag{2.3}$$

and

$$\sum_{i=m}^{n} r_{i-1} |y_i|^p \leq \sum_{i=t}^{s} r_{i-1} = \sum_{i=t-1}^{s-1} r_i. \tag{2.4}$$

From (1.13), (2.2), (2.3) and (2.4), we get

$$\sum_{i=t-1}^{s-1} w_i \leq C \left(\left(\sum_{i=\alpha-1}^{t-1} \rho_i^{1-p'} \right)^{1-p} + \left(\sum_{i=s}^{\beta} \rho_i^{1-p'} \right)^{1-p} + \sum_{i=\alpha-1}^{\beta-1} r_i \right) =$$

$$= C \left(\left(\sum_{i=\alpha-1}^{t-1} \rho_i^{1-p'} \right)^{1-p} + \left(\sum_{i=s}^{\beta} \rho_i^{1-p'} \right)^{1-p} + \sum_{i=\alpha-1}^{t-2} r_i + \sum_{i=t-1}^{s-1} r_i + \sum_{i=s}^{\beta-1} r_i \right) \leq$$

$$\leq C \left(\left(\left(\sum_{i=\alpha-1}^{t-1} \rho_i^{1-p'} \right)^{1-p} + \sum_{i=\alpha-1}^{t-2} r_i \right) + \sum_{i=t-1}^{s-1} r_i + \right.$$

$$\left. + \left(\left(\sum_{i=s}^{\beta} \rho_i^{1-p'} \right)^{1-p} + \sum_{i=s}^{\beta-1} r_i \right) \right)$$

or

$$\sum_{i=t}^{s} w_i \leq C \left(\varphi_r^-(\alpha-1, t-1) + \sum_{i=t}^{s} r_i + \varphi_r^+(s, \beta) \right). \tag{2.5}$$

Since the left-hand side of (2.5) is independent of α, $m < \alpha \leq t$, and β, $s \leq \beta < n$, and the constant C is independent of t, s, $m < t \leq s < n$, we have

$$\left(\sum_{i=t-1}^{s-1} w_i \right) \left(\varphi_r^-(m, t-1) + \sum_{i=t-1}^{s-1} r_i + \varphi_r^+(s, n) \right)^{-1} \leq C$$

or

$$B_{r,w} \leq C. \tag{2.6}$$

Sufficiency. Let $B_{r,w} < \infty$. Without loss of generality, we assume that $y = \{y_i\} \in \overset{\circ}{Y}(m, n)$ and $y_i \geq 0$ for $i = 0, 1, 2, \ldots$. Let $\lambda > 1$. For any integer k we define the set $T_k := \{i :$

$y_i > \lambda^k\}$. Since the set $\{y_i\}$ is bounded, there exists an integer number $\tau = \tau(y,\lambda)$ such that $T_\tau \neq \varnothing$ and $T_{\tau+1} = \varnothing$. Let $\Delta T_k = T_k \backslash T_{k+1}$. Then

$$[m,n] = \bigcup_{k=-\infty}^{\tau} T_k = \bigcup_{k=-\infty}^{\tau} \Delta T_k. \tag{2.7}$$

Remember that $[m,n] = [m,\infty)$ for $n = \infty$. From the definition of T_k and the condition $T_\tau \neq \varnothing$, we have $T_k \neq \varnothing$ for all $k \leq \tau$. Let $k < \tau$. We present the set T_k as $T_k = \bigcup\limits_j [t_k^j, s_k^j]$, where $[t_k^j, s_k^j] \bigcap [t_k^i, s_k^i] = \varnothing$ for $i \neq j$. Let $M_k^j = T_{k+1} \bigcap [t_k^j, s_k^j]$ and $\Omega_k = \{j : M_k^j \neq \varnothing\}$. For $j \in \Omega_k$ we define $x_k^j = \min M_k^j$ and $z_k^j = \max M_k^j$. It is obvious that $t_k^j \leq x_k^j$ and $z_k^j \leq s_k^j$. Moreover,

$$T_{k+1} \subset \bigcup_{j\in\Omega_k} [x_k^j, z_k^j] \qquad \text{and} \qquad \Delta T_k \supset \bigcup_{j\in\Omega_k} \left([t_k^j, x_k^i - 1] \bigcup [z_k^j + 1, s_k^j]\right). \tag{2.8}$$

Let $t_k^j < x_k^j$. Then

$$y_{t_k^j - 1} \leq \lambda^k \qquad \text{and} \qquad y_{x_k^j} > \lambda^{k+1}.$$

Hence,

$$\lambda^k(\lambda - 1) = \lambda^{k+1} - \lambda^k \leq y_{x_k^j} - y_{t_k^j - 1} = \sum_{i=t_k^j}^{x_k^j - 1} \Delta y_i \leq \left(\sum_{i=t_k^j}^{x_k^j - 1} \rho_i^{1-p'}\right)^{\frac{1}{p'}} \left(\sum_{i=t_k^j}^{x_k^j - 1} \rho_i |\Delta y_i|^p\right)^{\frac{1}{p}}. \tag{2.9}$$

From (2.9), we obtain

$$\lambda^{pk} \left(\sum_{i=t_k^j}^{x_k^j - 1} \rho_i^{1-p'}\right)^{1-p} \leq \frac{1}{(\lambda-1)^p} \sum_{i=t_k^j}^{x_k^j - 1} \rho_i |\Delta y_i|^p. \tag{2.10}$$

Similarly, for $z_k^j < s_k^j$, we have

$$\lambda^{pk} \left(\sum_{i=z_k^j}^{s_k^j} \rho_i^{1-p'}\right)^{1-p} \leq \frac{1}{(\lambda-1)^p} \sum_{i=z_k^j}^{s_k^j} \rho_i |\Delta y_i|^p. \tag{2.11}$$

Let $z_k^j = s_k^j$. Since $y_{s_k^j} = y_{z_k^j} > \lambda^{k+1}$ and $y_{s_k^j + 1} = y_{z_k^j + 1} \leq \lambda^k$, then

$$\lambda^k(\lambda - 1) \leq y_{z_k^j} - y_{z_k^j + 1} = -\Delta y_{z_k^j} = \sum_{i=z_k^j}^{s_k^j} (-\Delta y_i). \tag{2.12}$$

From (2.12) it follows that

$$\lambda^{pk} \rho_{z_k^j} = \lambda^{pk} \left(\sum_{i=z_k^j}^{s_k^j} \rho_i^{1-p'}\right)^{1-p} \leq \frac{\rho_{z_k^j}}{(\lambda-1)^p} |\Delta y_{z_k^j}|^p = \frac{1}{(\lambda-1)^p} \sum_{i=z_k^j}^{s_k^j} \rho_i |\Delta y_i|^p. \tag{2.13}$$

Again similarly, for $t_k^j = x_k^j$, we have

$$\lambda^{pk}\rho_{x_k^j-1} = \lambda^{pk}\left(\sum_{i=t_k^j-1}^{x_k^j-1}\rho_i^{1-p'}\right)^{1-p} \leq \frac{\rho_{x_k^j-1}}{(\lambda-1)^p}|\Delta y_{x_k^j-1}|^p = \frac{1}{(\lambda-1)^p}\sum_{i=t_k^j-1}^{x_k^j-1}\rho_i|\Delta y_i|^p. \tag{2.14}$$

If we join inequalities (2.10) and (2.14), we get

$$\lambda^{pk}\left(\sum_{i=\tilde{t}_k^j}^{x_k^j-1}\rho_i^{1-p'}\right)^{1-p} \leq \frac{1}{(\lambda-1)^p}\sum_{i=\tilde{t}_k^j}^{x_k^j-1}\rho_i|\Delta y_i|^p, \tag{2.15}$$

where $\tilde{t}_k^j = t_k^j$ for $t_k^j < x_k^j$ and $\tilde{t}_k^j = t_k^j - 1$ for $t_k^j = x_k^j$.

If we join inequalities (2.11) and (2.13), we obtain

$$\lambda^{pk}\left(\sum_{i=z_k^j}^{s_k^j}\rho_i^{1-p'}\right)^{1-p} \leq \frac{1}{(\lambda-1)^p}\sum_{i=z_k^j}^{s_k^j}\rho_i|\Delta y_i|^p. \tag{2.16}$$

Let us estimate the left-hand side of inequality (1.13). For $\Delta T_{k+1} \neq \varnothing$, we have

$$\sum_{i\in\Delta T_{k+1}} w_{i-1}|y_i|^p \leq \lambda^{p(k+2)}\sum_{i\in\Delta T_{k+1}} w_{i-1}. \tag{2.17}$$

In view of the assumption that any sum with respect to an empty set is equal to zero, inequality (2.17) is valid also for $\Delta T_{k+1} = \varnothing$.

We need the following equality:

$$\lambda^{pk} = (1-\lambda^{-p})\sum_{t=-\infty}^{k}\lambda^{pt}. \tag{2.18}$$

By using (2.7), (2.8), (2.17) and (2.18), we get

$$F \equiv \sum_{k=m}^{n} w_{k-1}|y_k|^p = \sum_{k=-\infty}^{\tau-1}\sum_{i\in\Delta T_{k+1}} w_{i-1}|y_i|^p \leq \sum_{k=-\infty}^{\tau-1}\lambda^{p(k+2)}\sum_{i\in\Delta T_{k+1}} w_{i-1} =$$

$$= \lambda^{2p}\sum_{k=-\infty}^{\tau-1}\lambda^{pk}\sum_{i\in\Delta T_{k+1}} w_{i-1} = \lambda^{2p}(1-\lambda^{-p})\sum_{k=-\infty}^{\tau-1}\sum_{i\in\Delta T_{k+1}} w_{i-1}\sum_{t=-\infty}^{k}\lambda^{pt} \leq$$

$$\leq \lambda^p(\lambda^p-1)\sum_{t=-\infty}^{\tau-1}\lambda^{pt}\sum_{k\geq t}\sum_{i\in\Delta T_{k+1}} w_{i-1} = \lambda^p(\lambda^p-1)\sum_{t=-\infty}^{\tau-1}\lambda^{pt}\sum_{i\in T_{t+1}} w_{i-1} \leq$$

$$\leq \lambda^p(\lambda^p-1)\sum_{k=-\infty}^{\tau-1}\lambda^{pk}\sum_{j\in\Omega_k}\sum_{i=x_k^j-1}^{z_k^j-1} w_i. \tag{2.19}$$

By the condition $B_p < \infty$, we obtain

$$\sum_{i=x_k^j-1}^{z_k^j-1} w_i \le B_p\left(\varphi_r^-(m, x_k^j-1) + \sum_{i=x_k^j-1}^{z_k^j-1} r_i + \varphi_r^+(z_k^j, n)\right). \tag{2.20}$$

It is obvious that

$$\varphi_r^-(m, x_k^j-1) \le \left(\sum_{i=\widetilde{t}_k^j}^{x_k^j-1} \rho_i^{1-p'}\right)^{1-p} + \sum_{i=\widetilde{t}_k^j-1}^{x_k^j-2} r_i \tag{2.21}$$

and

$$\varphi_r^+(z_k^j, n) \le \left(\sum_{i=z_k^j}^{s_k^j} \rho_i^{1-p'}\right)^{1-p} + \sum_{i=z_k^j}^{s_k^j-1} r_i. \tag{2.22}$$

By using (2.20), (2.21) and (2.22), we have

$$\sum_{i=x_k^j-1}^{z_k^j-1} w_i \le B_p\left(\left(\sum_{i=\widetilde{t}_k^j}^{x_k^j-1} \rho_i^{1-p'}\right)^{1-p} + \sum_{i=\widetilde{t}_k^j-1}^{s_k^j-1} r_i + \left(\sum_{i=z_k^j}^{s_k^j} \rho_i^{1-p'}\right)^{1-p}\right). \tag{2.23}$$

If we substitute (2.23) into (2.19), we get

$$F \le \lambda^p(\lambda^p-1)B_p \sum_{k=-\infty}^{\tau-1} \lambda^{pk} \sum_{j\in\Omega_k}\left(\left(\sum_{i=\widetilde{t}_k^j}^{x_k^j-1} \rho_i^{1-p'}\right)^{1-p} + \sum_{i=\widetilde{t}_k^j-1}^{s_k^j-1} r_i + \left(\sum_{i=z_k^j}^{s_k^j} \rho_i^{1-p'}\right)^{1-p}\right) \le$$

$$\le \lambda^p(\lambda^p-1)B_p\left[\sum_{k=-\infty}^{\tau-1}\sum_{j\in\Omega_k}\left(\lambda^{pk}\left(\sum_{i=\widetilde{t}_k^j}^{x_k^j-1} \rho_i^{1-p'}\right)^{1-p} + \lambda^{pk}\left(\sum_{i=z_k^j}^{s_k^j} \rho_i^{1-p'}\right)^{1-p}\right) + \right.$$

$$\left. + \sum_{k=-\infty}^{\tau-1} \lambda^{pk} \sum_{j\in\Omega_k}\sum_{i=\widetilde{t}_k^j}^{s_k^j} r_{i-1}\right] = \lambda^p(\lambda^p-1)B_p[F_1+F_2]. \tag{2.24}$$

By using (2.15) and (2.16), we obtain

$$F_1 = \sum_{k=-\infty}^{\tau-1}\sum_{j\in\Omega_k}\left(\lambda^{pk}\left(\sum_{i=\widetilde{t}_k^j}^{x_k^j-1} \rho_i^{1-p'}\right)^{1-p} + \lambda^{pk}\left(\sum_{i=z_k^j}^{s_k^j} \rho_i^{1-p'}\right)^{1-p}\right) \le$$

$$\le \frac{1}{(\lambda-1)^p}\sum_{k=-\infty}^{\tau-1}\sum_{j\in\Omega_k}\left(\sum_{i=\widetilde{t}_k^j}^{x_k^j-1} \rho_i|\Delta y_i|^p + \sum_{i=z_k^j}^{s_k^j} \rho_i|\Delta y_i|^p\right). \tag{2.25}$$

Assume that

$$\omega_k^+ = \left\{ j \in \Omega_k : t_k^j < x_k^j,\ \ z_k^j < s_k^j \right\}, \qquad \omega_{k,1} = \left\{ j \in \Omega_k : t_k^j = x_k^j,\ \ z_k^j < s_k^j \right\},$$

$$\omega_{k,2} = \left\{ j \in \Omega_k : t_k^j < x_k^j,\ \ z_k^j = s_k^j \right\}, \qquad \omega_k^- = \left\{ j \in \Omega_k : t_k^j = x_k^j,\ \ z_k^j = s_k^j \right\}.$$

Moreover,

$$\Delta_{k,1}^+ = \omega_k^+ \bigcup \omega_{k,2}, \qquad \Delta_{k,2}^+ = \omega_k^+ \bigcup \omega_{k,1}, \qquad \Delta_{k,1}^- = \omega_k^- \bigcup \omega_{k,1}, \qquad \Delta_{k,2}^- = \omega_k^- \bigcup \omega_{k,2}.$$

It is obvious that $\Omega_k = \omega_k^+ \bigcup \omega_{k,1} \bigcup \omega_{k,2} \bigcup \omega_k^-$. By the relation for ΔT_k, from (2.8) we have

$$\Delta T_k \supset \left(\bigcup_{j \in \Delta_{k,1}^+} [t_k^j, x_k^j - 1] \right) \bigcup \left(\bigcup_{j \in \Delta_{k,2}^+} [z_k^j + 1, s_k^j] \right). \tag{2.26}$$

Since

$$\sum_{j \in \Omega_k} \left(\sum_{i=\tilde{t}_k^j}^{x_k^j - 1} \rho_i |\Delta y_i|^p + \sum_{i=z_k^j}^{s_k^j} \rho_i |\Delta y_i|^p \right) =$$

$$= \sum_{j \in \omega_k^+} \left(\sum_{i=t_k^j}^{x_k^j - 1} \rho_i |\Delta y_i|^p + \sum_{i=z_k^j+1}^{s_k^j} \rho_i |\Delta y_i|^p + \rho_{z_k^j} |\Delta y_{z_k^j}|^p \right) +$$

$$+ \sum_{j \in \omega_{k,1}} \left(\rho_{x_k^j - 1} |\Delta y_{x_k^j - 1}|^p + \sum_{i=z_k^j+1}^{s_k^j} \rho_i |\Delta y_i|^p + \rho_{z_k^j} |\Delta y_{z_k^j}|^p \right) +$$

$$+ \sum_{j \in \omega_{k,2}} \left(\sum_{i=t_k^j}^{x_k^j - 1} \rho_i |\Delta y_i|^p + \rho_{z_k^j} |\Delta y_{z_k^j}|^p \right) + \sum_{j \in \omega_k^-} \left(\rho_{x_k^j - 1} |\Delta y_{x_k^j - 1}|^p + \rho_{z_k^j} |\Delta y_{z_k^j}|^p \right) =$$

$$= \left(\sum_{j \in \Delta_{k,1}^+} \sum_{i=t_k^j}^{x_k^j - 1} \rho_i |\Delta y_i|^p + \sum_{j \in \Delta_{k,2}^+} \sum_{i=z_k^j+1}^{s_k^j} \rho_i |\Delta y_i|^p \right) +$$

$$+ \left(\sum_{j \in \Delta_{k,1}^-} \rho_{x_k^j - 1} |\Delta y_{x_k^j - 1}|^p + \sum_{j \in \Delta_{k,2}^- \bigcup \Delta_{k,2}^+} \rho_{z_k^j} |\Delta y_{z_k^j}|^p \right) = F_{k,1} + F_{k,2},$$

from (2.25), we have

$$F_1 \le \frac{1}{(\lambda - 1)^p} \sum_{k=-\infty}^{\tau} (F_{k,1} + F_{k,2}). \tag{2.27}$$

On the basis of (2.26) we have that $F_{k,1} \le \sum_{i\in\Delta T_k} \rho_i|\Delta y_i|^p$. Hence,

$$\sum_{k=-\infty}^{\tau} F_{k,1} \le \sum_{k=-\infty}^{\tau} \sum_{i\in\Delta T_k} \rho_i|\Delta y_i|^p = \sum_{i=m}^{n} \rho_i|\Delta y_i|^p. \tag{2.28}$$

Since $t_k^j - 1 = x_k^j \le \lambda^k$ for $j \in \Delta_{k,1}^-$ and $z_k^j > \lambda^{k+1}$ for $j \in \Delta_{k,2}^- \bigcup \Delta_{k,2}^+$, then there exist integers $k_1 = k_1(k,j) < k$ and $k_2 = k_2(k,j) > k$ such that $x_k^j - 1 \in \Delta T_{k_1}$ for $j \in \Delta_{k,1}^-$ and $z_k^j \in \Delta T_{k_2}$ for $j \in \Delta_{k,2}^- \bigcup \Delta_{k,2}^+$. We note that $\Delta T_\tau = T_\tau$. Therefore,

$$\sum_{k=-\infty}^{\tau} F_{k,2} \le \sum_{k=-\infty}^{\tau} \sum_{i\in\Delta T_k} \rho_i|\Delta y_i|^p = \sum_{i=m}^{n} \rho_i|\Delta y_i|^p. \tag{2.29}$$

Thus, from (2.27), (2.28) and (2.29), we have

$$F_1 \le \frac{2}{(\lambda-1)^p} \sum_{i=m}^{n} \rho_i|\Delta y_i|^p. \tag{2.30}$$

Let us estimate F_2:

$$F_2 \le \sum_{k=-\infty}^{\tau} \lambda^{pk} \sum_{j\in\Omega_k} \sum_{i=\tilde{t}_k^j}^{s_k^j} r_{i-1} \le 2 \sum_{k=-\infty}^{\tau} \lambda^{pk} \sum_{i\in T_k} r_{i-1} = 2 \sum_{k=-\infty}^{\tau} \lambda^{pk} \sum_{t=-\infty}^{\tau} \sum_{i=\Delta T_t} r_{i-1} =$$

$$= 2 \sum_{t=-\infty}^{\tau} \sum_{i=\Delta T_t} r_{i-1} \sum_{k=-\infty}^{t} \lambda^{pt} = \frac{2}{1-\lambda^{-p}} \sum_{t=-\infty}^{\tau} \lambda^{pt} \sum_{i=\Delta T_t} r_{i-1} \le$$

$$\le \frac{2\lambda^p}{\lambda^p - 1} \sum_{t=-\infty}^{\tau} \sum_{i=\Delta T_t} r_{i-1}|y_i|^p = \frac{2\lambda^p}{\lambda^p-1} \sum_{i=m}^{n} r_{i-1}|y_i|^p. \tag{2.31}$$

If we combine (2.30) and (2.31) with (2.24), we get

$$F \le 2\lambda^p(\lambda^p - 1) B_{r,w} \max\left\{\frac{1}{(\lambda-1)^p},\ \frac{\lambda^p}{\lambda^p-1}\right\} \sum_{i=m}^{n} (\rho_i|\Delta y_i|^p + r_{i-1}|y_i|^p). \tag{2.32}$$

Since the left-hand side of inequality (2.32) is independent of $\lambda > 1$, by Lemma 2.1 we have

$$\sum_{i=m}^{n} w_k|y_k|^p \le$$

$$\le 2B_{r,w} \inf_{\lambda>1} \left\{\lambda^p(\lambda^p-1) \max\left\{\frac{1}{(\lambda-1)^p},\ \frac{\lambda^p}{\lambda^p-1}\right\}\right\} \sum_{i=m}^{n} (\rho_i|\Delta y_i|^p + r_{i-1}|y_i|^p) =$$

$$= 2B_{r,w} \min\left\{\inf_{1<\lambda<\lambda_0} \frac{\lambda^p(\lambda^p-1)}{(\lambda-1)^p},\ \inf_{\lambda\ge\lambda_0} \lambda^{2p}\right\} \sum_{i=m}^{n} (\rho_i|\Delta y_i|^p + r_{i-1}|y_i|^p) =$$

$$= \gamma_p B_{r,w} \sum_{i=m}^{n} (\rho_i |\Delta y_i|^p + r_{i-1}|y_i|^p).$$

Thus, inequality (1.13) is valid with the estimate

$$C \leq 2\gamma_p B_{r,w}, \tag{2.33}$$

where C is the least constant in (1.13). Inequalities (2.6) and (2.33) give (2.1).

Theorem 2.1 is proved.

Let us turn to inequality (1.9). In the case $r_i = 0,\ i = 1, 2, \ldots,$ we have the following theorem.

Theorem 2.2. *Let* $0 \leq m < n \leq \infty$ *and* $1 < p < \infty$. *Inequality* (1.9) *holds if and only if* $B_w(m, n) < \infty$. *Moreover, the least constant in* (1.9) *satisfies*

$$B_w \leq C \leq 2\widetilde{\gamma}_p\, B_w,$$

where

$$B_w := B_w(m,n) = \sup_{m<t\leq s<n} \left(\sum_{i=t}^{s-1} w_i\right) \left(\left(\sum_{i=m}^{t} \rho_i^{1-p'} \right)^{1-p} + \left(\sum_{i=s}^{n} \rho_i^{1-p'} \right)^{1-p} \right)^{-1}$$

and

$$\widetilde{\gamma}_p = \inf_{1<\lambda} \frac{\lambda^p(\lambda^p - 1)}{(\lambda - 1)^p}.$$

3. Main results. First, we study the oscillatory properties of equation (1.11) that follow from Theorems 1.1 and 2.1. Relation (2.1) obviously gives the following corollary.

***Corollary* 3.1.** *Let* $0 \leq m < n \leq \infty$ *and* $1 < p < \infty$. *If* (1.12) *holds, then* $B_{r,w} \leq 1$, *and if* $2\gamma_p B_{r,w} \leq 1$, *then* (1.12) *holds.*

Applying Corollary 3.1 and Theorem 1.1 to the problem of the conjugacy and disconjugacy of equation (1.11) on the interval $[m, n]$, we get the following theorem.

Theorem 3.1. *Let* $0 \leq m < n \leq \infty$ *and* $1 < p < \infty$. *Then*:

(i) *for the disconjugacy of equation* (1.11) *on the interval* $[m, n]$ *the condition* $B_{r,w} \leq 1$ *is necessary and the condition* $2\gamma_p B_{r,w} \leq 1$ *is sufficient*;

(ii) *for the conjugacy of equation* (1.11) *on the interval* $[m, n]$ *the condition* $2\gamma_p B_{r,w} > 1$ *is necessary and the condition* $B_{r,w} > 1$ *is sufficient.*

Proof. The statements (i) and (ii) are equivalent. Thus, we will prove only the statement (i).

If equation (1.11) is disconjugate on the interval $[m, n]$, then by Theorem 1.1 inequality (1.12) holds. Hence, by Corollary 3.1 we have $B_{r,w} \leq 1$.

Conversely, if $2\gamma_p B_{r,w} \geq 1$, then by Corollary 3.1 inequality (1.12) holds. Hence, by Theorem 1.1 equation (1.11) is disconjugate on the interval $[m, n]$.

The statement (i) is proved. Thus, Theorem 3.1 is also proved.

***Corollary* 3.2.** *Let* $0 \leq m < n \leq \infty$ *and* $1 < p < \infty$. *Then*:

(i) *if there exist integers* t *and* s, $m \leq t < s < n$, *such that*

$$\sum_{i=t}^{s-1} w_i > \varphi_r^-(m, t) + \sum_{i=t}^{s-1} r_i + \varphi_r^+(s, n)$$

holds, then equation (1.11) *is conjugate on the interval* $[m, n]$;

(ii) *if equation* (1.11) *is conjugate on the interval* $[m, n]$, *then there exist integers* t *and* s, $m \le t < s < n$, *such that*

$$\sum_{i=t}^{s-1} w_i > \frac{1}{2\gamma_p}\left[\varphi_r^-(m,t) + \sum_{i=t}^{s-1} r_i + \varphi_r^+(s,n)\right]$$

holds;

(iii) *if equation* (1.11) *is disconjugate on the interval* $[m, n]$, *then there exist integers* t *and* s, $m \le t < s < n$, *such that*

$$\sum_{i=t}^{s-1} w_i \le \varphi_r^-(m,t) + \sum_{i=t}^{s-1} r_i + \varphi_r^+(s,n)$$

holds.

We will present oscillation and nonoscillation results of equation (1.11).

Theorem 3.2. *Let* $1 < p < \infty$.

(i) *For equation* (1.11) *to be nonoscillatory the condition* $B_{r,w}(m,\infty) \le 1$ *is necessary and the condition* $2\gamma_p B_{r,w}(m,\infty) \le 1$ *is sufficient for some* $m \ge 0$.

(ii) *For equation* (1.11) *to be oscillatory the condition* $2\gamma_p \lim_{m\to\infty}\sup B_{r,w}(m,\infty) > 1$ *is necessary and the condition* $\lim_{m\to\infty}\sup B_{r,w}(m,\infty) > 1$ *is sufficient*.

Proof. The statement (i) directly follows from the statement (i) of Theorem 3.1. We will prove the statement (ii).

Let equation (1.11) be oscillatory. Then there exists an integer k, $0 \le k < \infty$, such that for all $m > k$ equation (1.11) is conjugate on the interval $[m,\infty)$. Therefore, by Theorem 3.1 we have that $2\gamma_p B_{r,w}(m,\infty) > 1$ for all $m > k$. Hence, it follows that $2\gamma_p \lim_{m\to\infty}\sup B_{r,w}(m,\infty) > 1$.

Conversely, let $\lim_{m\to\infty}\sup B_{r,w}(m,\infty) > 1$. Then there exists an increasing sequence of natural numbers $\{m_k\}_{k=1}^{\infty}$ such that $m_k \to \infty$ for $k \to \infty$ and $B_{r,w}(m_k,\infty) > 1$ for all $k \ge 1$. Then by Theorem 3.1 equation (1.11) is conjugate on the interval $[m_k,\infty)$ for all $k \ge 1$, i.e., for all $k \ge 1$ there exists a nontrivial solution of equation (1.11) that has at least two generalized zeros on the interval $[m_k,\infty)$. Hence, there exists a sequence $\{\tilde{m}_k\} \subset \{m_k\}$ such that on all intervals $[\tilde{m}_k, \tilde{m}_{k+1}-1]$ some nontrivial solution of equation (1.11) has two zeros. Then by Sturm's separation theorem [18] (Theorem 3) there exists a nontrivial solution of equation (1.11) that has at least one generalized zero on each interval $[m_k, m_{k+1}-1]$, $k \ge 1$. Thus, this solution of equation (1.11) is oscillatory.

Theorem 3.2 is proved.

From Theorem 3.2 we have the following corollary.

***Corollary* 3.3.** *Let* $1 < p < \infty$.

(i) *If there exist sequences of integers* m_k, t_k, *and* s_k, $k \ge 1$, *such that* $0 < m_k \le t_k < s_k$, $m_k \to \infty$ *for* $k \to \infty$ *and*

$$\sum_{i=t_k}^{s_k-1} w_i > \varphi_r^-(m_k,t_k) + \sum_{i=t_k}^{s_k-1} r_i + \varphi_r^+(s_k,\infty)$$

holds for a sufficiently large k, *then equation* (1.11) *is oscillatory.*

(ii) *If equation* (1.11) *is oscillatory, then there exist sequences of integers* $m_k,\ t_k$, *and* $s_k,\ k \geq 1$, *such that* $0 < m_k \leq t_k < s_k,\ m_k \to \infty$ *for* $k \to \infty$ *and*

$$\sum_{i=t_k}^{s_k-1} w_i > \frac{1}{2\gamma_p}\left(\varphi_r^-(m_k,t_k) + \sum_{i=t_k}^{s_k-1} r_i + \varphi_r^+(s_k,\infty)\right)$$

holds.

Now we turn to equation (1.1). We will remind that equation (1.1) is a partial case of equation (1.11). If we replace w by v^+ and r by v^-, we get the following theorems and corollaries.

Theorem 3.3. *Let* $0 \leq m < n \leq \infty$ *and* $1 < p < \infty$. *Then*:

(i) *for the disconjugacy of equation* (1.1) *on the interval* $[m,n]$ *the condition* $B_{v^-,v^+} \leq 1$ *is necessary and the condition* $2\gamma_p B_{v^-,v^+} \leq 1$ *is sufficient*;

(ii) *for the conjugacy of equation* (1.1) *on the interval* $[m,n]$ *the condition* $2\gamma_p B_{v^-,v^+} > 1$ *is necessary and the condition* $B_{v^-,v^+} > 1$ *is sufficient.*

***Corollary* 3.4.** *Let* $0 \leq m < n \leq \infty$ *and* $1 < p < \infty$. *Then*:

(i) *if there exist integers* t *and* s, $m \leq t < s < n$, *such that*

$$\sum_{i=t}^{s-1} v_i^+ > \varphi_{v^-}^-(m,t) + \sum_{i=t}^{s-1} v_i^- + \varphi_{v^-}^+(s,n)$$

or

$$\sum_{i=t}^{s-1} v_i > \varphi_{v^-}^-(m,t) + \varphi_{v^-}^+(s,n)$$

holds, *then equation* (1.1) *is conjugate on the interval* $[m,n]$;

(ii) *if equation* (1.1) *is conjugate on the interval* $[m,n]$, *then there exist integers* t *and* s, $m \leq t < s < n$, *such that*

$$\sum_{i=t}^{s-1} v_i^+ > \frac{1}{2\gamma_p}\left[\varphi_{v^-}^-(m,t) + \sum_{i=t}^{s-1} v_i^- + \varphi_{v^-}^+(s,n)\right]$$

holds;

(iii) *if equation* (1.1) *is disconjugate on the interval* $[m,n]$, *then there exist integers* t *and* s, $m \leq t < s < n$, *such that*

$$\sum_{i=t}^{s-1} v_i \leq \varphi_{v^-}^-(m,t) + \varphi_{v^-}^+(s,n)$$

holds.

Theorem 3.4. *Let* $1 < p < \infty$.

(i) *For equation* (1.1) *to be nonoscillatory the condition* $B_{v^-,v^+}(m,\infty) \leq 1$ *is necessary and the condition* $2\gamma_p B_{v^-,v^+}(m,\infty) \leq 1$ *is sufficient for some* $m \geq 0$.

(ii) *For equation* (1.1) *to be oscillatory the condition* $2\gamma_p \lim_{m\to\infty} \sup B_{v^-,v^+}(m,\infty) > 1$ *is necessary and the condition* $\lim_{m\to\infty} \sup B_{v^-,v^+}(m,\infty) > 1$ *is sufficient.*

***Corollary* 3.5.** *Let* $1 < p < \infty$.

(i) *If there exist sequences of integers* m_k, t_k, *and* s_k, $k \geq 1$, *such that* $0 < m_k \leq t_k < s_k$, $m_k \to \infty$ *for* $k \to \infty$ *and*

$$\sum_{i=t_k}^{s_k-1} v_i > \varphi^-_{v^-}(m_k, t_k) + \varphi^+_{v^-}(s_k, \infty)$$

holds for a sufficiently large k, *then equation* (1.1) *is oscillatory.*

(ii) *If equation* (1.1) *is oscillatory, then there exist sequences of integers* m_k, t_k, *and* s_k, $k \geq 1$, *such that* $0 < m_k \leq t_k < s_k$, $m_k \to \infty$ *for* $k \to \infty$ *and*

$$\sum_{i=t_k}^{s_k-1} v_i^+ > \frac{1}{2\gamma_p}\left(\varphi^-_{v^-}(m_k, t_k) + \sum_{i=t_k}^{s_k-1} v_i^- + \varphi^+_{v^-}(s_k, \infty)\right)$$

holds.

If starting from some number $n > 1$ we have $v_i^+ = 0$ for any $i > n$, then equation (1.1) is nonoscillatory. Therefore, for equation (1.1) to be oscillatory, it is necessary that for each natural $n > 1$ there exists an index $i_n > n$ such that $v_{i_n}^+ \neq 0$. Since $\operatorname{supp} v^+ \cap \operatorname{supp} v^- = \varnothing$, then from Corollary 3.5 we get a statement that answers the question about the influence of the positive part of the sequence v on the oscillation of equation (1.1) posed in Introduction.

***Corollary* 3.6.** *Let* $1 < p < \infty$. *Let* $\operatorname{supp} v^+ = \{i_1, i_2, \ldots, i_n, \ldots\}$ *and* $\lim_{n\to\infty} i_n = \infty$. *If*

$$\lim_{n\to\infty} \sup \frac{v_{i_n}^+}{\rho_{i_n} + \varphi^+_{v^-}(i_n + 1, \infty)} > 1,$$

then equation (1.1) *is oscillatory.*

Corollary 3.6 follows from the first part (i) of Corollary 3.5 for $m_k = t_k = i_n$ and $s_k = t_k + 1$.

From Theorems 2.2, 3.1 and 3.2 we get the results for equation (1.8). We note that the values B_w and $\widetilde{\gamma}_p$ used below are defined in Theorem 2.2.

Theorem 3.5. *Let* $0 \leq m < n \leq \infty$ *and* $1 < p < \infty$. *Then*:

(i) *for the disconjugacy of equation* (1.8) *on the interval* $[m, n]$ *the condition* $B_{wv} \leq 1$ *is necessary and the condition* $2\widetilde{\gamma}_p B_w \leq 1$ *is sufficient*;

(ii) *for the conjugacy of equation* (1.8) *on the interval* $[m, n]$ *the condition* $2\widetilde{\gamma}_p B_w > 1$ *is necessary and the condition* $B_w > 1$ *is sufficient*.

Theorem 3.6. *Let* $1 < p < \infty$.

(i) *For equation* (1.8) *to be nonoscillatory the condition* $B_w(m, \infty) \leq 1$ *is necessary and the condition* $2\widetilde{\gamma}_p B_w(m, \infty) \leq 1$ *is sufficient for some* $m \geq 0$.

(ii) *For equation* (1.8) *to be oscillatory the condition* $2\widetilde{\gamma}_p \lim_{m\to\infty} \sup B_w(m, \infty) > 1$ *is necessary and the condition* $\lim_{m\to\infty} \sup B_w(m, \infty) > 1$ *is sufficient.*

References

1. A. Z. Alimagambetova, R. Oinarov, *Two-sided estimates for solutions of a class of second-order nonlinear difference equations* (in Russian), Mat. Zh., **8**, № 3(29), 12 – 21 (2008).
2. G. Bennett, *Some elementary inequalities*, Quart. J. Math. Oxford Ser. (2), **38**, 401 – 425 (1987).
3. G. Bennett, *Some elementary inequalities, II*, Quart. J. Math. Oxford Ser. (2), **39**, 385 – 400 (1989).
4. G. Bennett, *Some elementary inequalities, III*, Quart. J. Math. Oxford Ser. (2), **42**, 149 – 174 (1991).

5. M. S. Braverman, V. D. Stepanov, *On the discrete Hardy's inequality*, Bull. London Math. Soc., **26**, 283 – 287 (1994).
6. O. Došlý, S. Fišnarová, *Summation characterization of the recessive solution for half-linear difference equations*, Adv. Difference Equat., **2009**, Article 521058 (2009); https://doi.org/10.1155/2009/521058.
7. O. Došlý, P. Řehák, *Nonoscillation criteries for half-linear second-order difference equations*, Comput. Math. Appl., **42**, 453 – 464 (2001).
8. H. A. El-Morshedy, *Oscillation and nonoscillation criteria for half-linear second order difference equations*, Dynam. Systems and Appl., **15**, 429 – 450 (2006).
9. P. Hasil, M. Veselý, *Oscillation constants for half-linear difference equations with coefficients having mean values*, Adv. Differerence Equat., **2015**, Article 210 (2015); https://doi.org/10.1186/s13662-015-0544-1.
10. J. Jiang, X. Tang, *Oscillation of second order half-linear difference equations, I*, Appl. Math. Sci., **8**, № 40, 1957 – 1968 (2014).
11. J. Jiang, X. Tang, *Oscillation of second order half-linear difference equations, II*, Appl. Math. Lett., **24**, 1495 – 1501 (2011).
12. A. Kalybay, D. Karatayeva, R. Oinarov, A. Temirkhanova, *Oscillation of a second order half-linear difference equation and the discrete Hardy inequality*, Electron. J. Qual. Theory Different. Equat., № 43, 1 – 16 (2017); http://dx.doi.org/10.14232/ejqtde.2017.1.43.
13. A. Kufner, L. Maligranda, L.-E. Persson, *The Hardy inequality. About its history and some related results*, Vydavatelský servis, Pilsen (2007).
14. R. Oinarov, A. P. Stikharnyi, *Boundedness and compactness criteria for a certain difference inclusion*, Math. Notes, **50**, Issue 5, 1130 – 1135 (1991).
15. R. Oinarov, K. Ramazanova, A. Tiryaki, *An extension of the weighted Hardy inequalities and its application to half-linear equations*, Taiwanese J. Math., **19**, № 6, 1693 – 1711 (2015); http://dx.doi.org/10.11650/tjm.19.2015.5764.
16. P. Řehák, *Hartman – Winter type lemma, oscillation, and conjugacy criteria for half-linear difference equations*, J. Math. Anal. and Appl., **252**, 813 – 827 (2000).
17. P. Řehák, *Half-linear discrete oscillation theory*, Electron. J. Qual. Theory Different. Equat., № 24, 1 – 14 (2000).
18. P. Řehák, *Oscillatory properties of second order half-linear difference equations*, Czechoslovak Math. J., **51**, № 126, 303 – 321 (2001).
19. P. Řehák, *Comparision theorems and strong oscillation in the half-linear discrete oscillation theory*, Rocky Mountain J. Math., **33**, № 1, 333 – 352 (2003).
20. C. Sturm, *Sur les equations differentielles lineares du second ordre*, J. Math. Pures et Appl., № 1, 106 – 186 (1836).
21. Y. G. Sun, *Oscillation and nonoscillation for half-linear second order difference equations*, Indian J. Pure and Appl. Math., **35**, № 2, 133 – 142 (2004).
22. E. Thandapani, K. Rarc, J. R. Graef, *Oscillation and comparison theorems for half-linear second order difference equations*, Comput. Math. Appl., **42**, 953 – 910 (2001).
23. M. Veselý, P. Hasil, *Oscillation and nonoscillation of asymptotically almost periodic half-linear difference equations*, Abstr. Appl. Anal., **2013**, Article 432936 (2013); http://dx.doi.org/10.1155/2013/432936.

Received 09.01.20